Remarks on two special matrices*

Weiping Wang [a,†] Tianming Wang [a,b]
[a] Department of Applied Mathematics, Dalian University of Technology
Dalian 116024, P.R.China
[b] Department of Mathematics, Hainan Normal University
Haikou 571158, P.R.China

Abstract

The current paper deals with two special matrices T_n and W_n related to the Pascal, Vandermonde and Stirling matrices. As a result, various properties of the entries of T_n and W_n are obtained, including the generating functions, recurrence relations, and explicit expressions. Some additional results are also presented.

Keywords: Matrices; Stirling numbers; Generating functions; Recurrence relations; Combinatorial identities; Inverse relation

1. Introduction

Recently, the connection between the Pascal, Vandermonde and Stirling matrices has been studied by M.E.A. El-Mikkawy [4, 5]. As a result, a new matrix has been constructed, that is,

$$T_n = L_n D_n^{-1} s_n = P_n V_n^{-1},$$

where L_n is the $n \times n$ Pascal matrix, s_n is the $n \times n$ Stirling matrix of the first kind, P_n is the $n \times n$ Pascal symmetric matrix, V_n is the $n \times n$ Vandermonde matrix and D_n is a $n \times n$ diagonal matrix. We will first give the explicit definitions of these matrices. For $i, j = 1, 2, \cdots, n$:

$$(L_n)_{ij} = \binom{i-1}{j-1}, \quad (s_n)_{ij} = s(i,j), \quad (P_n)_{ij} = \binom{i+j-2}{i-1},$$

$$(V_n)_{ij} = j^{i-1}, \quad (D_n)_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ (i-1)!, & \text{if } i = j, \end{cases}$$

*It is supported by NSF of Hainan Province.

† *E-mail addresses:* wpingwang@yahoo.com (W.P. Wang), wangtm@dlut.edu.cn (T.M. Wang)

where $s(i,j)$ are the Stirling numbers of the first kind.

It is clear that the matrix T_n is a lower triangular one, and it's inverse matrix $W_n = T_n^{-1} = S_n D_n L_n^{-1}$, where S_n is the $n \times n$ Stirling matrix of the second kind. The explicit expressions of S_n and L_n^{-1} are as follows:

$$(S_n)_{ij} = S(i,j)\,, \quad (L_n^{-1})_{ij} = (-1)^{i-j}\binom{i-1}{j-1}\,, \quad \text{for } i,j = 1,2,\cdots,n\,,$$

where $S(i,j)$ are the Stirling numbers of the second kind. The matrices T_{10} and W_{10} will be listed in the Appendix.

In [4, 5], M.E.A. El-Mikkawy gave the algorithms to compute the elements of these two matrices T_n and W_n by using the theory of elementary symmetric functions. He also obtained some properties of T_n, W_n by MAPLE programming. However, [4, 5] didn't provide more information about T_n, W_n and that to find the recurrence relations satisfied by the elements of these two matrices has been put forward as an open question (see [5], p. 763).

A short commentary on the open question was written by us [7], from which we noticed that the elements of T_n and W_n would have some beautiful properties, which also inspirited us to make a further study on these elements.

In the sequel, we will denote the elements of the matrices T_n and W_n by $T(i,j) := (T_n)_{i+1,j+1}$ and $W(i,j) := (W_n)_{i+1,j+1}$, respectively, for $i,j = 0,1,\cdots,n-1$. It should be noticed that the indices of the elements of T_n, W_n are different from those of [4, 5, 7], where in [4, 5, 7] the elements were denoted by $T_{ij} := (T_n)_{ij}$ and $W_{ij} := (W_n)_{ij}$ for $i,j = 1,2,\cdots,n$. We find that the changes of the indices will bring great convenience for computation.

