Lower Bounds on Ramsey Numbers $R(6,8)$, $R(7,9)$ and $R(8,17)$†

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Abstract. For integers $s,t \geq 1$, the Ramsey number $R(s,t)$ is defined to be the least positive integer n such that every graph on n vertices contains either a clique of order s or an independent set of order t. In this note, we derive new lower bounds for the Ramsey numbers: $R(6,8) \geq 129$, $R(7,9) \geq 235$ and $R(8,17) \geq 937$. The new bounds are obtained with a constructive method proposed by Xu and Xie et al. and the help of computer algorithm.

1 Introduction

In this note, we only consider graphs without multiple edges or loops. If $G = (V,E)$ is a graph, then the set of vertices is denoted by $V(G)$, the set of edges of G is denoted by $E(G)$, and the cardinality of $V(G)$ is denoted by $n(G)$. $G[S]$ denotes the subgraph induced in G by a subset of

†Supported by National Natural Science Foundation of China (Grant Nos. 60674106, 60533010, and 60503002), Program for New Century Excellent Talents in University (NCET-05-0612), Ph.D. Programs Foundation of Ministry of Education of China (20060487014), Chenguang Program of Wuhan (200750731262), and HUST-SRF (2007Z015A).

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vertices $S \subset V(G)$. For a positive integer n, let $A_n = \{1, 2, \cdots, n\}$, and $S \subseteq \{1, 2, \cdots, \lfloor n/2 \rfloor\}$. If G is a graph with the vertex set $V(G) = A_n$ and the edge set $E(G) = \{(x, y) : \mathbf{dist}(x, y) \in S \text{ and } x, y \in A_n\}$, where $\mathbf{dist}(x, y)$=min$\{|i-j|, n-|i-j|\}$, then G is called a cyclic graph of order n, which is denoted by $G_n(S)$. Please refer to [1] for more notation of graph theory.

For integers $s, t \geq 1$, the Ramsey number $R(s, t)$ is defined to be the least positive integer n such that every graph on n vertices contains either a clique of order s or an independent set of order t. G is called a (p, q)-graph if G contains neither a complete graph on p vertices nor an independent set of order q. A (p, q)-graph on n vertices is called a $(p, q; n)$-graph. The existence of $R(s, t)$ is a well-known consequence of Ramsey's theorem. For an extensive survey for Ramsey numbers, please refer to [5].

In 2004, it was proved that

Theorem 1 *[6] Let G be a (k, p)-graph and H a (k, q)-graph such that G and H contain an induced subgraph isomorphic to a K_{k-1}-free graph M, then $R(k, p+q-1) \geq n(G) + n(H) + n(M) + 1$.*

Let G be a (k, p)-graph and H be a (k, q)-graph. A simple algorithm is developed to find an induced K_{k-1}-free subgraph M_1 in G and an induced K_{k-1}-free subgraph M_2 in H such that $M_1 \cong M_2$. In this note, with the help of computer we used this algorithm to obtain $R(6, 8) \geq 129$, $R(7, 9) \geq 235$ and $R(8, 17) \geq 937$, which improves the best known results $R(6, 8) \geq 127$, $R(7, 9) \geq 233$ and $R(8, 17) \geq 929$ (see [5]).

2 Lower bounds on Ramsey numbers $R(6, 8)$, $R(7, 9)$ and $R(8, 17)$

In order to obtain lower bounds of Ramsey numbers $R(6, 8)$, $R(7, 9)$ and $R(8, 17)$, the main point is to find the K_{k-1}-free induced subgraph M in Theorem 1 such that the order of M is as great as possible. This is done by a computer algorithm, which is available from the first author by E-mail. In the proofs of the following Theorems 2 and 3, we directly give the subgraph M omitting the description of the algorithm application, which will not cause any trouble for checking the correctness of Theorems 2 and 3.

Theorem 2 $R(6, 8) \geq 129$.

