CONICS CHARACTERIZING THE GENERALIZED FIBONACCI AND LUCAS SEQUENCES WITH INDICES IN ARITHMETIC PROGRESSIONS

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ABSTRACT. In this paper, we determine the conics characterizing the generalized Fibonacci and Lucas sequences with indices in arithmetic progressions, generalizing work of Melham and McDaniel.

1. INTRODUCTION

The second order recurrence $\{W_n(a,b;p,q)\}$ is defined for $n>0$ by

$$W_{n+1} = pW_n - qW_{n-1} \tag{1.1}$$

in which $W_0 = a, W_1 = b$, where a, b, p, q are arbitrary integers.

In [3], Horadam showed that

$$qW_n^2 + W_{n+1}^2 - pW_nW_{n+1} + eq^n = 0$$

and

$$W_nW_{n+2} - W_{n+1}^2 = eq^n$$

where $e = pab - qa^2 - b^2$.

In [2], the authors considered all subsequences of sequence $\{W_n\}$ of the form $\{W_{kn}\}$ for any positive integer k. They also derived the recurrence formula and trigonometric factorizations for them.

As some special cases of $\{W_n\}$, denote $W_n(0,1;p,-1)$,$W_n(2,p;p,-1)$, $W_n(0,1;p,1)$ and $W_n(2,p;p,1)$ by U_n, V_n, u_n and v_n, respectively. Now we consider these sequences with indices in arithmetic progression for a positive integer k. From [2, 1], the recurrence relations for these sequences are given for $k,n>0$ by

$$U_{k(n+1)} = V_kU_{kn} + U_{k(n-1)}, \tag{1.2}$$

$$V_{k(n+1)} = V_kV_{kn} + V_{k(n-1)}, \tag{1.3}$$

$$u_{k(n+1)} = v_ku_{kn} - u_{k(n-1)},$$

$$v_{k(n+1)} = v_kv_{kn} - v_{k(n-1)}.$$

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The Binet forms of $\{U_{kn}\}$,$\{V_{kn}\}$, $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$U_{kn} = U_k \tfrac{\alpha^n-\beta^n}{a-\beta} \text{ and } V_{kn} = \alpha^n + \beta^n,$$
$$u_{kn} = u_k \tfrac{\gamma^n-\delta^n}{\gamma-\delta} \text{ and } v_{kn} = \gamma^n + \delta^n$$

where α, β and γ, δ are the roots of equations $x^2 - V_k x - 1 = 0$ and $x^2 - v_k x + 1 = 0$, respectively. Clearly $U_{kn} = W_n(0, U_k; V_k, -1)$,$V_{kn} = W_n(2, V_k; V_k, -1)$,$u_{kn} = W_n(0, u_k; v_k, 1)$ and $v_{kn} = W_n(2, v_k; v_k, 1)$.

From [8], we know that Lucas proved that if x and y are consecutive Fibonacci numbers, then (x, y) is a lattice point on one of the hyperbolas $y^2 - xy - x^2 = \pm 1$ and Wasteels proved the converse. Some authors [4, 7, 9] discussed the conics whose equations are satisfied by pairs of successive terms of Lucas sequences. In [5], McDaniel proved converses to several of the results of these writers. For example, he proved the following.

Theorem 1. *Let x and y be positive integers. The pair (x, y) is a solution of $y^2 - pxy - x^2 = \pm 1$ iff there exists a positive integer n such that $x = U_n$, $y = U_{n+1}$.*

In [6], the author generalized McDaniel' s results and gave some new results. For example, he proved the following.

Theorem 2. *If m is even, then the points with integer coordinates on the conics $y^2 - V_m xy + x^2 \mp U_m^2 = 0$ are precisely the pairs $\mp(U_n, U_{n+m})$.*

In this paper, we consider all given results on special conics mentioned below and then give more general results, generalizing work of Melham and McDaniel.