This article is organized as follows. $T(i,j)$, i.e., the elements of the matrix T_n will be studied in Section 2. We will show that

$$T(i,j) = \frac{1}{i!}c(i,j) = (-1)^{i+j}\frac{1}{i!}s(i,j)\,,$$

where $c(i,j) = (-1)^{i+j}s(i,j)$ are the unsigned Stirling numbers of the first kind. Some properties of $T(i,j)$ will be also found there. In Section 3, we will study $W(i,j)$ in a similar way, and we will prove that

$$W(i,j) = (-1)^{i+j}j!S(i,j)\,.$$

Finally, in Section 4, some additional results related to $T(i,j)$ and $W(i,j)$ are derived.

2. Properties satisfied by $T(i,j)$

Since $T_n = L_n D_n^{-1} s_n$, we can find that

$$T(i,j) = \sum_{k=j+1}^{i+1} \binom{i}{k-1} \frac{1}{(k-1)!} s(k, j+1) = \sum_{k=j}^{i} \binom{i}{k} \frac{1}{k!} s(k+1, j+1) .$$

Then, the 'horizontal' generating function of $T(i,j)$ can be reached:

$$\sum_{j \geq 0} T(i,j) x^j = \sum_{j=0}^{i} (\sum_{k=j}^{i} \binom{i}{k} \frac{1}{k!} s(k+1, j+1)) x^j$$

$$= \sum_{k=0}^{i} \binom{i}{k} \frac{1}{k!} \sum_{j=0}^{k} s(k+1, j+1) x^j = \sum_{k=0}^{i} \binom{i}{k} \frac{1}{k!} \sum_{l=1}^{k+1} s(k+1, l) x^{l-1}$$

$$= \sum_{k=0}^{i} \binom{i}{k} \frac{1}{k! x} \sum_{l=0}^{k+1} s(k+1, l) x^l = \sum_{k=0}^{i} \binom{i}{i-k} \frac{(x)_{k+1}}{k! x}$$

$$= \sum_{k=0}^{i} \binom{i}{i-k} \binom{x-1}{k} = \binom{i+x-1}{i},$$

where the Vandermonde convolution formula (cf., e.g., [6], Chapter 1) has been used in the last step, and $(x)_k = x(x-1)\cdots(x-k+1)$.

We can get more information from the results above. In fact, since

$$\sum_{j \geq 0} T(i,j) x^j = \binom{i+x-1}{i} = \frac{\langle x \rangle_i}{i!} = \frac{1}{i!} \sum_{j \geq 0} c(i,j) x^j ,$$

we have

$$T(i,j) = \frac{1}{i!} c(i,j) , \quad i, j \geq 0 ,$$

that is,

$$\sum_{k=j}^{i} \binom{i}{k} \frac{1}{k!} s(k+1, j+1) = (-1)^{i+j} \frac{1}{i!} s(i,j) ,$$

where $\langle x \rangle_i = x(x+1)\cdots(x+i-1)$. Then, the following theorem holds.

Theorem 1. (see [7]) *The $T(i,j)$ have the following 'horizontal' generating function:*

$$\sum_{j \geq 0} T(i,j) x^j = \binom{i+x-1}{i} , \tag{1}$$

and then

$$T(i,j) = \frac{1}{i!}c(i,j) = (-1)^{i+j}\frac{1}{i!}s(i,j)\,. \tag{2}$$

Note that by setting $x = 1$ in (1), we obtain

$$\sum_{j\geq 0} T(i,j) = 1\,,$$

which indicates that the matrix T_n is a stochastic one (see [4], p. 29) since $0 \leq T(i,j) \leq 1$, and by setting $x = -1$ in (1), we obtain

$$\sum_{j\geq 0}(-1)^j T(i,j) = 0\,, \quad \text{for } i \geq 2\,.$$

More important, by considering the combinatorial meaning of $c(n,k)$, we find that $T(i,j)$ can be interpreted as the proportion of the number of n-permutations with k-cycles to the total number of n-permutations.

Additionally, we find some other generating functions of $T(i,j)$, as shown in the next theorem.

