G-decomposition of λK_v, where G has six vertices and nine edges*

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Abstract

Let λK_v be the complete multigraph with v vertices. Let G be a finite simple graph. A G-decomposition of λK_v, denoted by G-$GD_\lambda(v)$, is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. In this paper, nine graphs G_i with six vertices and nine edges are discussed, and the existence of G_i-$GD_\lambda(v)$ is given, $1 \leq i \leq 9$.

Key words: G-decomposition; G-design; holey G-design; quasigroup

1 Introduction

A *complete graph* of order v with index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. Let G be a finite simple graph. A *G-design or G-decomposition* of λK_v, denoted by G-$GD_\lambda(v)$, is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. The necessary conditions for the existence of G-$GD_\lambda(v)$ are

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$v \geq |V(G)|$, $\lambda v(v-1) \equiv 0 \pmod{2|E(G)|}$, $\lambda(v-1) \equiv 0 \pmod{d}$, where $V(G)$ and $E(G)$ denotes the set of vertices and edges of G respectively, d is the greatest common divisor of the degrees of all vertices in G.

Let $\lambda K_{n_1,n_2,\cdots,n_t}$ be a complete multipartite graph with index λ, consisting of t parts with vertex set $X = \bigcup_{i=1}^{t} X_i$, where these X_i are disjoint and $|X_i| = n_i$, $1 \leq i \leq t$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G} = \{X_1, X_2, \cdots, X_t\}$. For a given graph G, if the edges of $\lambda K_{n_1,n_2,\cdots,n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to G and is called as *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by G-$HD_\lambda(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the type of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. For $\lambda = 1$, the index 1 is often omitted. A G-$HD_\lambda(1^{v-w} w^1)$ is called an *incomplete G-design*, denoted by G-$ID_\lambda(v, w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a G-$GD_\lambda(v)$ is a G-$HD_\lambda(1^v)$ or a G-$ID_\lambda(v, w)$ with $w = 0$ or 1. It is easy to see that, if there exists a G-$GD_\lambda(v)$, or a G-$HD_\lambda(T)$, or a G-$ID_\lambda(v, w)$, then there exists a G-$GD_{\lambda\mu}(v)$, or a G-$HD_{\lambda\mu}(T)$, or a G-$ID_{\lambda\mu}(v, w)$ for any positive integer μ.

For the path P_k, the star $K_{1,k}$ and the cycle C_k, the existence problem of P_k-$GD(v)$, $K_{1,k}$-$GD(v)$ and C_k-$GD(v)$ have been solved[1,5,10]. The graph design problem for some of other graphs, e.g. k-cube[18], cycle with one chord[4,17] and so on[19], have been already researched. In other hand, for the graphs with less vertices and less edges, the existence of their graph design has already been solved[2,3,6,11–15,20]. For the graphs with six vertices and nine edges, there are twenty graphs without isolated vertices (see the Appendix I in [9]). The designs of nine graphs among the twenty graphs with six vertices and nine edges for $\lambda = 1$ have been given by the following lemma.

Lemma 1.[16] *There exists a G_k-$GD(v)$ if and only if $v \equiv 0, 1 \pmod 9$ for $4 \leq k \leq 9$ or $v \equiv 1, 9 \pmod{18}$ for $k = 1$ or $v \equiv 1 \pmod 9$ for $k = 2, 3$ with the exceptions $(v, k) \in \{(9,4), (9,5), (9,6), (9,8), (9,9), (10,2)\}$.*

In this paper, we give several methods to construct the nine graph designs for $\lambda > 1$. We shall prove that the necessary conditions for the existence of G_i-$GD_\lambda(v)$ with $\lambda > 1$ are also sufficient for any G_i $(1 \leq i \leq 9)$ with some exceptions. The main way to get all constructions is the following lemma.

Lemma 2.[12] *For given graph G and positive integers h, w, m, λ, if there exist a G-$HD_\lambda(h^m)$, a G-$ID_\lambda(h+w, w)$ and a G-$GD_\lambda(w)$ (or a G-$GD_\lambda(h+$*

w)), then there exists a $G\text{-}GD_\lambda(mh+w)$.

The discussed nine graphs are listed as follows. For convenience, as a block in graph design, each graph may be denoted by (a,b,c,d,e,f) according to the following vertex-labels.

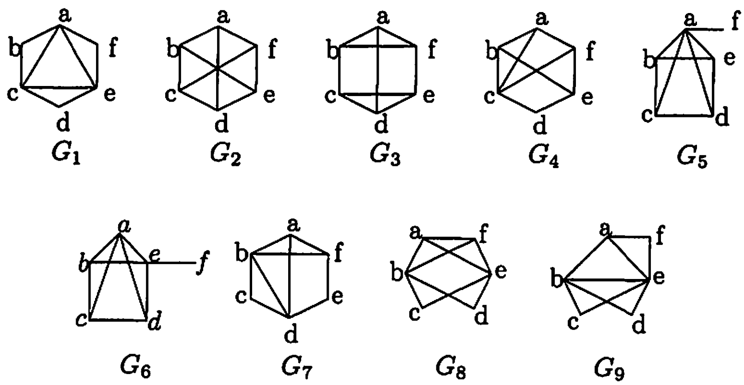

Figure 1: Nine graphs with six vertices and nine edges

The necessary conditions for the existence of $G_i\text{-}GD_\lambda(v)$ are

Table 1: Necessary conditions

i	λ	necessary conditions	i	λ	necessary conditions
1	1	$v \equiv 1,9 \pmod{18}$	2, 3	1	$v \equiv 1 \pmod 9$
	2	$v \equiv 0,1 \pmod 9$		3	$v \equiv 0,1,3,4,6,7 \pmod 9$
	3	$v \equiv 1,3 \pmod 6$		9	any v
	6	$v \equiv 0,1 \pmod 3$	4, 5	1	$v \equiv 0,1 \pmod 9$
	9	$v \equiv 1,3,5 \pmod 6$	6, 7	3	$v \equiv 0,1,3,4,6,7 \pmod 9$
	18	any v	8, 9	9	any v

2 Construction of *HD*

A *pairwise balanced design* $B[K,1;v]$ is a pair $(V,\mathcal{B})$, where V is a v-set (*point set*) and $\mathcal{B}$ is a family of subsets (*blocks*) of V with block sizes from K such that every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$. When $K=\{k\}$, a $B[K,1;v]$=$B[k,1;v]$ is just a balanced incomplete block design.

Lemma 3.[12] *Let K be a set of positive integers, m, v, λ be positive integers, and G be a finite simple graph, if there exist a $B[K,1;v]$ and a $G\text{-}HD_\lambda(m^k)$ for any $k \in K$, then there exists a $G\text{-}HD_\lambda(m^v)$.*

A *quasigroup* is a set Q with a binary operation "$\cdot$", denoted by $(Q,\cdot)$, such that the equations $a\cdot x=b$ and $y\cdot a=b$ are uniquely solvable for every pair of elements $a,\ b\in Q$. It is well known that the multiplication table of a quasigroup defines a Latin square. On the contrary, a quasigroup can be obtained from a Latin square. A quasigroup is said to be *idempotent* (resp. *symmetric*) if the identity $x\cdot x=x$ (resp. $x\cdot y=y\cdot x$) holds for all $x\in Q$ (resp. $x,y\in Q$). Let S be a finite set and $H=\{S_1,S_2,\cdots,S_n\}$ be a partition of S. A *holey Latin square* with holes H is a $|S|\times|S|$ array L on S such that:

(1) every cell of L either contains an element of S or is empty;

(2) every element of S occurs at most once in any row or column of L;

(3) the subarrays indexed by $S_i\times S_i$ are empty for $1\le i\le n$;

(4) element $s\in S$ occurs in row (or column) t if and only if $(s,t)\in(S\times S)\backslash\bigcup_{i=1}^{n}(S_i\times S_i)$.

The type of L is the multiset $T=\{|S_i|:1\le i\le n\}$ and will be denoted by exponential notation. A holey symmetric quasigroup corresponding a holey symmetric Latin square with type T is denoted by $HSQ(T)=(S,\mathcal{H},\cdot)$. Two Latin squares L_1 and L_2 on a set S are said to be *orthogonal* if their superposition yields every ordered pair in $S\times S$. A Latin square is called *self-orthogonal* if it is orthogonal to its transpose. A self-orthogonal quasigroup corresponding to a self-orthogonal Latin square of order v is denoted by $SOQ(v)$. An idempotent SOQ is denoted by $ISOQ$.

Lemma 4.[7,8]

(1) *There exists an idempotent quasigroup of order v if and only if $v\neq 2$;*

(2) *There exists an idempotent symmetric quasigroup of order v if and only if v is odd;*

(3) *There exists an $HSQ(2^n)$ for all $n\ge 3$;*

(4) *There exists an $ISOQ(v)$ for $v\neq 2,3,6$.*

Let G be a finite simple graph, n,k,λ be positive integers, $\lambda|ke$, $e=|E(G)|$. In order to construct a holey graph design G-$HD_\lambda((\frac{ke}{\lambda})^n)$, we may take $Z_{\frac{ke}{\lambda}}\times I_n$ as the vertex set and $Z_{\frac{ke}{\lambda}}$ as the automorphism group of the block set, where $(I_n,\cdot)$ be an idempotent quasigroup on the set $I_n=\{1,2,\cdots,n\}$. A G-$HD_\lambda((\frac{ke}{\lambda})^n)$ consists of $\frac{\lambda\binom{n}{2}(\frac{ke}{\lambda})^2}{e}=\frac{ke}{\lambda}kn(n-1)/2$ blocks. For our methods, the range of the subscripts of $n(n-1)/2$ base block collections $A_{i,j}$ is taken as $1\le i<j\le n$, every $A_{i,j}$ contains k base blocks.

On the other hand, in order to construct a holey graph design G-$HD_\lambda((\frac{2ke}{\lambda})^n)$, we may take $Z_{\frac{ke}{\lambda}}\times I_{2n}$ as the vertex set, $Z_{\frac{ke}{\lambda}}$ as the automorphism group of the block set, where $I_{2n}=\{1,2,\cdots,2n\}$ and $(I_{2n},\mathcal{H},\cdot)$ forms an $HSQ(2^n)$ with holes $\mathcal{H}=\{\{2r-1,2r\}:1\le r\le n\}$, which exists for $n \geq 3$ by Lemma 2(3). In fact, for the original $G\text{-}HD_\lambda((\frac{2ke}{\lambda})^n)$, the vertex set $Z_{\frac{2ke}{\lambda}} \times I_n$ contains n holes with size $\frac{2ke}{\lambda}$: $H_i = Z_{\frac{2ke}{\lambda}} \times \{i\}$, $1 \leq i \leq n$. Now, halve each hole H_i into $\overline{H}_{2i-1}$ and $\overline{H}_{2i}$, where each $\overline{H}_j = Z_{\frac{ke}{\lambda}} \times \{j\}$ has size $\frac{ke}{\lambda}$, $1 \leq j \leq 2n$. Then, equivalently, the holes of the $G\text{-}HD_\lambda((\frac{2ke}{\lambda})^n)$ can be regarded as $\overline{H}_1, \overline{H}_2, \cdots, \overline{H}_{2n}$ with such restriction that there is no edge between $\overline{H}_{2i-1}$ and $\overline{H}_{2i}$, $1 \leq i \leq n$. A $G\text{-}HD_\lambda((\frac{2ke}{\lambda})^n)$ consists of $\lambda \frac{\binom{n}{2}(\frac{2ke}{\lambda})^2}{e} = 2kn(n-1)\frac{ke}{\lambda}$ blocks. For our methods, the range of the subscripts of $2n(n-1)$ base block collections $A_{i,j}$ is taken as $1 \leq i < j \leq 2n$ and $\{i,j\} \notin \mathcal{H}$, every $A_{i,j}$ contains k base blocks. Below, it is enough to construct only one $A_{i,j}$ for constructing $G\text{-}HD_\lambda((\frac{ke}{\lambda})^n)$ and $G\text{-}HD_\lambda((\frac{2ke}{\lambda})^n)$, where i, j are variable in the given range.

Let $x, d \in Z_{\frac{ke}{\lambda}}$ and i, j be in the given range for $A_{i,j}$ in above-mentioned constructions $G\text{-}HD_\lambda((\frac{ke}{\lambda})^n)$ and $G\text{-}HD_\lambda((\frac{2ke}{\lambda})^n)$. Each vertex in the base block collection may be labelled as one among four forms: $(x,i), (x,j), (x, i\cdot j)$ and $(x, j \cdot i)$, where $(x, i\cdot j)$ and $(x, j\cdot i)$ are same for the symmetric quasigroup. Each unordered edge in the base block collection may be one among six forms:

$\{(x,i),(x+d,j)\}$, $\{(x,i),(x+d,i\cdot j)\}$, $\{(x,j\cdot i),(x+d,j)\}$,
$\{(x,i),(x+d,j\cdot i)\}$, $\{(x,i\cdot j),(x+d,j)\}$, $\{(x,i\cdot j),(x+d,j\cdot i)\}$.

For given $d \in Z_{\frac{ke}{\lambda}}$, $u, v \in \{i, j, i\cdot j, j\cdot i\}$ and $u \neq v$, the edge joining vertices (x,u) and $(x+d,v)$ in $A_{i,j}$ is denoted by $d(u,v)$, which represents a mixed difference orbit $\{\{(x,u),(x+d,v)\} : x \in Z_e\}$. And, for any 2-subset $\{u,v\} \subset \{i,j,i\cdot j, j\cdot i\}$, denote $D(u,v) = \{d : d(u,v) \in A_{i,j}\}$.

When $k = 1, \lambda = 1$, there are two lemmas as follows.

Lemma 5A.[12] *Let $(I_n, \cdot)$ be an idempotent quasigroup on the set $I_n = \{1, 2, \cdots, n\}$ and G be a graph with e edges, then $\mathcal{A} = \{A_{i,j} : 1 \leq i < j \leq n\}$ can be taken as a base of a $G\text{-}HD(e^n)$ under the action of automorphism group Z_e if the following conditions hold.*

(1) $D(i, i\cdot j) = D(j, j\cdot i)$, $D(i, j\cdot i) = D(j, i\cdot j)$, $D(i,j) = D(j,i)$;
(2) $D(i\cdot j, j\cdot i) = D(j\cdot i, i\cdot j)$ *when $(I_n, \cdot)$ is self-orthogonal;*
(3) $D(i,j) \cup D(i, i\cdot j) \cup D(j\cdot i, j) \cup D(i, j\cdot i) \cup D(i\cdot j, j) \cup D(i\cdot j, j\cdot i) = Z_e$.