Proof. Let $G = G_{101}(S)$, where S is the set of quadratic residues, and H be a graph shown in Fig. 1. Then, G is a $(6, 6; 101)$-graph and H is a

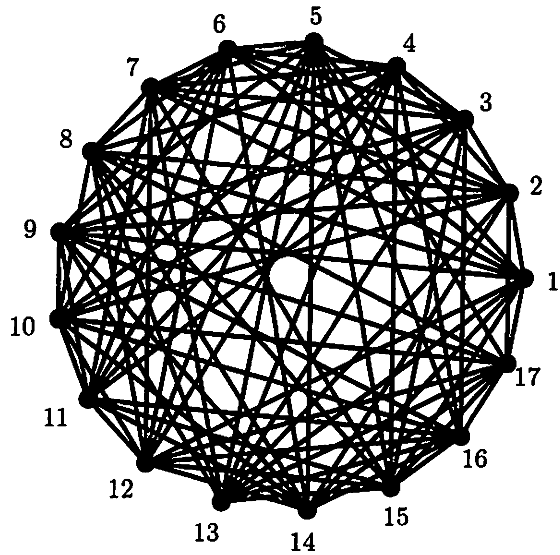

Figure 1: a (6, 3; 17)-graph

(6, 3; 17)-graph. We find a K_5-free graph M_1 in G, where

$$M_1 = G[S_1], S_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 25, 28\} \subset V(G),$$

and a K_5-free graph M_2 in H, where

$$M_2 = H[S_2], S_2 = \{1, 2, 3, 4, 5, 7, 10, 14, 16, 17\} \subset V(H).$$

We can verify that $M_1 \cong M_2$, where the isomorphic mapping f maps the vertices in S_1 to those in S_2 in the following way,

$$f(1) = 4, f(2) = 10, f(3) = 14, f(4) = 1, f(5) = 7,$$
$$f(6) = 3, f(7) = 16, f(8) = 17, f(25) = 2, f(28) = 5.$$

By the inequality $R(k, p+q-1) \geq n(G) + n(H) + n(M) + 1$ mentioned in section 1, we have $R(6, 8) \geq n(G)+n(H)+n(M_1)+1 = 101+17+10+1 = 129$.

Theorem 3 $R(7, 9) \geq 235$.

Proof. Let $G = G_{204}$ be a (7, 7; 204)-graph [8]. Let $n = 21$, $S = \{1, 3, 8\}$, $H' = G_n(S)$, then the cyclic graph H' is a (3, 7; 21)-graph. Suppose H is the complementary graph of H', then H is a (7, 3; 21)-graph. We find a K_6-free graph M_1 in G, where

$$M_1 = G[S_1], S_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 24\} \subset V(G),$$

and a K_6-free graph M_2 in H, where

$$M_2 = H[S_2], S_2 = \{1, 2, 3, 4, 5, 6, 8, 14, 19\} \subset V(H).$$

We can verify that $M_1 \cong M_2$, where the isomorphic mapping f maps the vertices in S_1 to those in S_2 in the following way,

$$f(1) = 3, f(2) = 14, f(3) = 2, f(4) = 6, f(5) = 1,$$
$$f(6) = 5, f(7) = 19, f(8) = 4, f(24) = 8.$$

By the inequality $R(k, p+q-1) \geq n(G) + n(H) + n(M) + 1$ mentioned in section 1, we have $R(7, 9) \geq n(G) + n(H) + n(M_1) + 1 = 204 + 21 + 9 + 1 = 235$.

By Theorem 3, we obtain the following corollary.

Corollary 1 $R(8, 17) \geq 937$.

Proof. In [7], Xu and Xie et al. gave the following inequality: $R(2k-1, l) \geq 4R(k, l-1) - 3$, for $l \geq 5, k \geq 2$. By this inequality, we can have $R(8, 17) \geq 4R(7, 9) - 3 = 4 \times 235 - 3 = 937$.

Acknowledgements

The kind and valuable comments and suggestion from Xiaodong Xu, S. P. Radziszowski, and the anonymous referees are greatly acknowledged.

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