Theorem 2. *The $T(i,j)$ have the following 'double' generating function:*

$$\sum_{i,j\geq 0} T(i,j)t^i x^j = (1-t)^{-x}\,, \tag{3}$$

and the following 'vertical' generating function:

$$\sum_{i\geq j} T(i,j)t^i = \frac{(-1)^j}{j!}\log^j(1-t)\,, \tag{4}$$

and the following 'horizontal' generating function:

$$\sum_{j\geq 0} T(i,j)u^{i-j} = \frac{1}{i!}\prod_{k=1}^{i-1}(1+ku)\,. \tag{5}$$

Proof. For (3), a direct computation will deduce that $\sum_{i,j\geq 0} T(i,j)t^i x^j = \sum_{i\geq 0}(\sum_{j=0}^{i} T(i,j)x^j)t^i = \sum_{i\geq 0}\binom{i+x-1}{i}t^i = (1-t)^{-x}$. For (4), equate the coefficients of x^j in $\sum_{j\geq 0}(\sum_{i\geq j} T(i,j)t^i)x^j = (1-t)^{-x} = \exp\{(-x)\log(1-t)\} = \sum_{j\geq 0}(-1)^j x^j \log^j(1-t)/j!$. For (5), replace x by u^{-1} in (1) and simplify. □

With the recurrence relation satisfied by the $c(i,j)$:

$$c(i,j) = c(i-1,j-1) + (i-1)c(i-1,j), \quad i,j \geq 1\,,$$
$$c(i,0) = c(0,j) = 0, \quad \text{except } c(0,0) = 1\,.$$

we can find the recurrence relation of $T(i,j)$ by means of (2).

Theorem 3. (see [7]) *The $T(i,j)$ satisfy the 'triangular' recurrence relation:*

$$T(i,j) = \frac{1}{i}T(i-1,j-1) + \frac{i-1}{i}T(i-1,j), \quad i,j \geq 1, \tag{6}$$
$$T(i,0) = T(0,j) = 0, \quad \text{except } T(0,0) = 1.$$

Thus Theorem 3 enables us to generate the elements of the stochastic lower triangular matrix T_n, for specific n, by using the following recurrence relation:

$$(T_n)_{i,j} = \frac{1}{i-1}(T_n)_{i-1,j-1} + \frac{i-2}{i-1}(T_n)_{i-1,j}, \quad i,j \geq 2,$$
$$(T_n)_{i,1} = (T_n)_{1,j} = 0, \quad \text{except } (T_n)_{1,1} = 1,$$

which solved the open question put forward in [5] (see also [7]).

Theorem 3 can be also proved according to the meaning of $T(i,j)$. And by virtue of this theorem, we can get some special values:

$$T(i,1) = \frac{1}{i}, \quad T(i,i) = \frac{1}{i!}, \quad T(i,i-1) = \frac{1}{2(i-2)!}.$$

Besides the 'triangular' recurrence relation (6), the $T(i,j)$ also satisfy some other recurrence relations, which are given in the next two theorems.

Theorem 4. *The $T(i,j)$ satisfy the 'vertical' recurrence relations:*

$$T(i,j) = \frac{1}{j}\sum_{l=1}^{i-j+1}\frac{1}{l}T(i-l,j-1), \tag{7}$$

$$T(i,j) = \frac{1}{i}\sum_{l=0}^{i-j}T(i-l-1,j-1). \tag{8}$$

Proof. Let $\Psi(t,x) = \sum_{i,j\geq 0}T(i,j)t^i x^j = (1-t)^{-x}$. For (7), equate the coefficients of $t^i x^{j-1}$ in $\partial\Psi/\partial x = -\log(1-t)\Psi$. For (8), use in an analogous way $\partial\Psi/\partial t = x(1-t)^{-1}\Psi$. □