Lemma 5B.[12] *Let $(I_{2n}, \mathcal{H}, \cdot)$ be an $HSQ(2^n)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \leq r \leq n\}$ and G be a graph with e edges. Then $\{A_{i,j} : 1 \leq i < j \leq 2n, \{i,j\} \notin \mathcal{H}\}$ can be taken as a base of a $G\text{-}HD((2e)^n)$ under the action of automorphism group Z_e if the following conditions hold.*

(1) $D(i, i\cdot j) = D(j, i\cdot j)$;
(2) $D(i,j) \cup D(i, i\cdot j) \cup D(i\cdot j, j) = Z_e$.

It is easy to generalize Lemma 5A and Lemma 5B to the following two lemmas.

Lemma 5C. *Let $(I_n, \cdot)$ be an idempotent quasigroup on the set $I_n = \{1, 2, \cdots, n\}$ and G be a graph with e edges, then $\mathcal{A} = \{A_{i,j} : 1 \le i < j \le n\}$ can be taken as a base of a G-$HD_\lambda((\frac{ke}{\lambda})^n)$ under the action of automorphism group $Z_{\frac{ke}{\lambda}}$ if the following conditions hold.*

(1) $D(i, i \cdot j) = D(j, j \cdot i)$, $D(i, j \cdot i) = D(j, i \cdot j)$, $D(i, j) = D(j, i)$;

(2) $D(i \cdot j, j \cdot i) = D(j \cdot i, i \cdot j)$ *when $(I_n, \cdot)$ is self-orthogonal;*

(3) $D(i,j) \cup D(i, i \cdot j) \cup D(j \cdot i, j) \cup D(i, j \cdot i) \cup D(i \cdot j, j) \cup D(i \cdot j, j \cdot i) = \lambda Z_{\frac{ke}{\lambda}}$.

Lemma 5D. *Let $(I_{2n}, \mathcal{H}, \cdot)$ be an $HSQ(2^n)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \le r \le n\}$ and G be a graph with e edges. Then $\{A_{i,j} : 1 \le i < j \le 2n, \{i, j\} \notin \mathcal{H}\}$ can be taken as a base of a G-$HD_\lambda((\frac{2ke}{\lambda})^n)$ under the action of automorphism group $Z_{\frac{ke}{\lambda}}$ if the following conditions hold.*

(1) $D(i, i \cdot j) = D(j, i \cdot j)$;

(2) $D(i, j) \cup D(i, i \cdot j) \cup D(i \cdot j, j) = \lambda Z_{\frac{ke}{\lambda}}$.

Lemma 6.[16] *There exists a G_2-$HD(9^{t+1})$ for any $t \ge 1$; There exists a G_i-$HD(9^{2t+1})$ for $1 \le i \le 9$ and $t \ge 1$; There exists a G_j-$HD(9^4)$ for $j = 3, 7, 8, 9$; There exists a G_k-$HD(18^{t+2})$ for $k = 1, 4$ and $t \ge 1$.*

Lemma 7. *There exists a G_s-$HD_3(6^{t+2})$ for $s = 1, 8, 9$ and $t \ge 1$.*

Proof. By Lemma 4(2), there exists an idempotent symmetric quasigroup $(I_{2t+1}, \cdot)$ on the set $I_{2t+1} = \{1, 2, \ldots, 2t+1\}$ for $t \ge 1$. On the set $X = Z_3 \times I_{2t+1}$ define

$A_1(i,j) = (0_{i \cdot j}, 1_j, 0_i, 1_{i \cdot j}, 0_j, 1_i)$; $A_8(i,j) = (0_i, 1_{i \cdot j}, 2_i, 2_j, 0_{i \cdot j}, 0_j)$;
$A_9(i,j) = (0_j, 2_{i \cdot j}, 1_i, 1_j, 0_i, 0_{i \cdot j})$.

It is not difficult to verify that each $A_s(i,j)$ satisfies the conditions in Lemma 5C(taking $k = 2, \lambda = 3$). Thus, each $\mathcal{A}_s = \{A_s(i,j) \bmod (3, -) : 1 \le i < j \le 2t+1\}$ forms a G_s-$HD_3(3^{2t+1})$, where $s = 1, 8, 9$. Furthermore, let us consider the $HSQ(2^{t+2}) = (I_{2t+4}, \mathcal{H}, \cdot)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \le r \le t+2\}$, which exists for $t \ge 1$ by Lemma 3(3). Then $\mathcal{A}_s = \{A_s(i,j) \bmod (3, -) : 1 \le i < j \le 2t+4, \{i,j\} \notin \mathcal{H}\}$ will form a G_s-$HD_3(6^{t+2})$ by Lemma 5D, where $s = 1, 8, 9$. ■

In what follows, I_n will denote the set $\{0, 1, \cdots, n-1\}$.

Lemma 8. *There exist a G_3-$HD_3(6^m)$ and a G_7-$HD_3(6^m)$ for $m \ge 3, m \equiv 0, 1 \ (mod\ 3)$ and $m \ne 6$; There exist a G_5-$HD_3(6^n)$ and a G_6-$HD_3(6^n)$ for $n \ge 3$ and $n \ne 6, 8$.*

Proof. First, we give the following direct constructions.

G_3-$HD_3(6^3)$ on the set $Z_6 \times Z_3$:

$(4_1, 0_0, 5_1, 1_2, 5_0, 0_2)$, $(1_1, 0_0, 3_1, 0_2, 2_0, 2_2)$, mod $(6, 3)$.

G_3-$HD_3(6^4)$ on the set $X = Z_6 \times I_4$:

$(0_0, 0_2, 2_3, 1_1, 1_2, 0_3)$, $(0_1, 0_3, 2_0, 1_2, 1_3, 0_0)$, $(0_2, 0_0, 2_1, 1_3, 1_0, 0_1)$,
$(0_3, 0_1, 2_2, 1_0, 1_1, 0_2)$, $(0_0, 1_1, 5_3, 3_2, 2_1, 1_3)$, $(2_1, 0_0, 1_2, 4_3, 1_0, 5_2)$,

$(0_0,4_2,5_3,4_1,1_2,3_3)$, $(5_1,1_3,5_0,1_2,4_3,0_0)$, $(2_2,0_0,2_1,1_3,4_0,3_1)$,
$(3_3,0_1,4_2,5_0,2_1,5_2)$, $(0_0,4_1,0_3,3_2,5_1,2_3)$,
$(3_1,0_0,3_2,1_3,5_0,1_2)$, mod $(6,-)$.

G_7-$HD_3(6^3)$ on the set $X = Z_6 \times Z_3$:
$(0_1,0_0,1_1,5_2,2_1,4_2)$, $(0_1,0_0,5_1,4_2,3_1,3_2)$, mod $(6,3)$.

G_7-$HD_3(6^4)$ on the set $X = Z_6 \times Z_4$:
$(1_2,4_1,4_2,5_3,3_1,0_0)$, $(2_2,0_1,3_2,2_3,1_1,0_0)$, mod $(6,4)$;
$(0_2,5_1,1_2,5_3,2_1,0_0)+i_j$, $(3_0,5_3,1_0,5_1,2_3,0_2)+i_j$,
$0 \le i \le 5,\ 0 \le j \le 1.$

However, there exists a $B[\{3,4\},1;m]$ for all $m \ge 3, m \equiv 0,1 \pmod 3$ and $m \ne 6$ by [7]. So, there exist a G_3-$HD_3(6^m)$ and a G_7-$HD_3(6^m)$ for $m \ge 3, m \equiv 0,1 \pmod 3$ and $m \ne 6$ by Lemma 3.

Now, let us construct a G_5-$HD_3(6^{2t+1})$ and a G_6-$HD_3(6^{2t+1})$ for $t \ge 1$ by using directed product of automorphism groups, which is first introduced in [17].

G_5-$HD_3(6^{2t+1}) = (Z_6 \times Z_{2t+1}, \mathcal{A})$, $\mathcal{A} = \{A \mod (6, 2t+1)\}$.
$A = \{(0_0,3_x,4_{-x},0_x,5_{-x},0_{-x}),\ (0_0,5_x,0_{-x},4_x,3_{-x},3_x) : 1 \le x \le t\}$.
G_6-$HD_3(6^{2t+1}) = (Z_6 \times Z_{2t+1}, \mathcal{A})$, $\mathcal{A} = \{A \mod (6, 2t+1)\}$.
$A = \{(0_0,3_x,4_{-x},0_x,5_{-x},5_0),\ (0_0,3_{-x},4_x,0_{-x},5_x,2_0) : 1 \le x \le t\}$.

Direct constructions of a G_5-$HD_3(6^4)$ and a G_6-$HD_3(6^4)$ are listed as follows.

G_5-$HD_3(6^4)$ on the set $X = Z_6 \times Z_4$:
$(0_0,1_3,2_2,3_1,1_2,0_2)$, $(0_0,3_3,5_2,1_1,3_2,2_2)$, mod $(6,4)$;
$(0_0,3_3,2_2,0_1,1_2,0_2)+i_j$, $(0_2,3_1,2_0,0_3,1_0,3_0)+i_j$,
$0 \le i \le 5,\ 0 \le j \le 1.$

G_6-$HD_3(6^4)$ on the set $X = Z_6 \times Z_4$:
$(0_0,1_3,2_2,3_1,1_2,1_0)$, $(0_0,3_3,5_2,1_1,3_2,1_0)$, mod $(6,4)$;
$(0_0,3_3,2_2,0_1,1_2,1_0)+i_j$, $(0_2,3_1,2_0,0_3,1_0,4_2)+i_j$,
$0 \le i \le 5,\ 0 \le j \le 1.$

We know that there exists a $B[\{3,4,5\},1;n]$ for all $n \ge 3, n \ne 6,8$ by [7]. Thus, there exist a G_5-$HD_3(6^n)$ and a G_6-$HD_3(6^n)$ for $n \ge 3$ and $n \ne 6,8$ by Lemma 3. ∎

3 Some results of G-design for certain orders

Lemma 9. *There exist a G_2-$GD_\lambda(10)$ and a G_k-$GD_\lambda(9)$ for $k = 4,5,6,8,9$ $\Longleftrightarrow \lambda > 1$.*

Proof. We can prove the necessary condition by Lemma 1. Now, let us construct a G_2-$GD_\lambda(10)$ and a G_k-$GD_\lambda(9)$, where $k = 4,5,6,8,9$ and $\lambda > 1$.

(1) $k = 4,8,9$, G_k-$GD_2(9) = (Z_8 \bigcup \{x\}, A_k \mod 8)$,

$A_4 : (3,2,0,x,1,5)$; $A_8 : (1,0,x,3,-1,2)$; $A_9 : (0,4,6,2,1,x)$.

(2) G_5-$GD_2(9)$ on the set $(Z_2 \times I_4) \bigcup \{x\}$:

$(x,1_0,0_0,0_3,0_2,0_1)$, $(0_1,x,0_2,1_2,1_3,0_3)$, $(0_0,1_1,0_2,1_3,0_3,0_2)$, $(0_0,0_1,0_2,1_3,1_1,1_2)$, mod $(2,-)$;

G_6-$GD_2(9)$ on the set $(Z_4 \times I_2) \bigcup \{x\}$:

$(x,0_0,1_1,2_1,2_0,1_0)$, $(0_0,2_1,3_1,1_0,0_1,3_0)$, mod $(4,-)$.

(3) $k = 4,6,9$, G_k-$GD_3(9) = ((Z_3 \times I_2) \bigcup \{x_1,x_2,x_3\}, B_k \text{ mod } (3,-))$,

$B_4 : (x_2,x_1,0_0,x_3,1_1,0_1)$, $(x_3,x_2,0_1,x_1,1_1,0_0)$, $(x_1,x_3,1_0,0_0,1_1,2_0)$, $(1_1,0_0,1_0,x_2,2_1,0_1)$;

$B_6 : (x_2,x_1,1_1,1_0,0_0,x_3)$, $(x_1,x_3,2_1,1_0,0_1,1_1)$, $(0_1,0_0,2_0,1_1,1_0,x_3)$, $(x_2,x_3,1_1,0_1,2_0,x_1)$;

$B_9 : (x_2,x_1,2_1,1_1,0_1,2_0)$, $(1_1,x_3,x_2,2_0,0_1,1_0)$, $(0_1,x_3,x_1,2_0,0_0,1_0)$, $(x_2,1_0,x_1,2_1,0_0,0_1)$.

(4) $k = 5,8$, G_k-$GD_3(9) = (Z_3 \times I_3, B_k \text{ mod } (3,-))$,

$B_5 : (0_0,2_1,2_2,1_2,1_1,0_1)$, $(0_1,2_2,2_0,1_0,1_2,0_2)$, $(0_2,2_0,2_1,1_1,1_0,0_0)$, $(0_0,1_1,2_1,1_2,2_2,1_0)$;

$B_8 : (1_1,1_0,0_2,0_0,2_1,1_2)$, $(1_2,1_1,0_0,0_1,2_2,1_0)$, $(1_0,1_2,0_1,0_2,2_0,1_1)$, $(1_1,0_0,1_0,1_2,0_2,2_1)$.

(5) G_2-$GD_2(10)$ on the set $Z_5 \times I_2$:

$(1_0,0_0,4_0,2_1,0_1,3_1)$, $(3_0,0_0,2_0,4_1,0_1,1_1)$, mod $(5,-)$;

G_2-$GD_3(10)$ on the set $Z_5 \times I_2$:

$(0_0,1_0,3_1,4_1,2_1,2_0)$, $(0_1,1_1,3_0,2_0,0_0,2_1)$, $(0_0,2_0,2_1,3_1,1_0,0_1)$, mod $(5,-)$.

For $\lambda > 1$, denote $\lambda = 2 \cdot \frac{\lambda}{2}$ (λ even), or $\lambda = 2 \cdot \frac{\lambda-3}{2} + 3$ (λ odd). So, for any $\lambda > 1$, there exist a G_2-$GD_\lambda(10)$ and a G_k-$GD_\lambda(9)$ for $k = 4,5,6,8,9$. ■

Lemma 10. *There exists a* G_2-$GD_\lambda(6) \iff 6|\lambda$.

Proof. It is obvious $3|\lambda$ is the necessary condition for the existence of G_2-$GD_\lambda(6)$. Let $\lambda = 3k, k \geq 1$. If a G_2-$GD_\lambda(6)$ exists, the block number of this design is $5k$. Let $X = Z_5 \bigcup \{x\}$ be the vertex set of a G_2-$GD_\lambda(6)$. Then all the ten possible block types are as follows.

