[1, 1, 2]-COLORINGS OF K_5 AND K_6

Changqing Xu[1] Xiaojun Wang[1] Yatao Du[2]
1 Department of Applied Mathematics, Hebei University of Technology, Tianjin 300401, China
2 Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China

ABSTRACT. Given non-negative integers r, s, and t, an $[r, s, t]$-coloring of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring. We prove that $\chi_{1,1,2}(K_5) = 7$ and $\chi_{1,1,2}(K_6) = 8$.

Keywords coloring, $[r, s, t]$-coloring, $[r, s, t]$-chromatic number.
MCS: 05C15

1. INTRODUCTION

Vertex coloring, edge coloring and total coloring are three classical colorings of graphs. Kemnitz and Marangio generalized these colorings to $[r, s, t]$-colorings ([1]).

Definition 1. ([1]) Given non-negative integers r, s, and t, an $[r, s, t]$-coloring of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring.

It is easy to see that a $[1, 0, 0]$-coloring is an ordinary vertex coloring, a $[0, 1, 0]$-coloring is an edge coloring, and a $[1, 1, 1]$-coloring is an ordinary total coloring. $[r, s, t]$-colorings are discussed in [1]-[4], and many results are

This research was supported by HENSF(A2007000002) and NNSF(10871058) of China.

obtained. It is difficult to determine $\chi_{r,s,t}(G)$ of a graph without constraint on r, s, t. In [1], the authors discuss $[r, s, t]$-colorings of complete graphs K_p, and determine $\chi_{r,s,t}(K_p)$ in many cases with $\min\{r, s, t\} \geq 1$. The smallest case with $\min\{r, s, t\} \geq 1$ which is not covered is $\chi_{1,1,2}(K_p)$. And they get that

Proposition 2. [1] *If p is odd, then $p + 1 \leq \chi_{1,1,2}(K_p) \leq 2p - 1$. If p is even, then $p + 1 \leq \chi_{1,1,2}(K_p) \leq 2p$.*

If $p = 3$ or 4, they have $\chi_{1,1,2}(K_3) = 5$ and $\chi_{1,1,2}(K_4) = 7$ where the upper bound is achieved.

In this paper we show that $\chi_{1,1,2}(K_5) = 7$ and $\chi_{1,1,2}(K_6) = 8$.

2. Our Main Results

In a vertex and edge coloring of a graph G, we write $i(c_1, c_2, \cdots, c_k)$ to denote a vertex with color i and its incident edges with colors from the set $\{c_1, \cdots, c_k\}$.

Theorem 3. *If $p \geq 2$, then $\chi_{1,1,2}(K_p) \geq p + 2$.*

Proof. If $p = 2$, obviously a $[1, 1, 2]$-coloring of K_2 needs at least 4 colors.

Now suppose that $p \geq 3$. Let c be a coloring of vertices and edges of K_p, whose color set is $\{0, 1, \cdots, p\}$. If c is a $[1, 1, 2]$-coloring of K_p, then the possible colors of the vertices are: $0(2, 3, \cdots, p)$, $1(3, 4, \cdots, p)$, $2(0, 4, \cdots, p)$, $\cdots$, $(p - 1)(0, 1, \cdots, p - 3)$, $p(0, 1, \cdots, p - 2)$. Since K_p is a complete graph, each vertex receives distinct colors. So at least $p - 2$ colors in $\{1, 2, \cdots, p - 1\}$ are used. Without loss of generality, suppose that one vertex v receives color 1. The color set for its incident edges is $\{3, 4, \cdots, p\}$, but there are $p - 1$ edges incident to v, a contradiction. It follows that c isn't a $[1, 1, 2]$-coloring of K_p. Thus $\chi_{1,1,2}(K_p) \geq p + 2$.

□

Theorem 4. $\chi_{1,1,2}(K_5) = 7$*, and there exist* 4 *different* $[1, 1, 2]$*-colorings of* K_5.

