Second factorization of the Generalized Lah Matrix

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Abstract

In [3], we gave a factorization of the generalized Lah matrix. In this short note, we show its another factorization. From this factorization, several interesting combinatorial identities involving the Fibonacci numbers are obtained.

Key Words: Generalized Lah matrix, Fibonacci matrix, Factorization of matrix

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In this note, we shall still apply the definitions of the generalized Lah number $L_{n,k}(x, y)$ and the generalized Lah matrix $\mathcal{L}[x, y]$ in [3].

Let x and y be two nonzero real numbers. the generalized Lah numbers are defined by $L_{n,k}(x, y) = x^n y^k \binom{n-1}{k-1} \frac{n!}{k!}$. The $n \times n$ gener-

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alized Lah matrix $\mathcal{L}[x,y]$ is defined by

$$\mathcal{L}[x,y] = [L_{i,j}(x,y)]_{i,j=1,2,\dots,n} = \begin{cases} x^i y^j \binom{i-1}{j-1} \frac{i!}{j!}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let F_n be the n-th Fibonacci number. The $n \times n$ Fibonacci matrix $\mathcal{F} = [f_{i,j}]$ $(i,j = 1,2,\dots,n)$ is defined by

$$f_{i,j} = \begin{cases} F_{i-j+1}, & \text{if } i-j+1 \geq 0, \\ 0, & \text{if } i-j+1 < 0. \end{cases} \tag{2}$$

In [1], Lee, etc. gave the inverse of $\mathcal{F}$ as follows: if $\mathcal{F}^{-1} = [f'_{i,j}]$ $(i,j = 1,2,\dots,n)$, then

$$f'_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i-2 \leq j \leq i-1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

By using the inverse of $\mathcal{F}$, in [2], Lee, etc. studied the Pascal matrix and the Stirling matrices of the first kind and of the second kind via Fibonacci matrix.

Recently, in [3], we gave the power formula of the generalized Lah matrix $\mathcal{L}[x,y]$ and showed $\mathcal{L}[x,y] = \mathcal{F}\mathcal{Q}[x,y]$, where $\mathcal{Q}[x,y] = [q_{i,j}(x,y)]_{i,j=1,2,\dots,n}$ was the lower triangular matrix which was defined by

$$q_{i,j}(x,y) = \left(x^i \binom{i-1}{j-1} i! - x^{i-1} \binom{i-2}{j-1} (i-1)! - x^{i-2} \binom{i-3}{j-1} (i-2)! \right) \frac{y^j}{j!}. \tag{4}$$

From this matrix representation, several combinatorial identities involving the Fibonacci numbers were obtained.

The purpose of this note is to provide another factorization of the generalized Lah matrix: $\mathcal{L}[x,y] = \mathcal{R}[x,y]\mathcal{F}$, where $\mathcal{F}$ is the Fibonacci matrix and $\mathcal{R}[x,y]$ is the lower triangular matrix. Finally, we give several interesting combinatorial identities involving the Fibonacci numbers. These results were apparently missed in [3].

We define the $n \times n$ matrix $\mathcal{R}[x,y] = [r_{i,j}(x,y)]$ $(i,j = 1,2,\ldots,n)$ as follows:

$$r_{i,j}(x,y) = x^i y^j \left(\binom{i-1}{j-1} \frac{1}{j!} - \binom{i-1}{j} \frac{y}{(j+1)!} - \binom{i-1}{j+1} \frac{y^2}{(j+2)!} \right) i!, \qquad (5)$$

i.e,

$$r_{i,j}(x,y) = L_{i,j}(x,y) - L_{i,j+1}(x,y) - L_{i,j+2}(x,y). \qquad (6)$$

Then we have

Theorem 1.

$$\mathcal{L}[x,y] = \mathcal{R}[x,y]\mathcal{F}. \qquad (7)$$

Proof. Since the matrix $\mathcal{F}$ is invertible, then it suffices to prove $\mathcal{L}[x,y]\mathcal{F}^{-1} = \mathcal{R}[x,y]$. Let $\mathcal{F}^{-1} = [f'_{i,j}]$ be the inverse of $\mathcal{F}$. By (3) and the definition of $\mathcal{L}[x,y]$, we have

$$\begin{aligned} & \sum_{k=1}^{n} L_{i,k}(x,y) f'_{k,j} = \sum_{k=j}^{i} L_{i,k}(x,y) f'_{k,j} \\ = \; & L_{i,j}(x,y) f'_{j,j} + L_{i,j+1}(x,y) f'_{j+1,j} + L_{i,j+2}(x,y) f'_{j+2,j} \\ = \; & L_{i,j}(x,y) - L_{i,j+1}(x,y) - L_{i,j+2}(x,y) \\ = \; & q_{i,j}(x,y), \end{aligned}$$

as desired. □

We have the following interesting identities involving the Fibonacci numbers from Theorem 1.