Theorem 5. *The $T(i,j)$ satisfy the 'horizontal' recurrence relations:*

$$(i+1)T(i+1,j+1) = \sum_{k=j}^{i}\binom{k}{j}T(i,k), \tag{9}$$

$$T(i-1,j-1) = i\sum_{k=j}^{i}(1-i)^{k-j}T(i,k). \tag{10}$$

Proof. For (9), equate the coefficients of x^j in the expressions to the right of $(*)$ and $(**)$:

$$\frac{i+1}{x}\binom{i+x}{i+1} = \frac{i+1}{x}\sum_j T(i+1,j)x^j \overset{*}{=} (i+1)\sum_j T(i+1,j)x^{j-1}$$

$$= \binom{i+x}{i} = \sum_j T(i,j)(x+1)^j = \sum_j T(i,j)\sum_{l=0}^{j}\binom{j}{l}x^l$$

$$\overset{**}{=} \sum_{l\geq 0}\sum_{k\geq l}\binom{k}{l}T(i,k)x^l\,.$$

And equating the coefficients of u^{i-j} in

$$\sum_{j\geq 0} T(i-1,j)u^{i-1-j} = \frac{i}{1+(i-1)u}\sum_{j\geq 0} T(i,j)u^{i-j}$$

will yield (10). □

Making use of (2), we can obtain the value of $T(n,k)$ from the value of $s(n,k)$.

Theorem 6. *The 'exact' value of* $T(n,k)$ *is*

$$T(n,k)$$

$$=(-1)^{n+k}\frac{1}{n!}\sum_{h=0}^{n-k}(-1)^h\binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h}S(n-k+h,h) \tag{11}$$

$$=(-1)^{n+k}\frac{1}{n!}\sum_{h=0}^{n-k}\sum_{j=0}^{h}(-1)^{j+h}\binom{h}{j}\binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h}\frac{(h-j)^{n-k+h}}{h!}\,.$$

And then,

$$T(n,n-k) = \frac{1}{n!}\sum_{h=0}^{k}\binom{k-n}{k+h}\binom{k+n}{k-h}S(k+h,h)\,. \tag{12}$$

Proof. By means of ([3], p. 216)

$$s(n,k) = \sum_{0\leq h\leq n-k}(-1)^h\binom{n-1+h}{n-k+h}\binom{2n-k}{n-k-h}S(n-k+h,h)$$

and ([3], p. 204)

$$S(n,k) = \frac{1}{k!}\sum_{0\leq j\leq k}(-1)^j\binom{k}{j}(k-j)^n = \frac{1}{k!}\sum_{0\leq i\leq k}(-1)^{k-i}\binom{k}{i}i^n\,, \tag{13}$$

the exact value of $T(n,k)$ will be deduced. Then replace k by $n-k$ in (11), and the fact that

$$\binom{n-1+h}{k+h}\binom{k+n}{k-h} = (-1)^{k+h}\binom{k-n}{k+h}\binom{k+n}{k-h}$$

will lead us to (12). □

Theorem 7. *We have*

$$T(n,k) = \frac{1}{n(k-1)!}Y_{k-1}(\zeta_{n-1}(1), -1!\zeta_{n-1}(2), 2!\zeta_{n-1}(3), \cdots) \tag{14}$$

$$= \frac{1}{n!}B_{n,k}(0!, 1!, 2!, \cdots), \tag{15}$$

where Y_k stands for the exponential complete Bell polynomial ([3], p. 134), $\zeta_n(s) = \sum_{j=1}^{n} j^{-s}$, *and $B_{n,k}(x_1, x_2, \cdots)$ stands for the exponential partial Bell polynomial* ([3], p. 133).

Proof. In fact, by (1)

$$\sum_k T(n,k)x^k = \binom{n+x-1}{n} = \frac{(n+x-1)(n+x-2)\cdots(x+1)x}{n!}$$

$$= \frac{x}{n}(1+x)(1+\frac{x}{2})\cdots(1+\frac{x}{n-2})(1+\frac{x}{n-1}) = \frac{x}{n}\exp\{\sum_{j=1}^{n-1}\log(1+xj^{-1})\}$$

$$= \frac{x}{n}\exp\{\sum_{j=1}^{n-1}\sum_{s\geq 1}(-1)^{s-1}x^s s^{-1} j^{-s}\} = \frac{x}{n}\exp\{\sum_{s\geq 1}(-1)^{s-1}x^s s^{-1}\zeta_{n-1}(s)\}$$

$$= \frac{x}{n}\{1 + \sum_{k\geq 1} Y_k(\zeta_{n-1}(1), -1!\zeta_{n-1}(2), 2!\zeta_{n-1}(3), \cdots)\frac{x^k}{k!}\}.$$