Proof. By Theorem 3, $\chi_{1,1,2}(K_5) \geq 7$. Next we will show that $\chi_{1,1,2}(K_5) \leq 7$.

Let c be a coloring of vertices and edges of K_5 with color set $\{0, 1, \cdots, 6\}$. If c is a $[1, 1, 2]$-coloring of K_5, then the possible colors of vertices are: $0(2, 3, 4, 5, 6)$, $1(3, 4, 5, 6)$, $2(0, 4, 5, 6)$, $3(0, 1, 5, 6)$, $4(0, 1, 2, 6)$, $5(0, 1, 2, 3)$ and $6(0, 1, 2, 3, 4)$. According to the colors of vertices we consider the following cases.

Case 1. The coloring uses $1(3, 4, 5, 6)$, $2(0, 4, 5, 6)$, $3(0, 1, 5, 6)$, $4(0, 1, 2, 6)$. Each edge color must appear an even number of times in the five edge color sets. Considering color 1, the last vertex can't be colored with

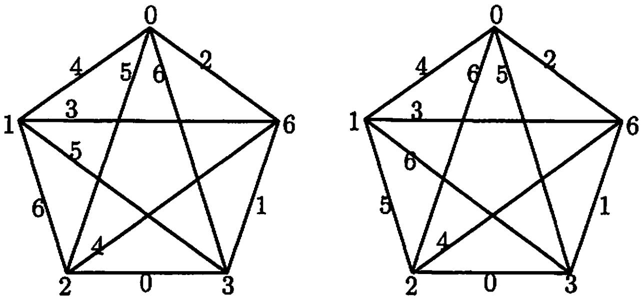

FIGURE 1. Two [1, 1, 2]-colorings of K_5.

5(0, 1, 2, 3). And considering color 0 and 1, the last vertex must be colored with 6(0, 2, 3, 4). Now color 4 appears 3 times in the 5 edge color sets, a contradiction.

Case 2. The coloring uses 1(3, 4, 5, 6), 2(0, 4, 5, 6), 3(0, 1, 5, 6), 5(0, 1, 2, 3). Considering color 0, the last vertex must be colored with 6. Now color 5 appears 3 times in the 5 edge color sets, a contradiction.

Case 3. The coloring uses 1(3, 4, 5, 6), 2(0, 4, 5, 6), 4(0, 1, 2, 6), 5(0, 1, 2, 3). Considering color 0, the last vertex must be colored with 6. Now color 6 appears 3 times in the 5 edge color sets, a contradiction. With the same argument, the coloring can't use 1(3, 4, 5, 6), 3(0, 1, 5, 6), 4(0, 1, 2, 6), 5(0, 1, 2, 3).

Case 4. The coloring uses 2(0, 4, 5, 6), 3(0, 1, 5, 6), 4(0, 1, 2, 6), 5(0, 1, 2, 3). Considering color 1, the last vertex must be colored with 6, while color 6 appears 3 times in the 5 edge color sets, a contradiction.

So if c is a [1, 1, 2]-coloring of K_5, two vertices must be colored with 0(2, 3, 4, 5, 6) and 6(0, 1, 2, 3, 4).

Case 5. The coloring uses 0(2, 3, 4, 5, 6), 1(3, 4, 5, 6), 2(0, 4, 5, 6), 3(0, 1, 5, 6), 6(0, 1, 2, 3, 4). Considering color 0 and 3, the vertices with color 6 and 0 have edge color sets $\{1, 2, 3, 4\}$ and $\{2, 4, 5, 6\}$, respectively. In this case we get two [1, 1, 2]-colorings of K_5 (see Figure 1).