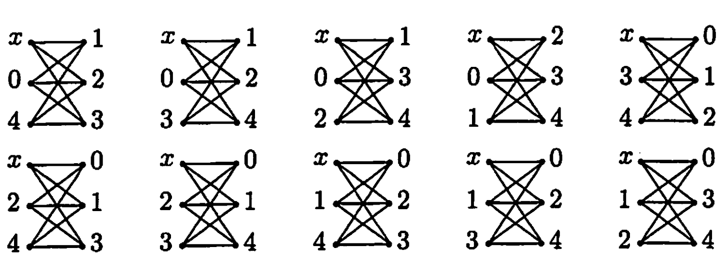

Figure 2: Block types

There are two differences(1 and 2) in Z_5. The edges corresponding to the difference "1" of Z_5 occur even times in every case above. So all edges corresponding to the difference "1" in $5k$ blocks of this design, must also occur even times. But the number of these edges is $\lambda \cdot 5 = 3k \cdot 5 = 15k$. Thus we get: $2|15k$, i.e., $2|k$. But $2|k \longrightarrow 6|\lambda$. On the other hand, if $6|\lambda$ and a G_2-$GD_6(6)$ exists, then a G_2-$GD_\lambda(6)$ exists. List a G_2-$GD_6(6)$ as follows.

G_2-$GD_6(6)$ on the set $Z_5 \bigcup \{x\}$: $(x,1,0,2,3,4)$, $(x,1,0,2,4,3)$, mod 5.

■

Let graph G have m_i vertices with degree $d_i, 1 \le i \le r$, and $\sum_{i=1}^{r} m_i = 6$. Suppose there exists a G-$GD_\lambda(v)$ on a v-set V, with b blocks. If some element α of v-set appears in s_i blocks as r_i-degree vertices, $1 \le i \le t$, we call the element α have the degree-type $r_1^{s_1} r_2^{s_2} \cdots r_t^{s_t}$. The proof will consist of the following steps.

1° Find nonnegative integer solutions for equations

$$\sum_{i=1}^{r} d_i x_i = \lambda(v-1) \text{ with restrictions } \sum_{i=1}^{r} x_i \le b \text{ and } x_i \ge 0. \qquad (*)$$

Its one solution $(x_1, x_2, \cdots, x_r) = (a_{1j}, a_{2j}, \cdots, a_{rj})$ means that some element α of V may have the degree-type $d_1^{a_{1j}} d_2^{a_{2j}} \cdots d_r^{a_{rj}}$, $1 \le j \le s$.

2° Solve the further equations

$$\sum_{j=1}^{s} y_j = v \text{ and } \sum_{j=1}^{s} a_{ij} y_j = m_i b, \ 1 \le i \le r. \qquad (**)$$

Each solution $(y_1, y_2, \cdots, y_s)$ means a possible structure of G-$GD_\lambda(v)$: y_j elements of V have degree-type $d_1^{a_{1j}} d_2^{a_{2j}} \cdots d_r^{a_{rj}}, 1 \le j \le s$.

3° For each solution obtained above, discuss the existence of such structure.

Lemma 11. *There exists a* G_8-$GD_\lambda(6) \Longleftrightarrow \lambda > 3$ *and* $3|\lambda$.

Proof. It is obvious $3|\lambda$ is the necessary condition for the existence of G_8-$GD_\lambda(6)$. In order to prove the necessary conditions, we must get the nonexistence of G_8-$GD_3(6)$. For a G_8-$GD_3(6)$, $v = 6$, $b = 5$, $\lambda = 3$, $(d_1, m_1) = (2,2)$, $(d_2, m_2) = (3,3)$, $(d_3, m_3) = (4,4)$. There are four solutions for $(*)$, which are listed as a matrix $A = (a_{ij})_1^{3,4}$. And the equations $a_{ij} y_j = m_i b$, $1 \le i \le r$ will be in this form.

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 5 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \\ 20 \end{pmatrix}.$$

Furthermore, the $(**)$ has only two solutions

$$(y_1, y_2, y_3, y_4) = (0,4,2,0), (0,5,0,1).$$

(1) For the first solution, let the two elements having degree-type $2^1 3^3 4^1$ be x, y. First, we arrange x and y in locations of 3^3. There will be the unique case.

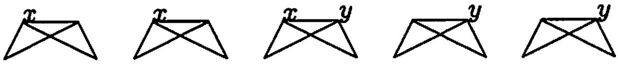

Figure 3: The unique case

Thus, we can't arrange x and y in locations of 4^1.

(2) For the second solution, let the element having degree-type 3^5 be x. If a G_8-$GD_3(6) = (X, \mathcal{A})$ satisfying this solution exists, $(X \setminus \{x\}, \{A \setminus x : A \in \mathcal{A}\})$ will be a $K_{2,3}$-$GD_3(5)$. But a $K_{2,3}$-$GD_3(5)$ does not exist by [4]. So a G_8-$GD_3(6)$ does not exist by (1) and (2).

On the other hand, we give the following constructions.

G_8-$GD_6(6)$ on the set $Z_5 \bigcup \{x\}$: $(0,x,2,4,3,1)$, $(3,0,x,2,1,4)$, mod 5;

G_8-$GD_9(6)$ on the set $Z_5 \bigcup \{x\}$:

$$(4,x,2,0,3,1),\ (x,0,2,4,3,1),\ (3,0,x,2,1,4),\ \text{mod } 5.$$

For $\lambda > 3$ and $3|\lambda$, denote $\lambda = 6 \cdot \frac{\lambda}{6}$ (λ even), or $\lambda = 6 \cdot \frac{\lambda-9}{6} + 9$ (λ odd). So, for any $\lambda > 3$ and $3|\lambda$, there exists a G_8-$GD_\lambda(6)$. ■

4 Main constructions

First, we list the following tables for the desired designs of the nine graphs.

Table 2: $t \geq 1$

	λ	2	3	6	9	18
G_1	v (mod 18)	0,10	3,7,13,15	4,6,12,16	5,11,17	2,8,14
	HD	$18^{t+2}, 9^{2t+1}$	6^{t+2}	6^{t+2}	18^{t+2}	$18^{t+2}, 9^{2t+1}$
	ID		(9,3)	(10,4)	(23,5) (29,11),(35,17)	(20,2) (26,8),(14,5)
	GD	10,18,36	7,9,13 15	6,10 12,16	11,17,23 41,47,53	8,14,20 38,44

Table 3: $t \geq 1, s \geq 3, s \neq 6, 8, u \geq 1, u \neq 2$

	λ	3	9
G_2	v (mod 9)	0,3,4,6,7	2,5,8
	HD	9^{t+1}	9^{t+1}
	ID	(12,3),(13,4),(15,6)(16,7)	(11,2),(14,5),(17,8)
	GD	7,9,12,13,15	8,11,14
G_3	v (mod 18)	0 (mod 9); 3,4,6,7,12,13,15,16	2,5,8,11,14,17
	HD	$9^{2t+1}, 9^4, 6^{3u}, 6^{3t+1}$	$9^{2t+1}, 9^4, 6^{3u}, 6^{3t+1}$
	ID	(9,3),(10,4),(12,3) (13,4),(15,6),(16,7)	(11,2),(14,5) (17,8),(8,2),(11,5)
	GD	6,7,9,10,12,13,18,	8,11,14
G_4	v (mod 18)	3,4,6,7,12,13,15,16	2,5,8,11,14,17
	HD	9^{2t+1}	$9^{2t+1}, 18^{t+2}$
	ID	(12,3),(13,4),(15,6),(16,7) (21,12),(22,13),(24,15),(25,16)	(11,2),(14,5),(17,8) (20,2),(23,5),(26,8)
	GD	6,7,12,13	8,11,14,20,23,38,41,44
G_5	v (mod 18)	3,4,6,7,12,13,15,16	2,5 (mod 6)
	HD	$9^{2t+1}, 6^{3t+1}$	6^s
	ID	(12,3),(13,4),(15,6) (16,7),(21,12),(22,13)	(8,2),(11,5)
	GD	6,7,12,13	8,11,14,17,38,41,50,53
G_6	v	3,4,6,7,12,13,15,16 (mod 18)	2,5,8 (mod 9)
	HD	$9^{2t+1}, 6^{3t+1}$	Direct construction
	ID	(12,3),(13,4),(15,6) (16,7),(21,12),(22,13)	
	GD	6,7,12,13	
G_7	v (mod 9)	3,4,6,7	2,5,8
	HD	$9^{2t+1}, 9^4, 6^{3u}, 6^{3t+1}$	$9^{2t+1}, 9^4$
	ID	(12,3),(13,4),(15,6) (16,7),(9,3),(10,4)	(11,2),(14,5),(17,8)
	GD	6,7,9,10,12,13	8,11,14,20,23,26 56,59,62,74,77,80
G_8	v	3, 4, 6, 7(mod 9)	2, 5 (mod 6)
	HD	9^4, $9^{2t+1}, 6^{t+2}$	6^{t+2}
	ID	(12,3),(13,4),(15,6) (16,7),(21,12),(22,13)	(8,2),(11,5) (8,2),(11,5)
	GD	7,12,13,15,24,60,78	8,11,14,17
G_9	v	3, 4, 6, 7(mod 9)	2, 5 (mod 6)
	HD	9^4, $9^{2t+1}, 6^{t+2}$	6^{t+2}
	ID	(12,3),(13,4),(15,6) (16,7),(21,12),(10,4)	(8,2),(11,5) (8,2),(11,5)
	GD	6,7,10,12,13	8,11,14,17

Now, let us construct the desired designs in the tables.

4.1 A constructing method for $\lambda = |E(G)|$ or $2|E(G)|$

Let G be a connected graph, $|V(G)| = m$ and $|E(G)| = e$. Consider the graph design G-$GD_e(v) = (X, \mathcal{B})$. Let $n = 2\lceil \frac{v}{2} \rceil - 1$, which is odd. The vertex set X is denoted by Z_n (odd v) or $Z_n \cup \{\infty\}$ (even v). The block set consists of $n \cdot \frac{n-1}{2}$ (odd v) or $n \cdot \frac{n+1}{2}$ (even v) blocks. Let us construct $\frac{n-1}{2}$ (for odd v) or $\frac{n+1}{2}$ (for even v) base blocks as follows.

Step 1. Define a bijection from Z_n to $\{1, 2, \cdots, \frac{n-1}{2}\} : a \mapsto \langle 2a \rangle$, where $\langle t \rangle = t$ (if $t \leq \frac{n-1}{2}$) or $n - t$ (if $t > \frac{n-1}{2}$). Then, the integers $1, 2, \cdots, \frac{n-1}{2}$ are partitioned into equivalent classes, each of which forms a cycle. The cycle containing integer a ($1 \le a \le \frac{n-1}{2}$) and its length is denoted by (a) and $l(a)$ respectively. The length $s = l(a)$ is the minimal positive integer satisfying $a \cdot 2^s \equiv \pm a \pmod{n}$. Obviously, $l(a) \le l(1)$ for $1 \le a \le \frac{n-1}{2}$. All the cycles form a graph H_n, which is 2-regular.

Step 2. For any $a \in [1, \frac{n-1}{2}]$ and $l(a) \ge 3$, take an injection f from $V(G)$ to $M = \{ma : -\frac{n-1}{2} \le m \le \frac{n-1}{2}\}$ such that the integer $\langle f(x) - f(y) \rangle$ is in the cycle (a) for any edge $xy \in E(G)$. Note that f is an injection if and only if $f(x) \ne f(y)$ for any $x \ne y \in V(G)$. When $|V(G)| \le 7$, the set M may be restricted to the 7-set: $\{-2a, -a, 0, a, 2a\} \bigcup T$, where $T = \{3a, 4a\}$, or $\{-3a, -4a\}$, or $\{3a, -3a\}$, or $\{4a, -4a\}$. Then, for $x \ne y \in V(G)$, the equation $f(x) = f(y)$ holds only to the following cases:

1° $0 = \pm 3a, \ \pm a = \pm 4a, \ \pm a = \mp 2a, \ \pm 2a = \mp 4a, \ 3a = -3a,$

$\Longrightarrow n = 3a, \ l(a) = 1$ and (a) is the unique 1-cycle;

2° $\pm a = \mp 4a, \ \pm 2a = \mp 3a,$

$\Longrightarrow n = 5a, \ l(a) = 2$ and $(a, 2a)$ is the unique 2-cycle.

Furthermore, there is another related case

3° $n = 15a$, there is the unique 1-cycle $(5a)$ and the unique 2-cycle $(3a, 6a)$.

Therefore, we only need to discuss the four cases:

Case 1 $gcd(n, 15) = 1$, and the length $l(a) \ge 3$ for any cycle (a) in H_n. The injection f here gives a base block B_a. But the base blocks $\mathcal{A} = \{B_a : 1 \le a \le \frac{n-1}{2}\}$ will cover all differences in Z_n e times. In fact, let the cycle (a) be $(a, 2a, 4a, \cdots, 2^{s-1}a)$ and each $2^j a$, as edge-value $\langle f(x) - f(y) \rangle$, appear i_j times in the base block B_a, where $0 \le j \le s-1$ and $\sum_{j=0}^{s-1} i_j = e$. Then, the all edges in B_a, B_{2a}, $\cdots, B_{2^{s-1}a}$ will take edge-values as follows.

Table 4: Edge-values

	a	$2a$	2^2a	$\cdots$	$2^{s-2}a$	$2^{s-1}a$
B_a	i_0	i_1	i_2	$\cdots$	i_{s-2}	i_{s-1}
B_{2a}	i_{s-1}	i_0	i_1	$\cdots$	i_{s-3}	i_{s-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{2^{s-1}a}$	i_1	i_2	i_3	$\cdots$	i_{s-1}	i_0

Thus, the base blocks B_a, B_{2a}, $\cdots$, $B_{2^{s-1}a}$ corresponding a, $2a$, $\cdots$, $2^{s-1}a$ in the cycle (a) cover the differences a, $2a$, $\cdots$, $2^{s-1}a$ e times.

Case 2 $n = 3b$ and $b \ne 0 \pmod 5$, there is the unique 1-cycle (b).

Case 3 $n = 5b$ and $b \ne 0 \pmod 3$, there is the unique 2-cycle $(b, 2b)$.

Case 4 $n = 15b$, there are the unique 1-cycle $(5b)$ and the unique 2-cycle $(3b, 6b)$.