Then equating the coefficients of x^k will derive (14). And (15) is a direct consequence of the fact that ([3], p. 135) $c(n,k) = B_{n,k}(0!, 1!, 2!, \cdots)$. □

With (14), we can obtain the following special values:

$$T(n,2) = \frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) = \frac{1}{n}H_{n-1},$$

$$T(n,3) = \frac{1}{2n}\{H_{n-1}^2 - (1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2})\},$$

$$T(n,4) = \frac{1}{6n}\{H_{n-1}^3 - 3H_{n-1}(1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2})$$
$$+ 2(1 + \frac{1}{2^3} + \cdots + \frac{1}{(n-1)^3})\},$$

where H_n denotes the harmonic number.

Theorem 8. *We have*

$$T(n,k)=\frac{1}{n!}\sum_{1\le i_1<i_2<\cdots<i_{n-k}\le n-1} i_1 i_2\cdots i_{n-k}$$
$$=\frac{1}{n}\sum_{1\le j_1<j_2<\cdots<j_{k-1}\le n-1}\frac{1}{j_1 j_2\cdots j_{k-1}}=\frac{1}{k!}\sum_{\substack{r_1+r_2+\cdots+r_k=n\\ r_1,r_2,\cdots,r_k\ge 1}}\frac{1}{r_1 r_2\cdots r_k}$$
$$=\sum_{\substack{k_1+2k_2+\cdots+nk_n=n\\ k_1+k_2+\cdots+k_n=k,\ k_1,k_2,\cdots,k_n\ge 0}}\frac{1}{k_1!k_2!\cdots k_n!}\left(\frac{1}{1}\right)^{k_1}\left(\frac{1}{2}\right)^{k_2}\cdots\left(\frac{1}{n}\right)^{k_n}.$$

Proof. These are direct consequences of the following equations:

$$c(n,k)=\sum_{1\le i_1<i_2<\cdots<i_{n-k}\le n-1} i_1 i_2\cdots i_{n-k}$$
$$=(n-1)!\sum_{1\le j_1<j_2<\cdots<j_{k-1}\le n-1}\frac{1}{j_1 j_2\cdots j_{k-1}}=\frac{n!}{k!}\sum_{\substack{r_1+r_2+\cdots+r_k=n\\ r_1,r_2,\cdots,r_k\ge 1}}\frac{1}{r_1 r_2\cdots r_k}$$
$$=\sum_{\substack{k_1+2k_2+\cdots+nk_n=n\\ k_1+k_2+\cdots+k_n=k,\ k_1,k_2,\cdots,k_n\ge 0}}\frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{1}{1}\right)^{k_1}\left(\frac{1}{2}\right)^{k_2}\cdots\left(\frac{1}{n}\right)^{k_n}$$

(see, e.g., [1], p. 280, equation (8.11), equation (8.12); p. 291, equation (8.22); p. 292, equation (8.24)). □

3. Properties satisfied by $W(i,j)$

Because $T_nW_n=W_nT_n=I_n$ and $T(i,j)=(-1)^{i+j}\frac{1}{i!}s(i,j)$, then

$$W(i,j)=(-1)^{i+j}j!S(i,j).\tag{16}$$

In fact, we can find this from the following computation:

$$\sum_{k=j}^{i}(-1)^{(i-1)+(k-1)}\frac{1}{(i-1)!}s(i-1,k-1)(-1)^{(k-1)+(j-1)}(j-1)!S(k-1,j-1)$$
$$=(-1)^{i+j}\frac{(j-1)!}{(i-1)!}\sum_{k=j}^{i}s(i-1,k-1)S(k-1,j-1)=(-1)^{i+j}\frac{(j-1)!}{(i-1)!}\delta_{ij}=\delta_{ij},$$

In the computation, the well known fact that $s_nS_n=S_ns_n=I_n$ has been made use of and δ_{ij} is the Kronecker delta ($\delta_{ii}=1$, $\delta_{ij}=0$, $i\neq j$).