2. Some Preliminary Results

In this section, we give some results related to the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ for further steps. Throughout this paper, we denote V_k^2+4 and v_k^2-4 by D_1 and D_2, respectively. Note that

$$U_k^2\left(V_{km}^2 - 4\right) = D_1 U_{km}^2, \text{ if } m \text{ is even} \tag{2.1}$$
$$U_k^2\left(V_{km}^2 + 4\right) = D_1 U_{km}^2, \text{ if } m \text{ is odd} \tag{2.2}$$
$$U_{kn}V_{km} + V_{kn}U_{km} = 2U_{k(m+n)}, \tag{2.3}$$
$$U_{kn}V_{km} - V_{kn}U_{km} = \begin{cases} 2U_{k(n-m)} & \text{if } m \text{ is even,} \\ -2U_{k(n-m)} & \text{if } m \text{ is odd,} \end{cases} \tag{2.4}$$
$$U_k^2 V_{kn}V_{km} + \left(V_k^2+4\right)U_{kn}U_{km} = 2U_k^2 V_{k(m+n)}, \tag{2.5}$$
$$U_k^2 V_{kn}V_{km} - \left(V_k^2+4\right)U_{kn}U_{km} = \begin{cases} 2U_k^2 V_{k(n-m)} & \text{if } m \text{ is even,} \\ -2U_k^2 V_{k(n-m)} & \text{if } m \text{ is odd.} \end{cases} \tag{2.6}$$

Lemma 1. *The integer solutions of $D_1 x^2 + 4U_k^2 = y^2 U_k^2$ are precisely the pairs $(\pm U_{2kn}, \pm V_{2kn})$.*

Proof. Taking $D_1 = V_k^2 + 4$, $x = U_{2kn}$ in the equation $D_1 x^2 + 4U_k^2 = y^2 U_k^2$, we write

$$\begin{aligned}
\left(V_k^2 + 4\right) U_{2kn}^2 + 4U_k^2 &= \left(\left(\alpha^2 + \beta^2\right)^2 + 4\right) U_k^2 \left(\frac{\alpha^{2n} - \beta^{2n}}{a - \beta}\right)^2 + 4U_k^2 \\
&= U_k^2 \left(\alpha^{4n} + \beta^{4n} + 2\right) = U_k^2 V_{2kn}^2.
\end{aligned}$$

So one can see that $y = V_{2kn}$. The proof is also valid for the pair $\left(-U_{2kn}, -V_{2kn}\right)$. Thus the theorem is proven. □

Using the technique in Lemmas 1, the proofs of Lemmas 2, 3 and 4 can be easily obtained.

Lemma 2. *The integer solutions of $D_1 x^2 - 4U_k^2 = y^2 U_k^2$ are precisely the pairs $\left(\pm U_{k(2n+1)}, \pm V_{k(2n+1)}\right)$.*

Lemma 3. *If D_1 is square free, then the integer solutions of $D_1 U_k^2 \left(x^2 - 4\right) = y^2$ and $D_1 U_k^2 \left(x^2 + 4\right) = y^2$ are precisely the pairs $\left(\pm V_{2kn}, \pm D_1 U_{2kn}\right)$ and $\left(\pm V_{k(2n+1)}, \pm D_1 U_{k(2n+1)}\right)$, respectively.*

Lemma 4. *The integer solutions of $U_k^2 y^2 - D_1 x^2 = \pm 4U_k^2$ are precisely the pairs $\left(\pm U_{kn}, \pm V_{kn}\right)$.*

Similar to the above results, here we give some basic results related to $\{u_{kn}\}$ and $\{v_{kn}\}$:

$$u_k^2 \left(v_{km}^2 - 4\right) = D_2 u_{km}^2, \tag{2.7}$$

$$u_{kn} v_{km} + v_{kn} u_{km} = 2u_{k(m+n)}, \tag{2.8}$$

$$u_{kn} v_{km} - v_{kn} u_{km} = 2u_{k(n-m)}, \tag{2.9}$$

$$u_k^2 v_{kn} v_{km} + D_2 u_{kn} u_{km} = 2u_k^2 v_{k(m+n)}, \tag{2.10}$$

$$u_k^2 v_{kn} v_{km} - D_2 u_{kn} u_{km} = 2u_k^2 v_{k(n-m)}. \tag{2.11}$$

By the Binet forms of $\{u_{kn}\}$ and $\{v_{kn}\}$, we have the following results without proof.