Case 6. The coloring uses 0(2, 3, 4, 5, 6), 1(3, 4, 5, 6), 2(0, 4, 5, 6), 4(0, 1, 2, 6), 6(0, 1, 2, 3, 4). Considering color 0 and 5, the vertices with color 6 and 0 have edge color sets $\{1, 2, 3, 4\}$ and $\{2, 3, 4, 6\}$, respectively. Therefore color 2 appears an odd number of times in the 5 edge color sets, a contradiction. Similarly, in the coloring the vertex color set can't be: $\{0, 1, 2, 5, 6\}$; $\{0, 1, 3, 4, 6\}$; $\{0, 1, 3, 5, 6\}$; $\{0, 1, 4, 5, 6\}$; $\{0, 2, 3, 4, 6\}$; $\{0, 2, 3, 5, 6\}$; $\{0, 2, 4, 5, 6\}$.

Case 7. The coloring uses 0(2, 3, 4, 5, 6), 3(0, 1, 5, 6), 4(0, 1, 2, 6), 5(0, 1, 2, 3), 6(0, 1, 2, 3, 4). Considering color 6 and 3, the vertices with color 0 and

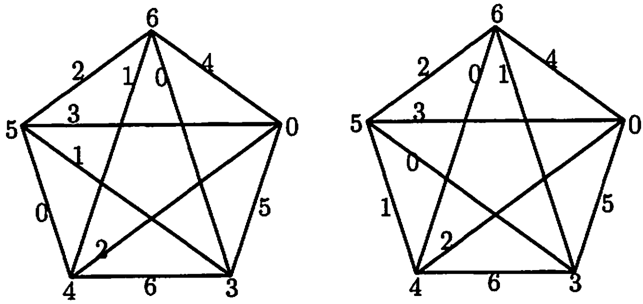

FIGURE 2. The other two [1, 1, 2]-colorings of K_5.

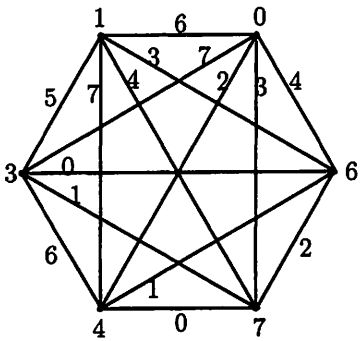

FIGURE 3. A [1, 1, 2]-coloring of K_6.

6 have edge color sets $\{2,3,4,5\}$ and $\{0,1,2,4\}$. In this case we get two $[1,1,2]$-colorings of K_5 (see Figure 2).

□

In the two colorings of Figure 1, changing each color i to $|6-i|$, we can get the two colorings of Figure 2.

Theorem 5. $\chi_{1,1,2}(K_6) = 8$.

Proof. By Theorem 3, $\chi_{1,1,2}(K_6) \geq 8$. The following coloring shows that $\chi_{1,1,2}(K_6) = 8$. The corresponding coloring is: $0(2,3,4,6,7)$, $1(3,4,5,6,7)$, $3(0,1,5,6,7)$, $4(0,1,2,6,7)$, $6(0,1,2,3,4)$, $7(0,1,2,3,4)$ (see Figure 3).

□

Note. We have been informed that the $[1,1,2]$-chromatic numbers of K_5, K_6, K_7, K_8, were obtained by Juliane Lehmann in her diploma thesis

"$[r,s,t]$-colorings of stars and complete graphs", at the Technical University of Braunschweig in 2006.

Acknowledgments. The authors would like to thank the referee for his detailed and helpful suggestions.

References

[1] A. Kemnitz, M. Marangio. $[r,s,t]$-Coloring of graphs. Discrete Math., 2007, 307: 199-207.

[2] A. Kemnitz, M. Marangio, P. Mihok. $[r,s,t]$-Chromatic and hereditary properties of graphs. Discrete Math., 2007, 307: 916-922.

[3] L. Dekar, B. Effantin, H. Kheddouci. $[r,s,t]$-Coloring of trees and bipartite graphs. Discrete Math., 2008, doi: 10.1016/j.disc.2008.09.021.

[4] C. Xu, X. Ma, S. Hua. $[r,s,t]$-coloring of $K_{n,n}$. J. of Appl. Math. and Computing, 2009, 31(1): 45-52.

E-mail address: chqxu@hebut.edu.cn