Step 3. For the Cases 2, 3 and 4, the method stated in step 2 can not be used for 1-cycle or 2-cycle because, replacing a by b, $2b$, $3b$, $5b$ or $6b$, the number of the available integers in the set M are less than six. We may change a few base blocks in $\mathcal{A}$ corresponding to cycle (1) (or b when $n = 15b$) and add some base blocks relating to the elements b, $2b$, $3b$, $5b$, $6b$. Note that the edges in these changed and added base blocks belong not yet to one cycle (but two or three cycles).

Step 4. For odd order v, the graph design G-$GD_e(v)$ will be obtained after Steps 2 and 3. For even order $v = n + 1$, we need to add one vertex ∞ to the vertex set Z_n, to change some base blocks in $\mathcal{A}$ corresponding to cycle (1), and to add some base blocks containing ∞.

Remark: The method can be used to construct G-$GD_{2e}(v)$, too.

Lemma 12. *There exists a G_6-$GD_9(v)$ for $v \equiv 2, 5, 8$ (mod 9).*

Proof. Using the method mentioned above, we list Table 5. First, the base block B_a for odd v and $l(a) \geq 3$, i.e., case 1 (odd), is given in the first row. The given vertex-values $f(x)$ are obviously in $\{0, \pm a, \pm 2a\} \bigcup T$ pointed in Step 2. We denote $\mathcal{A} = \{B_a : 1 \leq a \leq \frac{n-1}{2}\}$ for the B_a listed in the first row, then the base blocks for other cases will be uniformly denoted as $(\mathcal{A} \backslash \mathcal{C}) \bigcup \mathcal{C}' \bigcup \mathcal{D}$, where $\mathcal{C}$ is a few base blocks in $\mathcal{A}$ like B_1, B_2, B_4 or $B_b, B_{2b} \cdots$, which is changed to $\mathcal{C}'$ (denoted by $\rightarrow$), and $\mathcal{D}$ is a few added base blocks. We need not consider case 2 because $v \not\equiv 0$ (mod 3) and $v - 1 \not\equiv 0$ (mod 3) for $v \equiv 2, 5, 8$ (mod 9).

Table 5: Constructions for G_6

odd v(case 1)	$\mathcal{A}$	$(a, 2a, 3a, -a, 0, 4a)$
odd v(case 3)	$\mathcal{C}'$	$B_1 : 4 \longrightarrow b, B_2 : 8 \longrightarrow 2b$
$n = 5b$	$\mathcal{D}$	$(1, 2b+1, b+1, -b+1, -2b+1, -2b-3) \times 2$
odd v(case 4)	$\mathcal{C}'$	$B_{2^t b} : 2^t 4b \rightarrow 5b, t = 0, 1, 2, 3$
$n = 15b$	$\mathcal{D}$	$(3b, 6b, 9b, 12b, 0, 5b) \times 2$, $(5b, -5b, 2b, 4b, 0, 5b)$
even v	$\mathcal{C}'$	$B_1 : -1 \rightarrow \infty, B_2 : -2 \rightarrow \infty$
(case 1)	$\mathcal{D}$	$(4, -4, \infty, 2, 0, 1)$
even v(case 3)	$\mathcal{C}'$	$B_1 : 4 \longrightarrow 2b$
$n = 5b$	$\mathcal{D}$	$(1, 2b+1, b+1, \infty, -2b+1, -b+1)$
		$(1, 2b+1, b+1, \infty, -2b+1, -2b-3)$
		$(2, -b+2, 2b+2, \infty, b+2, 3b+2)$
even v(case 4),$n = 15b$		even v(case 1)+odd v(case 4)

■

Lemma 13. *There exists a G_7-$GD_9(v)$ for $v = 26, 56$.*

Proof. There exists a G_7-$HD_3(6^m)$ for $m = 4, 9$ by Lemma 8. There exists a G_7-$ID_3(6+2, 2)$ by the construction in §4.3 below. So, there exists a G_7-$GD_9(v)$ for $v = 26, 56$ by Lemma 2. ■

Below, we list all the other desired G_1-$GD_{18}(v)$ and G_i-$GD_9(v)$ for $i = 3, 4, 5, 7, 8, 9$.

Table 6: Constructions for G_1 and G_3

v		G_1	G_3
11,17,23,41,47,53	$\mathcal{A}$	$(2a, -2a, 0, -a, a, 3a)$	$(0, a, 2a, 4a, 3a, -a)$
8,14,20,38,44,50	$\mathcal{C}'$	$2 \times B_2 : 0 \to \infty$ $B_4 : -4 \to \infty$	$B_1 : 1 \to \infty$ $B_2 : 2 \to \infty$
	$\mathcal{D}$	$(0, 2, \infty, -8, -4, 4)$, $(6, 10, \infty, 0, 2, 4)$	$(0, 4, 3, \infty, 1, 2)$

Table 7: Constructions for G_4 and G_5

v		G_4	G_5
11,17,23,41,47,53	$\mathcal{A}$	$(-2a, 2a, 0, a, 3a, -a)$	$(a, 2a, 3a, -a, 0, -3a)$
8,14,20,38,44,50	$\mathcal{C}'$	$B_1 : -1 \to \infty$ $B_2 : -2 \to \infty$	$B_1 : -1 \to \infty$ $B_2 : -2 \to \infty$
	$\mathcal{D}$	$(-2, 2, 0, 8, \infty, -1)$	$(0, 4, -4, \infty, 2, 1)$

Table 8: Constructions for G_7, G_8 and G_9

v		G_7	G_8	G_9
11,17 23,59,77	$\mathcal{A}$	$(a, 0, -2a, 2a, 3a, -a)$	$(a, 0, -2a, -4a, -3a, -a)$	$(-2a, 2a, 4a, a, 0, -a)$
8,14,20, 62,74,80	$\mathcal{C}'$	$B_1 : 1 \to \infty$ $B_2 : 2 \to \infty$	$B_2 : -4 \to \infty$ $B_4 : -4 \to \infty$	$B_2 : -4 \to \infty$ $B_4 : -8 \to \infty$
	$\mathcal{D}$	$(1, 0, -2, 2, 4, \infty)$,	$(4, 0, -8, 2, \infty, 8)$	$(\infty, 8, 4, -8, 0, 2)$

4.2 Constructions for *GD*

In this section, we construct all the left G_k-$GD_\lambda(v)$s listed in Tables 2 and 3, $1 \leq k \leq 9$. For the block $B, B \times n$ means *n times* of the block B. The vertex set is taken as—

Z_v for $v = 7, 11, 13, 16, 25$;
$Z_{v-1} \bigcup \{x\}$ for $v = 6, 8, 12, 14, 18, 24, 36, 60, 78$;
$Z_3 \times Z_3$ for $v = 9$; $Z_5 \times I_2$ for $v = 10$;
$(Z_7 \times I_2) \bigcup \{x\}$ for $v = 15$.

G_1-$GD_2(10)$: $(2_1, 0_1, 0_0, 3_1, 1_0, 1_1), (3_1, 4_1, 0_0, 2_0, 4_0, 0_1)$, mod $(5, -)$.
G_1-$GD_2(18)$: $(0, 7, 1, x, 3, 8), (0, 5, 1, 9, 3, 7)$, mod 17.
G_1-$GD_2(36)$: $(0, 14, 2, 22, 19, 29), (0, 21, 9, 16, 1, x)$,

$(0,1,3,22,13,5),(0,17,4,15,11,5)$, mod 35.
G_1-$GD_3(7)$: $(0,6,1,5,2,4)$ mod 7.
G_1-$GD_3(13)$: $(5,3,0,4,6,9),(3,4,0,12,5,11)$, mod 13.
G_1-$GD_3(15)$: $(6_0,2_1,0_0,2_0,0_1,3_1),(1_0,6_1,5_0,5_1,3_0,0_0)$,
$(1_0,x,0_1,3_1,1_1,2_0),(3_0,x,0_1,3_1,2_1,0_0)$,
$(4_0,x,2_1,6_0,0_1,1_1)$, mod $(7,-)$.
G_1-$GD_6(6)$: $(x,3,0,2,4,1),(0,1,2,3,4,x)$, mod 5.
G_1-$GD_6(12)$: $(4,7,2,0,x,6),(0,5,1,x,3,4),(0,5,1,6,3,4)\times 2$, mod 11.
G_1-$GD_6(16)$: $(0,7,6,8,1,3),(12,6,1,0,3,9),(0,5,1,9,3,7)\times 2$, mod 16.
G_2-$GD_3(7)$: $(0,6,5,2,4,1)$ mod 7.
G_2-$GD_3(9)$: $(0_0,1_2,2_1,2_2,1_1,0_1)$ mod $(3,3)$,
$(0_0,1_1,0_1,1_2,0_2,1_0)$ mod $(3,-)$.
G_2-$GD_3(12)$: $(x,4,0,5,2,9),(7,3,0,1,2,5)$, mod 11.
G_2-$GD_3(13)$: $(0,1,4,8,10,6)$, $(0,1,3,6,2,8)$, mod 13.
G_2-$GD_3(15)$: $(x,0_0,1_0,3_0,6_0,2_0),(x,0_1,2_1,3_1,6_1,4_1),(0_0,1_1,2_0,3_1,6_0,0_1)$,
$(0_0,2_0,4_0,0_1,5_0,2_1),(0_1,3_1,6_1,0_0,4_1,1_0)$, mod $(7,-)$.
G_2-$GD_9(8)$: $(x,0,3,6,5,2),(x,0,3,6,5,4)$,
$(x,0,3,6,5,1),(0,1,3,4,6,5)$, mod 7.
G_2-$GD_9(11)$: $(0,1,2,3,6,4),(0,1,3,2,5,8),(0,1,7,2,3,6)$,
$(0,4,8,1,6,2),(0,7,3,8,4,6)$, mod 11.
G_2-$GD_9(14)$: $(x,11,5,0,6,12),(x,5,4,0,9,11),(x,0,2,6,10,5)$,
$(0,1,2,3,6,4),(0,1,3,2,5,8),(0,1,7,2,3,6)$,
$(0,7,3,8,4,6)$, mod 13.
G_3-$GD_3(6)$: $(x,4,0,3,2,1)$ mod 5.
G_3-$GD_3(7)$: $(0,2,5,6,1,3)$ mod 7.
G_3-$GD_3(9)$: $(0_0,1_2,2_2,2_1,1_1,2_0)$ mod $(3,3)$,
$(0_0,0_1,1_2,1_1,1_0,0_2)$ mod $(3,-)$.
G_3-$GD_3(12)$: $(x,4,0,3,1,5),(6,0,1,3,9,4)$, mod 11.
G_3-$GD_3(13)$: $(9,3,1,4,0,5),(1,3,9,2,5,0)$, mod 13.
G_3-$GD_3(18)$: $(x,0,7,9,6,4),(0,8,1,6,15,13),(0,1,7,6,3,8)$, mod 17.
G_4-$GD_3(6)$: $(1,0,2,4,3,x)$ mod 5.
G_4-$GD_3(7)$: $(1,5,0,4,6,2)$ mod 7.
G_4-$GD_3(12)$: $(2,5,8,x,1,0),(0,2,1,3,9,4)$, mod 11.
G_4-$GD_3(13)$: $(3,2,5,0,4,12),(1,4,0,12,9,7)$, mod 13.
G_5-$GD_3(6)$: $(0,4,2,3,x,1)$ mod 5.
G_5-$GD_3(7)$: $(0,4,1,2,6,5)$ mod 7.
G_5-$GD_3(12)$: $(0,x,4,5,3,6),(0,3,1,5,2,4)$, mod 11.
G_5-$GD_3(13)$: $(0,3,8,6,2,10),(0,4,1,7,5,9)$, mod 13.
G_6-$GD_3(6)$: $(2,x,1,3,0,4)$ mod 5.
G_6-$GD_3(7)$: $(0,5,1,2,3,6)$ mod 7.
G_6-$GD_3(12)$: $(0,x,5,4,2,10),(0,1,5,4,9,6)$, mod 11.
G_6-$GD_3(13)$: $(0,4,6,5,3,7),(0,2,6,5,8,11)$, mod 13.
G_7-$GD_3(6)$: $(x,0,2,3,4,1)$ mod 5.

G_7-$GD_3(7)$: $(5,0,3,1,2,6)$ mod 7.
G_7-$GD_3(12)$: $(1,0,5,9,7,x),(8,0,1,6,10,7)$, mod 11.
G_7-$GD_3(13)$: $(10,4,1,0,5,6),(5,0,2,8,10,6)$, mod 13.
G_8-$GD_3(7)$: $(6,0,2,4,1,5)$ mod 7.
G_8-$GD_3(12)$: $(x,0,4,5,2,3),(10,0,2,5,1,4)$, mod 11.
G_8-$GD_3(13)$: $(3,6,12,8,4,0),(1,5,7,8,6,0)$, mod 13.
G_8-$GD_3(15)$: $(0_0,2_0,x,4_1,6_1,3_1),(1_0,3_0,0_1,6_1,1_1,0_0),(0_0,1_0,4_0,5_1,3_1,6_1)$,
$(0_0,3_0,2_0,3_1,4_1,5_1),(2_0,x,3_1,6_0,0_1,0_0)$, mod $(7,-)$.
G_8-$GD_3(24)$: $(x,0,5,4,2,3),(11,0,7,6,2,10)$,
$(7,0,11,9,1,2),(9,0,4,11,1,7)$, mod 23.
G_8-$GD_3(60)$: $(29,0,31,28,1,27),(29,0,24,22,2,26),(13,0,25,20,2,12)$,
$(17,0,20,18,16),(14,0,16,15,3,10),(14,0,16,15,1,9)$,
$(8,0,10,9,1,6),(x,0,6,4,3,2),(24,0,26,25,3,21)$,
$(19,0,25,21,2,7)$, mod 59.
G_8-$GD_3(78)$: $(37,0,38,36,1,39),(37,0,36,35,2,34),(32,0,34,33,3,29)$,
$(11,0,18,12,2,8),(29,0,33,31,1,25),(x,0,12,5,2,1)$,
$(18,0,20,19,3,9),(7,0,11,10,3,4),(26,0,28,27,2,21)$,
$(23,0,27,25,3,17),(23,0,31,28,1,16),(21,0,23,22,2,13)$,
$(18,0,14,15,1,13)$, mod 77.
G_9-$GD_3(6)$: $(x,0,2,4,1,3)$ mod 5.
G_9-$GD_3(7)$: $(6,0,3,5,1,2)$ mod 7.
G_9-$GD_3(12)$: $(x,0,4,5,3,1),(8,0,6,2,1,5)$, mod 11.
G_9-$GD_3(13)$: $(3,0,1,4,6,12),(7,2,0,11,5,6)$, mod 13.