Lemma 5. *The integer solutions of $D_2 x^2 + 4u_k^2 = u_k^2 y^2$ are precisely the pairs $\left(\pm u_{kn}, \pm v_{kn}\right)$.*

Lemma 6. *The integer solutions of $D_2 u_k^2 \left(x^2 - 4\right) = y^2$ are precisely the pairs $\left(\pm v_{kn}, \pm D_2 u_{kn}\right)$.*

Lemma 7. *The integer solutions of $u_k^2 y^2 - D_2 x^2 = 4u_k^2$ are precisely the pairs $\left(\pm u_{kn}, \pm v_{kn}\right)$.*

3. Conics Characterizing the Sequences $\{U_{kn}\}, \{V_{kn}\}, \{u_{kn}\}$ and $\{v_{kn}\}$

Conics characterizing the Fibonacci and Lucas sequences were given in [5] and [6]. Here we determine the conics characterizing the more generalized Fibonacci and Lucas sequences with indices in arithmetic progress.

Theorem 3. *If m is even, then the points with integer coordinates on the conics $y^2 - V_{km}xy + x^2 \mp U_{km}^2 = 0$ are precisely the pairs $\mp\left(U_{kn}, U_{k(n+m)}\right)$.*

Proof. First we consider the case $y^2 - V_{km}xy + x^2 + U_{km}^2 = 0$. Considering this equation as a quadratic equation in y, by (2.1), we get

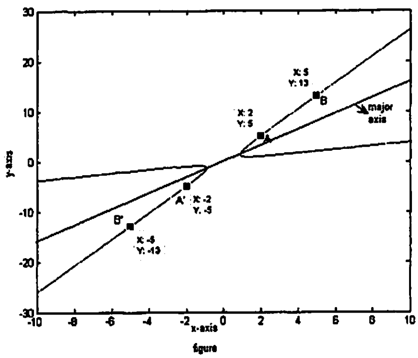
$$y = \left(V_{km}U_k x \pm U_{km}\sqrt{D_1x^2 - 4U_k^2}\right)/2U_k.$$

From Lemma 2, the integer points arise only when $x = U_{k(2n+1)}$. Thus

$$y = \left(V_{km}U_{k(2n+1)} \pm U_{km}V_{k(2n+1)}\right)/2. \tag{3.1}$$

By (2.3) and (2.4), we get the integers points $(x,y) = \mp\left(U_{k(2n+1)}, U_{k(2n+1+m)}\right)$ and $(x,y) = \mp\left(U_{k(2n+1)}, U_{k(2n+1-m)}\right)$ for all integer n. Similarly the integers points on the conic $y^2 - V_{km}xy + x^2 - U_{km}^2 = 0$ are $(x,y) = \mp\left(U_{2kn}, U_{k(2n+m)}\right)$. Thus the proof is complete. □

For $k = 1, m = 2$, the points $\mp(F_3, F_5) = \mp(2,5), \mp(F_5, F_7) = \mp(5,13)$ are on the conic $y^2 - 3xy + x^2 + 1 = 0$. We illustrate these points in the following figure:



Considering the proof method of Theorem 3, we give the following Theorems without proof.

Theorem 4. *For odd m, the points with integer coordinates on the conics $y^2 - V_{km}xy - x^2 \mp U_{km}^2 = 0$ are precisely the pairs $\mp\left(U_{kn}, U_{k(n+m)}\right)$.*

Theorem 5. *For even m and square free D_1, the points with integer coordinates on the conics $U_k^2y^2 - V_{km}U_k^2xy + U_k^2x^2 \mp D_1U_{km}^2 = 0$ are precisely the pairs $\mp\left(V_{kn}, V_{k(n+m)}\right)$.*

Theorem 6. *For odd m and square free D_1, the points with integer coordinates on the conics $U_k^2y^2 - V_{km}U_k^2xy - U_k^2x^2 \mp D_1U_{km}^2 = 0$ are precisely the pairs $\mp\left(V_{kn}, V_{k(n+m)}\right)$.*

Theorem 7. *For all m, the points with integer coordinates on the conics $y^2 - v_{km}xy + x^2 - u_{km}^2 = 0$ are precisely the pairs $\mp\left(u_{kn}, u_{k(n+m)}\right)$.*

Theorem 8. *For all m and square free D_2, the points with integer coordinates on the conics $u_k^2y^2 - v_{km}u_k^2xy + u_k^2x^2 + D_2u_{km}^2 = 0$ are precisely the pairs $\mp\left(v_{kn}, v_{k(n+m)}\right)$.*

Clearly Theorems 4-8 are the general cases of the result of Melham [6] for $k = 1$.

