A note on graphs containing all trees of a given size *

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Abstract

There are some results and many conjectures with the conclusion that a graph G contains all trees of given size k. We prove some new results of this type.

1 Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs $G = (V, E)$ of order $n = |V|$ and size $e(G) = |E|$. All graphs will be assumed to have neither loops nor multiple edges.

In this note we are interested in graphs which contain all trees of given size k. There are some results and many conjectures with this conclusion.
The following simple fact is often referred to as a 'folklore lemma'. Usually, it is presented in the following form:

Theorem 1 *Let T be a tree of size k. If $\delta(G) \geq k$, then G contains T.* ■

The most famous conjecture of this type is probably the well known Erdős-Sós conjecture.

Conjecture 2 (Erdős-Sós) *If G is a graph on n vertices and the number of edges of G is $e(G) > \frac{n(k-1)}{2}$ then G contains all trees of size k.*

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The conjecture below was firstly formulated by Loebl in 1995 in the case $k = \frac{n}{2}$ and then generalised by Komlós and Sós.

Conjecture 3 (Loebl-Komlós-Sós) *If G is a graph on n vertices with at least $\frac{n}{2}$ vertices of degree at least k, then G contains all trees of size k.*

These two conjectures are still open (for some special cases of them see for example [6] as well as [2] and [5] or [1] or [3]). We mention that using the Regularity Lemma, Ajtai, Komlós and Szemerédi proved an approximate form of the Loebl-Komlós-Sós conjecture (see [4]).

Observe that these two last conjectures can be formulated in a way which involves degrees of the vertices of the graph. Indeed, by dividing both sides of the size condition in the Erdős-Sós conjecture by $n/2$ and denoting by $\bar{d}(G)$ the average degree of graph G, we get the following statement.

Erdős-Sós conjecture (degree form) If $\bar{d}(G) > k - 1$, then G contains all trees of size k.

As remarked in [4], the Loebl-Komlós-Sós conjecture can be formulated as follows:

Loebl-Komlós-Sós conjecture (degree form) If the median degree of G is at least k. then G contains all trees of size k.

It seams to be natural to consider other conditions concerning degrees of the graph. In particular, we may ask if the conditions below imply the existence of all trees of size k as subgraphs of G.

(1) $d(x) + d(y) \geq 2k$ for pair of nonadjacent vertices of G.

(2) $\max\{d(x), d(y)\} \geq k$ for each pair of vertices of G with $\text{dist}(x, y) = 2$.

The above conditions are analogous to well-known conditions from hamiltonian problems, namely to Ore's condition and Fan's condition, respectively. Observe that the Fan-type condition (2) is weaker than Ore-type condition (1).

The answer in both cases is YES. Actually, we shall prove a somewhat stronger result. In order to formulate it, we need some additional definitions.

For a graph G, we define $B = \{v \in V(G) \mid d_G(v) \geq k\}$ and $S = V(G) - B$. The vertices of B and S will also be referred to as *big* vertices and *small* vertices, respectively. We denote by $N(x)$ the set of neighbours of x and by $N'(x)$ the **graph induced** by small neighbours of x and call it the *small neighbourhood* of x *i.e.*

$$N'(x) = G[N(x) \cap S].$$

We shall consider the following condition:

$(*)$ For each big vertex of G, its small neighbourhood is a clique.

Our main result can be now formulated as follows:

Theorem 4 *Let G be a graph of order n having at least one vertex of degree at least k. If $(*)$ holds, then G contains all trees of size k.*

Corollary 5 *If* (2) *holds, then G contains all trees of size k.*

Proof. We shall show that the condition (2) implies $(*)$. Let us observe first that the fact that there is a vertex of degree at least k follows trivially from (2). So, suppose there is a big vertex of G having two nonadjacent small neighbours.. It suffices to observe that these two small vertices are then of distance two which contradicts (2). ∎

Since, as remarked above, condition (2) is weaker than condition (1) we have also the following corollary.

Corollary 6 *If* (1) *holds, then G contains all trees of size at most k.* ∎

2 Proof of the main result

Let G be a graph satisfying the hypothesis of Theorem 4 and let T be a tree of size k. If T is a subgraph of G we are done. If not, we remove the pendent vertices of T one by one until the tree T' obtained in this way is a subgraph of the graph induced by big vertices of G. (Observe, that this is always possible since G contains at least one big vertex and the graph induced by this vertex contains K_1.) Suppose that T' has p edges, $p < k$.

Let $T^1, \ldots, T^q$ be the non-singleton components of $T - E(T')$. A root of a removed subtree of T would then be the unique vertex in a T^i that is also an end-point of an edge in T'. Denote these roots by $x_1, \ldots, x_q$, respectively.

Of course, we have

$$(**) \qquad \sum_{i=1}^{q} e(T^i) = k - p.$$

Let us consider now the root x_1 as a vertex of G. By definition of T', x_1 is a big vertex. This implies that it has at least $k-p$ neighbours outside of T'. All these vertices are small. For, otherwise by adding one edge joining x_1 with its big neighbour outside of T' we would get a bigger subtree of T as a subgraph of $G[B]$. By assumption, the small neighbourhood $N'(x_1)$ is complete in G. We choose arbitrary $e(T^1)$ vertices in $N'(x_1)$. Together with the vertex x_1 these vertices form a clique of size $e(T^1)+1$ in which we embed easily the tree T^1.

Similarly we proceed with other removed trees. By (**), it is always possible, even in the "worst" case when all small neighbourhoods of vertices x_i coincide. This finishes the proof of our theorem. ■

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