4.3 Constructions for *ID*

In this section, we construct all G_k-$ID_\lambda(v,w)$s listed in Tables 2 and 3 for $1 \le k \le 9$.

G_1-$ID_3(9,3)$: $Z_6 \bigcup \{x_1,x_2,x_3\}$
$(x_1,2,1,5,0,3),(x_1,0,2,x_3,4,1),(x_3,3,2,x_2,5,1),(x_1,5,3,4,0,1)$,
$(x_2,3,4,x_1,5,0),(x_2,2,0,x_3,4,1),(x_2,5,3,x_3,1,2),(x_2,1,3,2,4,0)$,
$(x_1,4,2,1,3,5),(x_3,0,1,4,5,2),(x_3,4,5,2,0,3)$.
G_1-$ID_3(23,5)$: $(Z_9 \times I_2) \bigcup \{x_1,x_2,\cdots,x_5\}$
$(x_1,0_0,0_1,x_2,1_0,2_1),(x_2,0_0,2_1,x_3,8_0,6_1),(x_3,0_0,2_1,x_4,8_0,6_1)$,
$(x_4,3_0,0_1,x_5,4_0,8_1),(x_5,3_0,0_1,x_1,4_0,8_1),(0_1,4_1,1_1,5_1,3_1,8_1)$,
$(0_0,4_0,1_0,5_0,3_0,8_0),(0_1,0_0,1_1,4_0,3_1,5_0)$,
$(0_0,0_1,1_0,4_1,3_0,5_1)$, mod $(9,-)$.
G_1-$ID_3(29,11)$: $(Z_9 \times I_2) \bigcup \{x_1,x_2,\cdots,x_{11}\}$
$(x_1,0_1,0_0,x_2,1_0,2_1),(x_3,3_1,0_0,x_4,3_0,1_1),(x_{11},3_0,0_1,x_1,4_0,8_1)$,
$(x_7,0_0,2_1,x_8,8_0,6_1),(x_4,7_1,3_0,x_5,0_1,3_1),(x_6,2_1,1_0,x_7,0_1,4_1)$,
$(x_8,3_0,0_1,x_9,4_0,8_1),(x_9,0_0,0_1,x_{10},1_0,2_1),(x_{10},0_0,2_1,x_{11},8_0,6_1)$,
$(x_5,5_1,0_0,x_6,4_0,4_1),(x_2,3_1,1_0,x_3,0_1,1_1)$,

$(0_i, 1_i, 2_i, 4_i, 7_i, 3_i), i = 0, 1$, mod $(9, -)$.

G_1-$ID_3(35, 17)$: $(Z_9 \times I_2) \bigcup \{x_1, x_2, \cdots, x_{17}\}$

$(x_{15}, 6_0, 8_1, x_{16}, 0_0, 1_1), (x_{10}, 2_1, 0_1, x_{11}, 1_1, 0_0), (x_{11}, 4_0, 0_0, x_{12}, 3_0, 3_1),$
$(x_{13}, 0_0, 2_1, x_{14}, 8_0, 6_1), (x_{14}, 3_0, 0_1, x_{15}, 4_0, 8_1), (x_7, 4_0, 0_0, x_8, 3_0, 6_1),$
$(x_3, 4_0, 0_0, x_4, 3_0, 2_1), (x_1, 2_0, 0_0, x_2, 1_0, 1_1), (x_2, 2_1, 0_1, x_3, 1_1, 0_0),$
$(x_8, 4_1, 0_1, x_9, 3_1, 7_0), (x_9, 2_0, 0_0, x_{10}, 1_0, 5_1), (x_4, 4_1, 0_1, x_5, 3_1, 1_0),$
$(x_5, 2_0, 0_0, x_6, 1_0, 8_1), (x_6, 2_1, 0_1, x_7, 1_1, 4_0), (x_{16}, 2_0, 0_1, x_{17}, 3_0, 6_1),$
$(x_{17}, 5_0, 5_1, x_{13}, 0_0, 4_1), (x_{12}, 4_1, 0_1, x_1, 3_1, 4_0)$, mod $(9, -)$.

G_1-$ID_6(10, 4)$: $(Z_2 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$

$(x_1, 1_2, 0_0, x_3, 0_1, 1_0), (x_3, 0_2, 0_0, 1_0, 1_1, 1_2), (x_1, 1_1, 0_1, 1_0, 1_2, 0_0),$
$(x_1, 1_2, 0_2, x_4, 0_0, 0_1), (x_4, 0_0, 0_1, x_1, 0_2, 1_1), (x_1, 1_0, 0_0, x_4, 1_1, 1_2),$
$(x_2, 1_1, 0_1, x_4, 1_2, 0_0), (x_2, 0_1, 0_2, x_4, 0_0, 1_2), (x_2, 1_0, 0_0, x_4, 0_1, 1_2),$
$(x_2, 1_1, 0_1, x_4, 0_2, 0_0), (x_3, 1_2, 0_2, x_4, 0_0, 1_1), (x_3, 0_0, 0_1, 1_2, 0_2, 1_0),$
$(x_3, 0_1, 1_2, x_2, 0_0, 1_1)$, mod $(2, -)$.

G_1-$ID_6(14, 5)$: $Z_9 \bigcup \{x_1, x_2, \cdots, x_5\}$

$(x_1, 1, 0, x_2, 2, 4), (x_2, 0, 2, x_3, 4, 1), (x_3, 0, 3, x_4, 6, 2),$
$(x_4, 0, 3, x_5, 6, 2), (x_5, 3, 0, x_1, 4, 8), (0, 1, 2, 3, 4, 5)$, mod 9.

G_1-$ID_6(20, 2)$: $Z_{18} \bigcup \{x_1, x_2\}$

$(x_1, 9, 0, x_2, 3, 7), (x_2, 8, 0, x_1, 3, 7), (0, 11, 1, 4, 8, 7), (0, 16, 1, 4, 3, 2),$
$(0, 13, 4, 8, 9, 5), (0, 12, 5, 16, 11, 6), (0, 12, 2, 10, 8, 6)$, mod 18.

G_1-$ID_6(26, 8)$: $Z_{18} \bigcup \{x_1, x_2, \cdots, x_8\}$

$(x_5, 0, 1, x_6, 2, 3), (x_6, 9, 0, x_7, 2, 5), (x_7, 9, 0, x_8, 2, 5), (x_8, 9, 0, x_1, 2, 5),$
$(x_1, 0, 4, x_2, 8, 12), (0, 12, 5, 16, 11, 6) \times 2, (x_3, 8, 0, x_4, 10, 2),$
$(x_4, 8, 0, x_5, 10, 2), (0, 16, 1, 4, 3, 2), (x_2, 0, 4, x_3, 8, 12)$, mod 18.

G_2-$ID_3(11, 2)$: $Z_9 \bigcup \{x_1, x_2\}$ $(x_1, 1, 0, 4, x_2, 3), (3, 1, 0, 4, 7, 5)$, mod 9.

G_2-$ID_3(12, 3)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3\}$

$(x_1, 2_0, 1_2, 0_0, x_3, 1_1), (x_1, 2_1, 1_0, 0_1, x_2, 0_2), (0_1, 1_1, 0_0, 2_1, x_3, 1_2),$
$(x_2, 0_0, 1_2, 1_1, 2_2, 0_2), (x_2, 0_0, 1_0, 1_2, 2_1, 2_0), (x_1, 0_2, 0_1, 2_2, x_3, 2_0),$
$(0_0, 0_1, 1_1, 1_0, 1_2, 0_2)$, mod $(3, -)$.

G_2-$ID_3(13, 4)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$

$(x_1, 2_0, 1_2, 0_0, x_3, 1_1), (x_1, 2_1, 1_0, 0_1, x_2, 0_2), (x_1, 0_2, 0_1, 2_2, x_3, 2_0),$
$(0_0, 0_1, 1_1, 1_0, 1_2, 0_2), (0_1, 1_1, 2_1, 1_2, 2_2, 0_0), (x_4, 1_1, 0_0, 2_1, x_3, 1_2),$
$(x_2, 0_0, 1_2, 1_1, x_4, 0_2), (x_2, 0_0, 1_0, 1_2, x_4, 2_0)$, mod$(3, -)$

G_2-$ID_3(14, 5)$: $Z_9 \bigcup \{x_1, x_2, \cdots, x_5\}$

$(x_1, 1, 0, 3, x_2, 2), (x_3, 2, 0, 4, x_4, 3), (x_5, 4, 0, 5, 2, 1)$, mod 9.

G_2-$ID_3(15, 6)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_6\}$

$(x_1, 2_0, 1_2, 0_0, x_3, 1_1), (x_1, 2_1, 1_0, 0_1, x_2, 0_2), (x_4, 1_1, 0_0, 2_1, x_3, 1_2),$
$(x_2, 0_0, 1_2, 1_1, x_4, 0_2), (x_2, 0_0, 1_0, 1_2, x_4, 2_0), (x_5, 0_1, 1_1, 1_0, 1_2, 0_2),$
$(x_6, 1_1, 2_1, 1_2, 2_2, 0_0), (x_5, 0_0, 0_1, 1_1, x_6, 1_2), (x_5, 1_0, 0_0, 0_1, x_6, 0_2),$
$(x_1, 0_2, 0_1, 2_2, x_3, 2_0)$, mod $(3, -)$.

G_2-$ID_3(16, 7)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_7\}$

$(x_1, 2_0, 1_2, 0_0, x_3, 1_1), (x_1, 2_1, 1_0, 0_1, x_2, 0_2), (x_4, 1_1, 0_0, 2_1, x_3, 1_2),$
$(x_2, 0_0, 1_2, 1_1, x_4, 0_2), (x_2, 0_0, 1_0, 1_2, x_4, 2_0), (x_5, 0_1, 1_1, 1_0, x_7, 0_2),$

$(x_6,1_1,2_1,1_2,2_2,0_0),(x_5,0_0,x_7,1_1,x_6,1_2),(x_5,1_0,0_0,0_1,x_6,0_2),$
$(x_7,0_0,0_1,2_1,0_2,1_2),(x_1,0_2,0_1,2_2,x_3,2_0),$ mod $(3,-)$.

G_2-$ID_3(17,8)$: $Z_9\bigcup\{x_1,x_2,\cdots,x_8\}$
$(x_1,1,0,3,x_2,2),(x_3,2,0,4,x_4,3),(x_5,3,0,1,x_6,4),$
$(x_7,1,0,4,x_8,2)$, mod 9.

G_3-$ID_3(11,2)$: $Z_9\bigcup\{x_1,x_2\}$ $(x_1,4,0,5,x_2,2),(0,1,4,5,2,3)$, mod 9.

G_3-$ID_3(12,3)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,x_3\}$
$(0_0,x_1,2_2,1_0,x_2,0_1),(0_1,x_1,1_0,1_1,x_2,1_2),(0_1,x_3,1_0,2_1,0_0,0_2),$
$(2_0,x_3,1_1,0_0,0_2,1_2),(2_0,x_3,2_2,1_2,0_1,1_1),(1_0,0_1,0_0,1_2,1_1,2_2),$
$(0_2,x_1,2_0,1_2,x_2,0_1)$, mod $(3,-)$.

G_3-$ID_3(13,4)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,x_3,x_4\}$
$(0_0,x_1,0_2,1_0,x_2,2_1),(1_1,x_1,0_2,0_1,x_2,0_0),(1_0,x_3,0_1,0_0,x_4,1_2),$
$(1_1,x_3,0_0,0_1,x_4,0_2),(1_2,x_3,2_0,1_1,0_2,2_1),(0_0,0_1,0_2,1_0,2_2,1_1),$
$(2_0,x_4,0_2,1_2,0_0,1_1),(1_2,x_1,0_0,0_2,x_2,0_1)$, mod $(3,-)$.

G_3-$ID_3(14,5)$: $Z_9\bigcup\{x_1,x_2,\cdots,x_5\}$
$(0,x_1,5,2,x_2,1),(0,x_3,7,3,x_4,2),(x_5,5,0,7,3,4)$, mod 9.

G_3-$ID_3(15,6)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,\cdots,x_6\}$
$(1_0,x_1,0_2,0_0,x_2,1_1),(1_1,x_1,0_2,0_1,x_2,0_0),(1_0,x_3,2_2,0_0,x_4,2_1),$
$(1_1,x_3,1_0,0_1,x_4,2_2),(1_2,x_3,0_0,2_2,x_4,0_1),(1_0,x_5,0_0,1_1,x_6,0_1),$
$(1_2,x_5,0_2,0_0,x_6,1_0),(0_1,x_5,2_1,2_2,x_6,1_2),(0_0,2_1,1_1,1_0,0_2,1_2),$
$(2_2,x_1,0_0,1_2,x_2,2_1)$, mod $(3,-)$.

G_3-$ID_3(16,7)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,\cdots,x_7\}$
$(1_0,x_1,0_2,0_0,x_7,1_1),(1_1,x_1,0_2,0_1,x_2,0_0),(1_0,x_3,2_2,0_0,x_4,2_1),$
$(1_1,x_3,1_0,0_1,x_4,2_2),(1_2,x_3,0_0,2_2,x_4,0_1),(1_0,x_5,0_0,1_1,x_6,0_1),$
$(1_2,x_5,0_2,0_0,x_6,1_0),(0_1,x_5,2_1,2_2,x_6,1_2),(0_0,x_7,1_1,1_0,0_2,1_2),$
$(0_1,x_7,0_0,2_1,x_2,2_2),(2_2,x_1,0_0,1_2,x_2,2_1)$, mod $(3,-)$.

G_3-$ID_3(17,8)$: $Z_9\bigcup\{x_1,x_2,\cdots,x_8\}$
$(x_1,2,0,1,x_2,5),(x_3,3,0,4,x_6,2),(x_5,2,0,4,x_4,5),$
$(x_7,2,0,4,x_8,1)$, mod 9.

G_3-$ID_3(8,2)$: $(Z_3\times I_2)\bigcup\{x_1,x_2\}$
$(1_0,0_1,2_0,1_1,2_1,0_0),(1_1,x_1,1_0,0_0,x_2,0_1),$
$(0_0,x_1,0_1,1_1,x_2,1_0)$, mod $(3,-)$.

G_3-$ID_3(9,3)$: $\{1,2,\cdots,6\}\bigcup\{x_1,x_2,x_3\}$
$(5,x_2,3,1,x_3,6),(4,x_1,1,2,x_2,3),(2,x_1,5,3,x_2,6),(x_3,1,3,5,6,4),$
$(x_3,2,6,4,1,5),(x_3,3,5,2,4,6),(x_3,4,1,3,6,5),(2,x_1,5,4,x_2,6),$
$(5,x_1,6,1,x_3,2),(3,x_1,1,2,x_2,4)\times 2$.