4. Diophantine Representations of the Sequences

The set of terms of any Lucas sequence is a recursively enumerable set, and such sets have been shown to be Diophantine [10] . That is, for each recursively enumerable set S, there exists a polynomial P with integral coefficients in variables $x_1, x_2, \ldots, x_n$, such that $x \in S$ iff there exist positive integers $y_1, y_2, \ldots, y_{n-1}$ such that $P(x, y_1, y_2, \ldots, y_{n-1}) = 0$. As a consequence, it is possible to construct a polynomial whose positive values are precisely the elements of S. The construction is due to Putnam [11], who observed that $x\left(1 - P^2\right)$ has the desired property. In [5], the authors considered the mentioned facts and obtained such polynomials for the set of the sequences $\{U_n\}, \{V_n\}, \{u_n\}$ and $\{v_n\}$. From the results of Theorems 4,6,7,8 and Lemmas 4, 7, we obtain such polynomials for the set of terms of sequences $\{U_{kn}\}, \{V_{kn}\}, \{u_{kn}\}$ and $\{v_{kn}\}$ as generalizations of results of [5].

Let $\mathcal{F}(V_k, -1), \mathcal{F}(v_k, 1), \mathcal{L}(V_k, -1)$ and $\mathcal{L}(v_k, 1)$ be the set of terms of sequences $\{U_{kn}\}, \{u_{kn}\}, \{V_{kn}\}$ and $\{v_{kn}\}$, respectively.

Theorem 9. *Then, if x and y assume all positive integral values, the set S is identical to the set of positive values of the polynomial*

(i) $x\left[2 - \frac{1}{U_k^4}\left(y^2 - V_kxy - x^2\right)^2\right]$ if $S = \mathcal{F}(V_k, -1)$,

(ii) $x\left[2 - \frac{1}{u_k^2}\left(y^2 - v_kxy + x^2\right)\right]$ if $S = \mathcal{F}(v_k, 1)$, $v_1 = p > 2$,

(iii) $y\left[1 - \left(\left(y^2 - D_1x^2\right)^2 - 16U_k^4\right)^2\right]$ if $S = \mathcal{L}(V_k, -1)$, $D_1 = V_k^2 + 4$,

(iv) $y\left[1 - \left(\left(y^2 - D_2x^2\right) - 4u_k^2\right)^2\right]$ if $S = \mathcal{L}(v_k, 1)$, $D_2 = v_k^2 - 4$.

Proof. From the special cases of Theorems 4, 7 for $m = 1$ and Lemmas 4, 7, the proof is obvious. We show that $y^2 - V_kxy - x^2$ and $y^2 - v_kxy + x^2$ are never 0 for x and y integers. However, if either equals 0, we write $y = \left(V_k \pm \sqrt{V_k^2 + 4}\right)/2$, $y = \left(v_k \pm \sqrt{v_k^2 - 4}\right)/2$, respectively. Since neither $V_k^2 + 4$ not $v_k^2 - 4$ is a square, y is irrational for all integral x values. □

By Lemmas 4, 7, the polynomials in (i) and (ii) can also be given:

$$x\left[1-\left(\left(y^2-D_1x^2\right)^2-16U_k^4\right)^2\right] \text{ for } D_1 = V_k^2+4,$$

$$x\left[1-\left(\left(y^2-D_2x^2\right)-4u_k^2\right)^2\right] \text{ for } D_2 = v_k^2-4,$$

respectively. By special cases of Theorems 6, 8 for $m=1$, the polynomials in (iii) and (iv) can be given, alternatively, if D_1 and D_2 are square free, as

$$x\left[1-\left(\left(y^2-V_kxy-x^2\right)^2-D_1^2\right)^2\right]$$

and

$$x\left[1-\left(y^2-v_kxy+x^2+D_2\right)^2\right],$$

respectively.

When $k=1$ in given results throughout Section 4, the results of [5] can be derived.

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