Theorem 2.

$$y^r \binom{n-1}{r-1} \frac{n!}{r!} = y^n F_{n-r+1} + y^{n-1} \left(n^2 - n - y\right) F_{n-r} + n! \sum_{k=r}^{n-2} \binom{n-1}{k-1} \frac{y^k}{k!} \left(1 - \frac{n-k}{k(k+1)} y - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)} y^2\right) F_{k-r+1}. \qquad (8)$$

In particular,

$$\binom{n-1}{r-1}\frac{n!}{r!} = F_{n-r+1} + \left(n^2-n-1\right)F_{n-r} + n!\sum_{k=r}^{n-2}\binom{n-1}{k-1}$$
$$\times\frac{1}{k!}\left(1-\frac{n-k}{k(k+1)}-\frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}\right)F_{k-r+1} \tag{9}$$

and

$$(-1)^r\binom{n-1}{r-1}\frac{n!}{r!} = (-1)^n F_{n-r+1} + (-1)^{n-1}\left(n^2-n+1\right)F_{n-r} +$$
$$+n!\sum_{k=r}^{n-2}\binom{n-1}{k-1}\frac{(-1)^k}{k!}\left(1+\frac{n-k}{k(k+1)}-\frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}\right)$$
$$\times F_{k-r+1}. \tag{10}$$

Proof. From (5), we have

$$r_{n,n}(x,y) = x^n y^n, \quad r_{n,n-1} = x^n y^{n-1}(n^2-n-y);$$

and for $k \le n-2$,

$$\begin{aligned} & r_{n,k}(x,y) \\ = \; & x^n y^k\left(\binom{n-1}{k-1}\frac{1}{k!} - \binom{n-1}{k}\frac{y}{(k+1)!} - \binom{n-1}{k+1}\frac{y^2}{(k+2)!}\right)n! \\ = \; & n!x^n y^k\binom{n-1}{k-1}\frac{1}{k!}\left(1-\frac{n-k}{k(k+1)}y - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}y^2\right). \end{aligned}$$

From $\mathcal{L}[x,y] = \mathcal{R}[x,y]\mathcal{F}$, it follows

$$\begin{aligned} & x^n y^r\binom{n-1}{r-1}\frac{n!}{r!} = L_{n,r}(x,y) = \sum_{k=r}^{n} r_{n,k}(x,y)F_{k-r+1} \\ = \; & r_{n,n}(x,y)F_{n-r+1} + r_{n,n-1}(x,y)F_{n-r} + \sum_{k=r}^{n-2} r_{n,k}(x,y)F_{k-r+1} \\ = \; & x^n y^n F_{n-r+1} + x^n y^{n-1}(n^2-n-y)F_{n-r} + \sum_{k=r}^{n-2} x^n y^k\binom{n-1}{k-1}\frac{n!}{k!} \\ & \times\left(1-\frac{n-k}{k(k+1)}y - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}y^2\right)F_{k-r+1}. \end{aligned}$$

Dividing both sides of the equation by x^n, the proof of (8) is completed. In (8), taking $y = 1$ or $y = -1$, (9) and (10) can be obtained respectively. □

Corollary 1.

$$n!y = y^n F_n + y^{n-1}\left(n^2 - n - y\right) F_{n-1} + n! \sum_{k=1}^{n-2} \binom{n-1}{k-1} \times \frac{y^k}{k!}\left(1 - \frac{n-k}{k(k+1)}y - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}y^2\right) F_k. \tag{11}$$

Specially,

$$n! = F_n + \left(n^2 - n - 1\right) F_{n-1} + n! \sum_{k=1}^{n-2} \binom{n-1}{k-1} \frac{1}{k!}\left(1 - \frac{n-k}{k(k+1)} - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}\right) F_k \tag{12}$$

and

$$-n! = (-1)^n F_n + (-1)^{n-1}\left(n^2 - n + 1\right) F_{n-1} + n! \sum_{k=1}^{n-2} \binom{n-1}{k-1} \times \frac{(-1)^k}{k!}\left(1 + \frac{n-k}{k(k+1)} - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}\right) F_k. \tag{13}$$

Proof. Take $r = 1$ in Theorem 2. □

Corollary 2.

$$\frac{1}{2}n!(n-1)y^2 = y^n F_{n-1} + y^{n-1}\left(n^2 - n - y\right) F_{n-2} + n! \sum_{k=r}^{n-2} \binom{n-1}{k-1} \frac{y^k}{k!}\left(1 - \frac{n-k}{k(k+1)}y - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}y^2\right) F_{k-1}. \tag{14}$$

Specially,

$$\frac{1}{2}n!(n-1) = F_{n-1} + \left(n^2 - n - 1\right) F_{n-2} + n! \sum_{k=r}^{n-2} \binom{n-1}{k-1} \times \frac{1}{k!}\left(1 - \frac{n-k}{k(k+1)} - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}\right) F_{k-1} \tag{15}$$

and

$$\frac{1}{2}n!(n-1) = (-1)^n F_{n-1} + (-1)^{n-1}\left(n^2 - n + 1\right) F_{n-2} + n! \sum_{k=r}^{n-2} \binom{n-1}{k-1} \frac{(-1)^k}{k!} \left(1 + \frac{n-k}{k(k+1)} - \frac{(n-k)(n-k-1)}{k(k+1)^2(k+2)}\right) F_{k-1}. \qquad (16)$$

Proof. Take $r = 2$ in Theorem 2. □

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