G_3-$ID_3(10,4)$: $\{1,2,\cdots,6\}\bigcup\{x_1,x_2,x_3,x_4\}$
$(3,x_1,1,2,x_2,4)\times 2,(4,x_1,1,2,x_2,3),(2,x_1,5,3,x_2,6),(2,x_1,5,4,x_2,6),$
$(5,x_1,2,4,x_3,6),(5,x_2,3,1,x_3,6),(x_4,3,1,5,4,6),(5,x_3,1,3,x_4,2),$
$(3,x_3,1,6,x_4,5),(6,x_3,4,1,x_4,3),(5,x_3,4,6,x_4,2),(x_4,2,6,4,1,5)$.

G_3-$ID_3(11,5)$: $(Z_3\times I_2)\bigcup\{x_1,x_2,\cdots,x_5\}$
$(0_1,x_1,1_0,0_0,x_2,1_1),(0_0,x_1,0_1,1_1,x_2,1_0),(0_1,x_4,1_1,0_0,x_5,1_0),$

$(0_0, x_3, 1_0, 0_1, x_5, 1_1), (0_1, x_3, 0_0, 1_0, x_4, 1_1)$, mod $(3, -)$.

G_4-$ID_3(11,2)$: $Z_9 \bigcup \{x_1, x_2\}$ $(x_1, 3, 0, 6, x_2, 4), (0, 1, 2, 7, 5, 3)$, mod 9.

G_4-$ID_3(12,3)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3\}$

$(x_1, 2_0, 0_2, 1_2, 0_1, 1_1), (x_1, 1_2, 1_1, 0_1, 1_0, 0_0), (x_2, 2_2, 0_0, 1_0, x_3, 0_1)$,
$(x_2, 2_0, 0_1, 1_1, x_3, 0_2), (x_2, 0_0, 0_2, 1_2, x_3, 2_1), (0_0, 0_1, 0_2, 1_1, 1_0, 2_2)$,
$(x_1, 1_2, 0_1, 2_2, 0_0, 1_0)$, mod $(3, -)$.

G_4-$ID_3(13,4)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$

$(x_1, 2_0, 0_2, 1_2, x_4, 1_1), (x_1, 1_2, 1_1, 0_1, x_4, 0_0), (x_2, 2_2, 0_0, 1_0, x_3, 0_1)$,
$(x_2, 2_0, 0_1, 1_1, x_3, 0_2), (x_2, 0_0, 0_2, 1_2, x_3, 2_1), (0_0, 0_1, 0_2, 1_1, 1_0, 2_2)$,
$(0_1, x_4, 1_0, 0_0, 0_2, 2_1), (x_1, 1_2, 0_1, 2_2, 0_0, 1_0)$, mod $(3, -)$.

G_4-$ID_3(14,5)$: $Z_9 \bigcup \{x_1, x_2, \cdots, x_5\}$

$(x_1, 1, 0, 3, x_2, 2), (x_3, 2, 0, 4, x_4, 3), (x_5, 5, 0, 4, 2, 1)$, mod 9.

G_4-$ID_3(15,6)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_6\}$

$(0_0, x_1, 1_1, 1_0, 1_2, x_2), (0_0, x_1, 0_2, 1_0, 0_1, x_2), (1_1, x_3, 2_2, 0_2, 1_0, x_4)$,
$(2_0, x_3, 0_1, 2_2, 0_2, x_4), (2_0, x_3, 0_2, 0_1, 1_1, x_4), (1_0, x_5, 0_1, 1_1, 1_2, x_6)$,
$(1_2, x_5, 0_0, 2_2, 0_1, x_6), (2_1, x_5, 0_2, 1_2, 1_0, x_6), (2_0, 0_0, 1_0, 2_2, 0_1, 2_1)$,
$(0_1, x_1, 1_2, 1_1, 2_0, x_2)$, mod $(3, -)$.

G_4-$ID_3(16,7)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_7\}$

$(2_0, x_1, 0_1, 0_0, 2_2, x_2), (0_0, x_1, 1_2, 1_0, 1_1, x_2), (0_2, x_3, 2_1, 0_0, 1_0, x_4)$,
$(0_0, x_3, 0_2, 0_1, 1_1, x_4), (1_1, x_3, 1_2, 0_2, 2_0, x_4), (2_0, x_5, 0_1, 1_0, 2_2, x_6)$,
$(0_0, x_5, 2_1, 0_2, 1_2, x_7), (2_1, x_6, 1_2, 0_2, 0_0, x_7), (2_1, x_5, 2_2, 0_0, 1_0, x_7)$,
$(0_0, x_6, 0_1, 2_1, 0_2, 1_1), (0_1, x_1, 2_2, 0_0, 1_0, x_2)$, mod $(3, -)$.

G_4-$ID_3(17,8)$: $Z_9 \bigcup \{x_1, x_2, \cdots, x_8\}$

$(x_1, 1, 0, 8, x_2, 2), (x_3, 2, 0, 7, x_4, 4), (x_5, 4, 0, 3, x_6, 1)$,
$(x_7, 3, 0, 6, x_8, 4)$, mod 9.

G_4-$ID_3(21,12)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{12}\}$

$(1_0, x_9, 0_1, 2_0, 0_2, x_{12}), (2_1, x_{11}, 0_2, 0_1, 0_0, x_{12}), (1_0, x_1, 0_1, 2_0, 0_2, x_2)$,
$(1_0, x_7, 0_0, x_2, 0_1, x_8), (1_2, x_{10}, 1_0, 0_2, 1_1, x_4), (0_0, x_9, 1_1, x_3, 0_2, x_{11})$,
$(1_2, x_9, 0_0, x_3, 2_1, 0_2), (1_1, 0_1, 0_2, 1_0, 0_0, x_4), (1_0, x_5, 0_0, x_1, 0_1, x_6)$,
$(1_1, x_5, 0_1, x_1, 0_2, x_6), (1_2, x_5, 0_2, x_1, 0_0, x_6), (1_2, x_{11}, 1_0, 0_2, 1_1, x_{12})$,
$(1_1, x_7, 0_1, x_2, 0_2, x_8), (1_2, x_7, 0_2, x_2, 0_0, x_8), (0_1, x_{10}, 0_0, x_3, 1_2, x_4)$,
$(2_1, x_3, 0_2, 0_1, 1_0, x_{10})$, mod $(3, -)$.

G_4-$ID_3(22,13)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{13}\}$

$(1_0, x_{11}, 0_1, 2_0, 0_2, x_{12}), (2_1, x_{11}, 0_2, 0_1, 0_0, x_{12}), (1_1, x_7, 0_1, x_2, 0_2, x_8)$,
$(1_2, x_7, 0_2, x_2, 0_0, x_8), (1_1, 0_1, 0_2, 1_0, 0_0, x_4), (1_0, x_1, 0_1, 2_0, 0_2, x_2)$,
$(2_1, x_3, 0_2, 0_1, 1_0, x_{13}), (1_2, x_{13}, 1_0, 0_2, 1_1, x_4), (1_0, x_5, 0_0, x_1, 0_1, x_6)$,
$(1_1, x_5, 0_1, x_1, 0_2, x_6), (1_2, x_5, 0_2, x_1, 0_0, x_6), (1_0, x_7, 0_0, x_2, 0_1, x_8)$,
$(0_0, x_9, 1_1, x_3, 0_2, x_{10}), (0_1, x_9, 1_2, x_3, 0_0, x_{10}), (0_1, x_{13}, 0_0, 0_2, 1_2, x_4)$,
$(1_2, x_9, 0_0, x_3, 2_1, x_{10}), (1_2, x_{11}, 1_0, 0_2, 1_1, x_{12})$, mod $(3, -)$.

G_4-$ID_3(24,15)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{15}\}$

$(1_0, x_{11}, 0_0, x_3, 0_1, x_{12}), (1_1, x_{11}, 0_1, x_3, 0_2, x_{12}), (2_0, x_{13}, 1_1, 0_0, 0_2, x_{14})$,
$(1_1, x_{13}, 0_2, 2_1, 2_0, x_{14}), (1_2, x_{13}, 0_0, 2_2, 2_1, x_{14}), (1_0, x_9, 0_0, x_2, 0_1, x_{10})$,
$(1_1, x_9, 0_1, x_2, 0_2, x_{10}), (1_2, x_9, 0_2, x_2, 0_0, x_{10}), (1_0, x_{15}, 0_1, 2_0, 2_2, x_5)$,

$(2_1, x_{15}, 0_2, 1_2, 0_0, x_5), (1_2, x_{15}, 0_0, 2_2, 0_1, x_5), (0_0, x_4, 0_1, x_6, 0_2, 1_0),$
$(0_1, x_4, 0_2, x_6, 0_0, x_3), (1_2, x_1, 0_1, x_6, 0_0, x_2), (0_0, x_7, 0_1, x_1, 0_2, x_8),$
$(0_1, x_7, 0_2, x_1, 0_0, x_8), (0_2, x_7, 0_0, x_1, 0_1, x_8), (1_2, x_{11}, 0_2, x_3, 0_0, x_{12}),$
$(0_0, x_6, 1_1, 0_1, 2_2, x_4)$, mod $(3, -)$.

G_4-$ID_3(25, 16)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{16}\}$
$(1_0, x_{11}, 0_0, x_3, 0_1, x_{12}), (1_1, x_{11}, 0_1, x_3, 0_2, x_{12}), (2_0, x_{13}, 1_1, 0_0, 0_2, x_{14}),$
$(1_1, x_{13}, 0_2, 2_1, 2_0, x_{14}), (1_2, x_{13}, 0_0, 2_2, 2_1, x_{14}), (1_0, x_{15}, 0_1, 2_0, 2_2, x_{16}),$
$(2_1, x_{15}, 0_2, 1_2, 0_0, x_{16}), (1_2, x_{15}, 0_0, 2_2, 0_1, x_{16}), (1_0, x_9, 0_0, x_2, 0_1, x_{10}),$
$(1_1, x_9, 0_1, x_2, 0_2, x_{10}), (1_2, x_9, 0_2, x_2, 0_0, x_{10}), (0_0, x_4, 0_1, x_6, 0_2, x_5),$
$(0_1, x_4, 0_2, x_6, 0_0, x_5), (1_2, x_1, 0_1, x_6, 0_0, x_2), (0_0, x_7, 0_1, x_1, 0_2, x_8),$
$(0_1, x_7, 0_2, x_1, 0_0, x_8), (0_2, x_7, 0_0, x_1, 0_1, x_8), (0_0, x_3, 1_1, 0_1, 2_2, x_4),$
$(0_2, x_5, 1_0, 0_0, 2_1, x_6), (1_2, x_{11}, 0_2, x_3, 0_0, x_{12})$, mod $(3, -)$.

G_4-$ID_3(20, 2)$: $(Z_9 \times I_2) \bigcup \{x_1, x_2\}$
$(8_0, x_1, 5_1, 0_0, 1_0, x_2), (6_1, x_1, 3_0, 0_1, 4_1, x_2), (1_1, 0_1, 3_1, 1_0, 0_0, 2_0),$
$(4_1, 0_1, 1_1, 7_0, 4_0, 0_0), (0_0, 3_0, 2_1, 0_1, 6_1, 4_0), (1_0, 0_0, 3_0, 1_1, 0_1, 2_1),$
$(4_1, 0_1, 4_0, 8_1, 2_0, 0_0)$, mod $(9, -)$.

G_4-$ID_3(23, 5)$: $(Z_9 \times I_2) \bigcup \{x_1, x_2, \cdots, x_5\}$ $(5_0, x_3, 0_1, 3_1, 0_0, x_4),$
$(7_0, x_1, 3_1, 0_0, 2_0, x_2), (3_1, x_1, 0_0, 3_0, 0_1, x_2), (1_1, x_3, 3_0, 0_1, 4_1, x_4),$
$(0_0, x_5, 0_1, 2_1, 6_0, 1_1), (4_0, x_5, 0_0, 5_0, 7_1, 2_1), (1_0, 0_0, 3_0, 1_1, 0_1, 2_1),$
$(1_1, 0_1, 3_1, 1_0, 0_0, 2_0), (4_0, 0_0, 1_0, 7_1, 4_1, 0_1)$, mod $(9, -)$.

G_4-$ID_3(26, 8)$: $(Z_9 \times I_2) \bigcup \{x_1, x_2, \cdots, x_8\}$
$(7_0, x_1, 3_1, 0_0, 2_0, x_2), (3_1, x_1, 0_0, 3_0, 0_1, x_2), (1_1, x_3, 3_0, 0_1, 4_1, x_4),$
$(5_0, x_3, 0_1, 3_1, 0_0, x_4), (0_0, x_5, 0_1, 2_1, 6_0, 1_1), (1_0, 0_0, 3_0, 1_1, 0_1, 2_1),$
$(4_0, x_5, 0_0, 5_0, 7_1, 2_1), (1_1, 0_1, 3_1, 1_0, 0_0, x_8), (4_0, 0_0, 1_0, 7_1, 4_1, x_8),$
$(2_0, x_6, 1_1, 0_0, 7_0, x_7), (3_1, x_6, 4_0, 0_1, 4_1, x_7)$, mod $(9, -)$.

G_5-$ID_3(12, 3)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3\}$
$(0_0, x_1, 2_2, x_2, 1_1, 1_0), (0_1, x_1, 1_0, x_2, 2_2, 1_1), (0_0, x_3, 1_2, 1_1, 0_1, 2_2),$
$(0_1, x_3, 1_0, 0_0, 1_2, 2_0), (0_2, x_3, 1_1, 1_2, 0_0, 2_1), (0_0, 2_1, 0_1, 2_2, 0_2, 1_0),$
$(0_2, x_1, 0_1, x_2, 0_0, 1_2)$, mod $(3, -)$.

G_5-$ID_3(13, 4)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$
$(0_0, x_1, 2_2, x_2, 2_1, 1_0), (0_1, x_1, 1_0, x_2, 2_2, 1_1), (0_0, x_3, 1_2, 1_1, 0_1, 2_2),$
$(0_1, x_3, 1_0, 0_0, 1_2, 2_0), (0_2, x_3, 1_1, x_4, 0_0, 2_1), (0_0, x_4, 0_1, 2_2, 0_2, 1_0),$
$(1_1, x_4, 0_0, 1_2, 2_2, 0_1), (0_2, x_1, 0_1, x_2, 0_0, 1_2)$, mod $(3, -)$.

G_5-$ID_3(15, 6)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_6\}$
$(0_0, x_1, 1_2, x_2, 1_1, 1_0), (0_1, x_1, 0_0, x_2, 0_2, 1_1), (0_0, x_3, 0_2, x_4, 2_1, 1_0),$
$(0_1, x_3, 1_0, x_4, 2_2, 1_1), (0_2, x_3, 2_1, x_4, 0_0, 1_2), (0_2, x_5, 1_1, x_6, 1_0, 1_2),$
$(0_1, x_5, 2_0, x_6, 0_2, 1_2), (0_2, x_5, 0_1, x_6, 2_0, 1_0), (0_0, 2_1, 1_1, 1_0, 1_2, 0_1),$
$(0_2, x_1, 2_1, x_2, 1_0, 1_2)$, mod $(3, -)$.

G_5-$ID_3(16, 7)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_7\}$
$(0_0, x_1, 1_2, x_2, 1_1, 1_0), (0_1, x_1, 0_0, x_2, 0_2, 1_1), (0_0, x_3, 0_2, x_4, 2_1, 1_0),$
$(0_1, x_3, 1_0, x_4, 2_2, 1_1), (0_2, x_3, 2_1, x_4, 0_0, 1_2), (0_2, x_5, 1_1, x_6, 1_0, 1_2),$
$(0_1, x_5, 2_0, x_6, 0_2, 2_2), (0_2, x_5, 0_1, x_7, 2_0, 1_0), (0_0, x_6, 1_1, x_7, 1_2, 0_1),$

$(0_0, x_7, 0_1, 2_1, 0_2, 1_0), (0_2, x_1, 2_1, x_2, 1_0, 1_2)$, mod $(3, -)$.

G_5-$ID_3(21, 12)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{12}\}$

$(0_0, x_1, 1_0, x_2, 0_1, x_{11}), (0_1, x_1, 1_1, x_2, 0_2, x_{11}), (0_0, x_3, 1_0, x_4, 0_1, x_{12}),$
$(0_1, x_3, 1_1, x_4, 0_2, x_{12}), (0_2, x_3, 1_2, x_4, 0_0, x_{12}), (0_0, x_5, 1_0, x_6, 0_1, x_{11}),$
$(0_1, x_5, 1_1, x_6, 0_2, x_{11}), (0_2, x_5, 1_2, x_6, 1_0, x_{11}), (2_2, x_9, 0_1, x_{10}, 1_0, 0_0),$
$(1_1, x_9, 0_0, x_{10}, 2_2, 0_2), (0_0, x_9, 1_2, x_{10}, 2_1, 2_2), (1_1, x_7, 0_0, x_8, 2_2, 2_0),$
$(1_2, x_7, 0_1, x_8, 1_0, x_{12}), (0_0, x_7, 1_2, x_8, 1_1, x_{12}), (2_1, x_{11}, 0_0, x_{12}, 1_2, x_{12}),$
$(0_2, x_1, 1_2, x_2, 0_0, x_{11})$, mod $(3, -)$.

G_5-$ID_3(22, 13)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{13}\}$

$(0_0, x_1, 1_0, x_2, 0_1, x_{11}), (0_1, x_1, 1_1, x_2, 0_2, x_{11}), (0_0, x_3, 1_0, x_4, 0_1, x_{12}),$
$(0_1, x_3, 1_1, x_4, 0_2, x_{12}), (0_2, x_3, 1_2, x_4, 0_0, x_{12}), (0_0, x_5, 1_0, x_6, 0_1, x_{13}),$
$(0_1, x_5, 1_1, x_6, 0_2, x_{13}), (0_2, x_5, 1_2, x_6, 1_0, x_{13}), (1_1, x_7, 0_0, x_8, 2_2, x_{13}),$
$(1_2, x_7, 0_1, x_8, 1_0, x_{13}), (0_0, x_7, 1_2, x_8, 1_1, x_{13}), (2_2, x_9, 0_1, x_{10}, 1_0, 0_0),$
$(1_1, x_9, 0_0, x_{10}, 2_2, 2_0), (0_0, x_9, 1_2, x_{10}, 2_1, x_{13}), (2_1, x_{11}, 0_0, x_{12}, 1_2, x_{13}),$
$(2_2, x_{11}, 0_1, x_{12}, 0_0, x_{13}), (0_2, x_1, 1_2, x_2, 0_0, x_{11})$, mod $(3, -)$.

G_5-$ID_3(8, 2)$: $(Z_3 \times I_2) \bigcup \{x_1, x_2\}$

$(0_0, x_1, 1_0, x_2, 2_1, 1_1), (2_1, x_2, 0_0, 1_0, 0_1, 2_0),$
$(1_1, x_1, 0_1, 0_0, 1_0, 2_1)$, mod$(3, -)$.

G_5-$ID_3(11, 5)$: $(Z_3 \times I_2) \bigcup \{x_1, x_2, \cdots, x_5\}$

$(1_0, x_1, 0_0, 0_1, 1_1, 2_1), (1_0, x_2, 0_0, x_3, 0_1, 2_1), (0_1, x_3, 1_1, x_4, 0_0, x_2),$
$(0_0, x_4, 1_0, x_5, 1_1, x_2), (0_1, x_5, 1_1, x_1, 1_0, x_2)$, mod $(3, -)$.

G_6-$ID_3(12, 3)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3\}$

$(0_0, x_1, 2_2, x_2, 1_1, 0_1), (0_1, x_1, 1_0, x_2, 2_2, 0_0), (0_0, x_3, 1_2, 1_1, 0_1, 2_0),$
$(0_1, x_3, 1_0, 0_0, 1_2, 0_2), (0_2, x_1, 0_1, x_2, 0_0, 1_0), (0_2, x_3, 1_1, 1_2, 0_0, 1_0),$
$(0_0, 2_1, 0_1, 2_2, 0_2, 2_1)$, mod $(3, -)$.

G_6-$ID_3(13, 4)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$

$(0_0, x_1, 2_2, x_2, 2_1, 0_1), (0_1, x_1, 1_0, x_2, 2_2, 0_2), (0_2, x_1, 0_1, x_2, 0_0, 1_2),$
$(0_2, x_3, 1_2, 1_1, 0_1, 2_1), (0_1, x_3, 1_2, 0_0, 1_0, 2_1), (0_2, x_3, 1_1, x_4, 0_0, 1_0),$
$(0_0, x_4, 0_1, 2_2, 0_2, 2_1), (1_1, x_4, 0_0, 1_2, 2_2, 0_0)$, mod $(3, -)$.

G_6-$ID_3(15, 6)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_6\}$

$(0_0, x_1, 1_2, x_2, 1_1, 0_1), (0_1, x_1, 0_0, x_2, 0_2, 1_2), (0_2, x_1, 2_1, x_2, 1_0, 0_0),$
$(0_0, x_3, 0_2, x_4, 2_1, 0_1), (0_1, x_3, 1_0, x_4, 2_2, 0_2), (0_2, x_3, 2_1, x_4, 0_0, 2_2),$
$(0_2, x_5, 1_1, x_6, 1_0, 0_0), (0_1, x_5, 2_0, x_6, 0_2, 2_1), (0_2, x_5, 0_1, x_6, 2_0, 2_1),$
$(0_0, 2_1, 1_1, 1_0, 1_2, 0_2)$, mod $(3, -)$.

G_6-$ID_3(16, 7)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_7\}$

$(0_0, x_1, 1_2, x_2, 1_1, 0_1), (0_1, x_1, 0_0, x_2, 0_2, 1_2), (0_2, x_1, 2_1, x_2, 1_0, 0_0),$
$(0_0, x_3, 0_2, x_4, 2_1, 0_1), (0_1, x_3, 1_0, x_4, 2_2, 0_2), (0_2, x_3, 2_1, x_4, 0_0, 1_0),$
$(0_2, x_5, 1_1, x_6, 1_0, 0_0), (0_1, x_5, 2_0, x_6, 0_2, 1_2), (0_2, x_5, 0_1, x_7, 2_0, 2_1),$
$(0_0, x_6, 1_1, x_7, 1_2, 2_0), (0_0, x_7, 0_1, 2_1, 0_2, 1_1)$, mod $(3, -)$.

G_6-$ID_3(21, 12)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{12}\}$

$(0_0, x_1, 1_0, x_2, 0_1, x_{11}), (0_1, x_1, 1_1, x_2, 0_2, x_{11}), (0_2, x_1, 1_2, x_2, 0_0, x_{11}),$
$(0_0, x_3, 1_0, x_4, 0_1, 2_2), (0_1, x_3, 1_1, x_4, 0_2, 1_0), (0_2, x_3, 1_2, x_4, 0_0, 2_1),$
$(0_0, x_5, 1_0, x_6, 0_1, x_{12}), (0_1, x_5, 1_1, x_6, 0_2, x_{12}), (0_2, x_5, 1_2, x_6, 0_0, x_{12}),$

$(1_1, x_7, 0_0, x_8, 2_2, x_{12}), (1_2, x_7, 0_1, x_8, 1_0, x_{12}), (0_0, x_7, 1_2, x_8, 1_1, x_{12}),$
$(1_1, x_9, 0_0, x_{10}, 2_2, x_{12}), (2_2, x_9, 0_1, x_{10}, 1_0, x_{12}), (0_0, x_9, 1_2, x_{11}, 2_1, x_{12}),$
$(2_1, x_{11}, 0_0, x_{10}, 1_2, 2_0)$, mod $(3, -)$.

G_6-$ID_3(22, 13)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{13}\}$
$(0_0, x_1, 1_0, x_2, 0_1, x_{11}), (0_1, x_1, 1_1, x_2, 0_2, x_{11}), (0_2, x_1, 1_2, x_2, 0_0, x_{11}),$
$(0_0, x_3, 1_0, x_4, 0_1, x_{12}), (0_1, x_3, 1_1, x_4, 0_2, x_{12}), (0_2, x_3, 1_2, x_4, 0_0, x_{12}),$
$(0_0, x_5, 1_0, x_6, 0_1, x_{13}), (0_1, x_5, 1_1, x_6, 0_2, x_{13}), (0_2, x_5, 1_2, x_6, 0_0, x_{13}),$
$(1_1, x_7, 0_0, x_8, 2_2, x_{13}), (1_2, x_7, 0_1, x_8, 1_0, x_{13}), (0_0, x_7, 1_2, x_8, 1_1, x_{13}),$
$(1_1, x_9, 0_0, x_{10}, 2_2, x_{13}), (2_2, x_9, 0_1, x_{10}, 1_0, x_{13}), (0_0, x_9, 1_2, x_{10}, 2_1, x_{13}),$
$(2_1, x_{11}, 0_0, x_{12}, 1_2, 2_0), (2_2, x_{11}, 0_1, x_{12}, 0_0, 2_1)$, mod $(3, -)$.

G_7-$ID_3(9, 3)$: $\{1, 2, \cdots, 6\} \bigcup \{x_1, x_2, x_3\}$
$(1, 2, 3, x_1, 4, x_2), (1, 2, 3, x_2, 4, x_1), (1, x_2, 5, 3, x_3, 6), (1, x_1, 2, 5, x_3, 6),$
$(4, 5, 6, 2, x_3, 1), (6, x_2, 3, 5, x_3, 2), (6, 3, x_3, 4, 5, x_1), (6, 4, x_3, 1, 5, x_2),$
$(3, 2, 4, 5, x_3, 1), (3, x_1, 5, 6, x_3, 4), (6, 4, 1, 3, x_3, 2)$.

G_7-$ID_3(10, 4)$: $\{1, 2, \cdots, 6\} \bigcup \{x_1, x_2, x_3, x_4\}$
$(4, x_3, 6, 2, x_4, 5), (1, x_4, 3, 6, x_3, 4), (4, 5, x_1, 6, x_2, 2), (1, 2, 3, x_1, 4, x_2),$
$(1, 2, 3, x_2, 4, x_1), (1, 2, 3, x_3, 4, x_4), (5, x_1, 6, 2, 4, 3), (x_1, 1, 3, 4, x_2, 5),$
$(x_2, 5, 1, 3, x_1, 6), (x_3, 1, 4, 6, x_4, 5), (3, x_2, 1, 6, 4, 5),$
$(1, 3, 4, x_4, 2, x_3), (x_4, 6, 2, 5, x_3, 3)$.

G_7-$ID_3(11, 2)$: $Z_9 \bigcup \{x_1, x_2\}$ $(x_1, 0, 1, 3, 5, 4), (x_2, 0, 8, 4, 1, 3)$, mod 9.

G_7-$ID_3(12, 3)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3\}$
$(x_1, 0_0, 0_1, 0_2, 1_0, 1_1), (2_0, x_1, 1_0, 0_2, 1_2, 1_1), (2_0, x_2, 0_0, 0_1, x_1, 1_2),$
$(x_2, 2_0, 1_0, 0_1, 2_2, 0_2), (1_1, 0_1, x_3, 2_1, x_2, 2_2), (1_0, x_3, 0_1, 1_2, 2_1, 0_0),$
$(x_3, 0_2, 0_1, 1_2, 1_0, 2_0)$, mod $(3, -)$.

G_7-$ID_3(13, 4)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$
$(0_1, 0_0, 1_0, x_1, 0_2, x_2), (1_0, 0_2, 2_0, 1_2, 0_1, x_4), (2_2, 1_2, 0_0, x_4, 2_1, 1_1),$
$(0_2, 0_0, 0_1, x_3, 2_1, 1_0), (0_0, 2_1, 2_2, 1_0, 1_2, x_3), (2_1, 0_1, x_4, 1_0, x_3, 1_2),$
$(0_1, 0_2, x_2, 1_1, x_1, 2_2), (0_0, 2_1, 2_0, x_2, 0_2, x_1)$, mod $(3, -)$.

G_7-$ID_3(15, 6)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_6\}$
$(0_0, 2_1, 2_0, x_2, 0_2, x_1), (0_1, 0_0, 1_0, x_1, 0_2, x_2), (0_2, 0_0, 0_1, x_3, 2_1, x_6),$
$(0_0, 2_1, 2_2, 1_0, 1_2, x_3), (2_1, 0_1, x_4, 1_0, x_3, 1_2), (2_2, 1_2, 0_0, x_4, 2_1, 1_1),$
$(1_0, 0_2, 2_0, x_5, 0_1, x_4), (2_2, 0_0, 1_0, x_6, 0_1, x_5), (0_0, 2_2, x_5, 1_1, x_6, 0_2),$
$(0_1, 0_2, x_2, 1_1, x_1, 2_2)$, mod $(3, -)$.

G_7-$ID_3(16, 7)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_7\}$
$(0_0, 2_1, 2_0, x_2, 0_2, x_1), (0_1, 0_0, 1_0, x_1, 0_2, x_2), (0_2, 0_0, 0_1, x_3, 2_1, x_6),$
$(0_0, 2_1, x_7, 1_0, 1_2, x_3), (2_1, 0_1, x_4, 1_0, x_3, 1_2), (2_2, 1_2, 0_0, x_4, 2_1, 1_1),$
$(1_0, 0_2, 2_0, x_5, 0_1, x_4), (2_2, 0_0, 1_0, x_6, 0_1, x_5), (x_7, 2_2, x_5, 1_1, x_6, 0_2),$
$(0_1, 2_0, 1_2, x_7, 0_0, 0_2), (0_1, 0_2, x_2, 1_1, x_1, 2_2)$, mod $(3, -)$.

G_7-$ID_9(14, 5)$: $Z_9 \bigcup \{x_1, x_2, \cdots, x_5\}$
$(1, 0, x_2, 2, 4, x_1), (2, 0, x_3, 4, 8, x_2), (3, 0, 4, x_1, 1, x_3),$
$(1, 0, x_5, 3, 4, x_4), (0, 1, 2, x_4, 4, 3), (1, 0, x_4, 4, 7, x_5),$
$(0, x_2, 5, 2, x_1, 4), (0, 2, 5, x_3, 7, 4), (0, 1, 5, x_5, 8, 4)$, mod 9.

G_7-$ID_9(17, 8)$: $Z_9 \bigcup \{x_1, x_2, \cdots, x_8\}$

$(1,0,x_2,2,4,x_1),(2,0,x_3,4,8,x_2),(3,0,4,x_1,1,x_3),$
$(1,0,x_5,3,4,x_4),(1,0,x_4,4,7,x_5),(0,x_2,5,2,x_1,4),$
$(4,0,5,x_7,1,x_8),(0,x_3,5,2,x_6,1),(0,x_4,7,4,x_7,2),$
$(0,x_5,4,1,x_8,5),(2,1,0,x_2,3,x_6),(2,0,3,x_6,1,x_7),$ mod 9.

G_8-$ID_3(8,2)$: $(Z_3\times I_2)\bigcup\{x_1,x_2\}$
$(0_0,x_1,1_1,1_0,x_2,0_1),(0_0,1_0,x_1,x_2,1_1,0_1),$
$(0_0,1_0,2_0,2_1,1_1,0_1),$ mod $(3,-)$.

G_8-$ID_3(11,5)$: $(Z_3\times I_2)\bigcup\{x_1,x_2,\cdots,x_5\}$
$(0_0,x_1,1_0,1_1,x_2,2_1),(0_0,x_3,0_1,1_0,x_4,1_1),(x_5,2_0,x_1,x_2,1_1,0_1),$
$(x_5,1_0,2_0,1_1,2_1,0_0),(0_0,1_0,x_3,x_4,1_1,0_1),$ mod $(3,-)$.

G_8-$ID_3(12,3)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,x_3\}$
$(1_0,x_1,1_1,1_2,2_0,0_0),(x_1,0_0,x_3,x_2,1_1,0_2),(0_0,x_2,0_1,1_2,1_1,2_2),$
$(1_2,x_3,2_2,0_1,0_0,1_1),(x_3,0_0,0_1,0_2,1_2,2_1),(0_2,0_1,1_1,2_1,0_0,1_2),$
$(1_0,1_1,0_2,x_1,2_2,x_2),$ mod $(3,-)$.

G_8-$ID_3(13,4)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,x_3,x_4\}$
$(0_0,x_1,0_2,1_1,1_2,2_2),(2_1,x_2,2_2,1_1,0_1,1_0),(2_2,x_4,1_1,0_2,0_0,1_0),$
$(x_1,0_0,1_2,x_4,1_1,2_2),(x_2,0_0,1_0,x_1,1_1,0_2),(x_3,0_0,2_1,x_2,1_2,0_1),$
$(x_4,1_1,2_2,x_3,0_2,0_0),(0_0,x_3,0_2,1_1,0_1,1_0),$ mod $(3,-)$.

G_8-$ID_3(15,6)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,\cdots,x_6\}$
$(0_0,x_1,1_0,0_2,x_2,0_1),(0_1,x_3,0_0,0_2,x_4,1_1),(1_1,0_0,1_0,0_2,0_1,2_2),$
$(x_1,0_1,1_0,x_2,0_2,2_0),(x_2,0_0,1_1,x_3,1_2,0_1),(x_3,2_1,x_4,1_2,0_0,0_2),$
$(x_4,1_2,x_5,0_2,2_2,0_0),(x_5,2_0,x_6,0_0,1_0,1_1),(x_6,1_1,x_1,0_1,2_2,1_2),$
$(0_2,x_5,0_1,1_1,x_6,0_0),$ mod $(3,-)$.

G_8-$ID_3(16,7)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,\cdots,x_7\}$
$(0_1,x_1,0_0,1_0,x_2,0_2),(0_1,x_3,0_0,0_2,x_4,1_1),(0_2,x_7,1_0,2_0,0_0,1_2),$
$(x_1,1_0,0_2,x_2,1_2,2_1),(x_2,0_1,0_0,x_3,2_2,1_1),(x_3,0_0,2_1,x_4,1_2,1_0),$
$(x_4,1_1,2_2,x_5,1_2,0_0),(x_5,0_0,0_2,x_6,1_0,0_1),(x_6,0_1,0_0,x7,1_1,0_2),$
$(x_7,2_1,1_1,x_1,1_2,0_0),(1_2,x_5,0_0,0_2,x_6,0_1),$ mod $(3,-)$.

G_8-$ID_3(21,12)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,\cdots,x_{12}\}$
$(0_1,x_1,0_0,0_2,x_2,1_1),(x_1,2_1,x_2,1_0,0_2,0_0),(x_8,2_1,x_7,x_{12},1_2,0_2),$
$(x_3,1_1,x_4,x_{10},2_2,0_1),(x_4,0_0,x_3,1_1,0_2,0_1),(0_0,x_5,0_2,1_2,x_6,0_1),$
$(x_5,1_0,x_6,x_{12},0_2,0_1),(x_6,1_1,x_5,2_2,0_0,0_1),(x_2,1_0,x_1,2_2,0_2,0_0),$
$(0_0,x_7,1_0,0_1,x_8,1_0),(x_7,0_1,x_8,x_9,0_2,2_2),(x_{12},2_0,x_{11},x_{10},0_1,0_2),$
$(x_9,1_0,x_{10},x_{11},2_2,0_0),(x_{11},0_0,x_{12},x_9,0_1,1_2),(0_0,x_3,1_0,0_2,x_4,1_1),$
$(x_{10},2_1,x_9,x_{11},0_2,0_0),$ mod $(3,-)$.

G_8-$ID_3(22,13)$: $(Z_3\times I_3)\bigcup\{x_1,x_2,\cdots,x_{13}\}$
$(0_0,x_1,0_1,0_2,x_2,1_0),(x_2,1_1,x_1,x_3,0_2,0_1),(x_1,0_0,x_2,x_4,1_2,0_1),$
$(x_8,0_0,x_7,x_{10},2_2,1_1),(x_{12},1_0,0_1,x_{13},2_0,0_2),(0_2,x_3,0_0,0_1,x_4,1_2),$
$(x_3,1_0,x_5,x_6,1_1,0_0),(x_4,1_1,x_5,x_6,1_0,0_1),(x_5,1_2,x_6,x_7,1_1,0_2),$
$(x_6,2_0,x_5,x_7,1_2,0_2),(x_{13},1_2,0_0,1_0,0_2,0_1),(0_0,x_{10},0_1,0_2,x_{11},1_0),$
$(0_0,x_8,1_2,1_1,x_9,0_2),(x_7,1_1,x_8,x_9,1_0,0_1),(x_9,0_1,x_{11},x_{12},2_2,0_0),$
$(x_{11},0_0,x_{12},x_{13},2_2,0_1),(x_{10},0_1,x_{12},x_{13},1_1,0_2),$ mod $(3,-)$.

G_9-$ID_3(8,2)$: $(Z_3\times I_2)\bigcup\{x_1,x_2\}$

$(0_1, 0_0, 2_0, 2_1, 1_1, 1_0), (0_1, x_1, 0_0, 1_0, 2_0, x_2),$
$(0_1, x_2, 1_0, 0_0, 1_1, x_1),$ mod $(3, -)$.

G_9-$ID_3(10, 4)$: $\{1, 2, \cdots, 6\} \bigcup \{A, B, C, D\}$
$(1, A, 2, 3, 4, B), (2, B, 3, 5, 6, A), (1, C, 2, 3, 5, D), (3, D, 2, 5, 1, C),$
$(4, C, 1, 3, 6, D), (4, D, 2, 3, 5, C), (3, 2, C, 1, 6, D), (2, 5, A, 1, 3, B),$
$(4, 6, A, B, 5, C), (4, 5, A, B, 6, D), (6, 1, D, 3, 2, C),$
$(1, A, 2, 3, 4, B), (1, B, 2, 3, 4, A)$.

G_9-$ID_3(11, 5)$: $(Z_3 \times I_2) \bigcup \{x_1, x_2, \cdots, x_5\}$
$(0_0, x_1, 1_1, 2_1, 1_0, x_2), (1_0, x_2, 1_1, 2_1, 0_1, x_1), (2_1, x_4, 0_1, 1_0, 0_0, x_5),$
$(2_0, x_5, 1_0, 1_1, 0_1, x_3), (0_1, x_3, 1_0, 1_1, 0_0, x_4),$ mod $(3, -)$.

G_9-$ID_3(12, 3)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3\}$
$(0_2, 2_1, 0_0, 1_0, 2_2, 1_2), (x_2, 0_0, x_1, x_3, 0_2, 1_1), (2_1, 1_0, x_1, 1_2, 0_1, 1_1),$
$(1_0, 1_2, 1_1, x_2, 2_0, 0_0), (0_0, 1_1, x_3, 1_0, 2_2, 0_1), (x_1, 0_0, x_2, x_3, 0_1, 2_2),$
$(x_3, 1_1, x_1, x_2, 1_2, 0_0),$ mod $(3, -)$.

G_9-$ID_3(13, 4)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, x_3, x_4\}$
$(0_2, 2_1, x_3, 0_0, 2_2, 1_2), (1_0, 0_0, x_4, 2_0, 0_1, 2_2), (x_4, 0_0, x_1, x_2, 2_2, 0_1),$
$(1_1, 0_0, x_1, 0_2, 2_1, 0_1), (0_0, 0_1, x_2, 2_0, 0_2, 1_0), (x_1, 0_0, x_2, x_3, 0_1, 0_2),$
$(x_2, 0_0, x_3, x_4, 0_2, 2_1), (x_3, 2_1, x_1, x_4, 1_2, 0_0),$ mod $(3, -)$.

G_9-$ID_3(15, 6)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_6\}$
$(1_0, 0_0, x_1, 0_2, 0_1, x_4), (1_0, 0_0, x_2, 2_0, 0_2, 1_1), (x_1, 0_0, x_2, x_3, 0_1, 0_2),$
$(x_5, 2_2, x_6, x_1, 0_0, 1_1), (x_6, 2_2, x_1, x_2, 0_1, 1_0), (x_4, 0_0, x_5, x_6, 1_1, 2_2),$
$(x_3, 2_1, x_4, x_5, 0_2, 2_0), (1_2, 0_0, x_5, 2_1, 2_2, 0_2), (x_2, 0_2, x_3, x_4, 0_0, 1_1),$
$(0_1, 1_2, x_3, x_6, 2_1, 1_1),$ mod $(3, -)$.

G_9-$ID_3(16, 7)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_7\}$
$(1_1, 2_0, 0_0, 2_1, 1_2, x_7), (x_7, 0_0, x_2, 2_0, 0_2, 1_1), (1_2, 0_0, x_5, x_7, 2_2, 0_2),$
$(x_6, 1_2, x_1, x_2, 0_1, 1_0), (x_3, 2_1, x_4, x_5, 0_2, 0_0), (1_0, 0_0, x_1, x_7, 0_1, x_4),$
$(x_4, 0_0, x_5, x_6, 1_1, 1_2), (x_5, 2_2, x_6, x_1, 0_0, 1_1), (x_1, 0_0, x_2, x_3, 0_1, 0_2),$
$(x_2, 0_2, x_3, x_4, 0_0, 1_1), (0_1, 1_2, x_3, x_6, 2_1, 1_1),$ mod $(3, -)$.

G_9-$ID_3(21, 12)$: $(Z_3 \times I_3) \bigcup \{x_1, x_2, \cdots, x_{12}\}$
$(2_2, 0_0, x_4, x_5, 1_1, x_6), (x_2, 1_1, x_1, x_3, 0_2, 1_2), (0_1, 0_0, x_8, x_9, 0_2, x_7),$
$(x_5, 0_1, x_4, x_6, 1_1, 1_2), (x_6, 0_2, x_5, x_4, 1_2, 0_0), (x_7, 2_0, x_8, x_9, 0_1, 1_0),$
$(x_8, 1_1, x_7, x_9, 0_2, 2_1), (x_9, 1_2, x_7, x_8, 0_0, 2_1), (0_2, 0_0, x_1, x_2, 0_1, x_3),$
$(x_1, 0_0, x_2, x_3, 0_1, 1_2), (x_4, 0_0, x_5, x_6, 1_0, 2_2), (x_{10}, 0_0, x_{11}, x_{12}, 1_0, 2_1),$
$(x_{11}, 0_1, x_{10}, x_{12}, 1_1, 2_1), (2_1, 0_0, x_{11}, x_{10}, 1_2, x_{12}), (x_3, 2_2, x_2, x_1, 0_0, 1_0),$
$(x_{12}, 0_2, x_{10}, x_{11}, 1_2, 2_0),$ mod $(3, -)$.

4.4 Conclusion

Theorem 1. *The necessary conditions for the existence of* G_i-$GD_\lambda(v)$ *are also sufficient for any* G_i $(1 \le i \le 9)$ *with the exceptions* $(v, i, \lambda) \in \{(10, 2, 1), (6, 8, 3)\} \bigcup \{(9, k, 1) : k = 4, 5, 6, 8, 9\} \bigcup \{(6, 2, u) : 3|u, 6 \not| u\}$.

Proof. Summarizing Lemmas 1, 2, 6–13 and the constructions in §4.1,

§4.2 and §4.3, the conclusions hold. ■

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