Since $W_n = T_n^{-1} = S_n D_n L_n^{-1}$, then

$$W(i,j) = \sum_{k=j+1}^{i+1} S(i+1,k)(k-1)!(-1)^{k-j-1}\binom{k-1}{j}$$
$$= \sum_{k=j}^{i}(-1)^{k-j}\binom{k}{j}k!S(i+1,k+1),$$

and we obtain an identity related to the Stirling numbers of the second kind

$$\sum_{k=j}^{i}(-1)^{k-j}\binom{k}{j}k!S(i+1,k+1) = (-1)^{i+j}j!S(i,j).$$

In addition to this, the expression (13) of $S(n,k)$ leads us at once to the value of $W(i,j)$:

$$W(i,j) = \sum_{k=0}^{j}(-1)^{i-k}\binom{j}{k}k^i.$$

And we can also obtain another expression of $W(i,j)$ from the fact that ([3], p. 135) $S(i,j) = B_{i,j}(1,1,1,\cdots)$, that is,

$$W(i,j) = (-1)^{i+j}j!B_{i,j}(1,1,1,\cdots).$$

And, by appealing instead to the following equations related to $S(i,j)$ (see, e.g., [1], p. 298, equation (8.34); p. 292, equation (8.23), equation (8.25)):

$$S(i,j) = \sum_{\substack{r_1+r_2+\cdots+r_j=i-j\\ r_1,r_2,\cdots,r_j\geq 0}} 1^{r_1}2^{r_2}\cdots j^{r_j} = \frac{i!}{j!}\sum_{\substack{r_1+r_2+\cdots+r_j=i\\ r_1,r_2,\cdots,r_j\geq 1}} \frac{1}{r_1!r_2!\cdots r_j!}$$
$$= \sum_{\substack{j_1+2j_2+\cdots+ij_i=i\\ j_1+j_2+\cdots+j_i=j,\ j_1,j_2,\cdots,j_i\geq 0}} \frac{i!}{j_1!j_2!\cdots j_i!}\left(\frac{1}{1!}\right)^{j_1}\left(\frac{1}{2!}\right)^{j_2}\cdots\left(\frac{1}{i!}\right)^{j_i},$$

we have

$$W(i,j) =(-1)^{i+j}j!\sum_{\substack{r_1+r_2+\cdots+r_j=i-j\\ r_1,r_2,\cdots,r_j\geq 0}} 1^{r_1}2^{r_2}\cdots j^{r_j}$$
$$=(-1)^{i+j}i!\sum_{\substack{r_1+r_2+\cdots+r_j=i\\ r_1,r_2,\cdots,r_j\geq 1}} \frac{1}{r_1!r_2!\cdots r_j!}$$
$$=(-1)^{i+j}i!\sum_{j_1+2j_2+\cdots+ij_i=i} \binom{j}{j_1,j_2,\cdots,j_i}\left(\frac{1}{1!}\right)^{j_1}\left(\frac{1}{2!}\right)^{j_2}\cdots\left(\frac{1}{i!}\right)^{j_i}.$$

Moreover, it's easy to find that $|W(i,j)| = |(-1)^{i+j} j! S(i,j)| = j! S(i,j)$ has an explicit combinatorial meaning, that is, the number of ordered j-partitions of $[i] = \{1, 2, \cdots, i\}$, and the matrix representation of $|W(i,j)|$, which is called the factorial Stirling matrix, has already been studied (cf., e.g., [2]).

By virtue of (16), we can compute the generating functions of the $W(i,j)$.

Theorem 9. *The $W(i,j)$ have the following 'vertical' generating function:*

$$\sum_{i\geq 0} W(i,j)\frac{t^i}{i!} = (1 - e^{-t})^j\,, \tag{17}$$

and the following 'double' generating functions:

$$\sum_{i,j\geq 0} W(i,j)\frac{t^i}{i!}x^j = \frac{1}{1 - x(1 - e^{-t})}\,, \tag{18}$$

$$\sum_{i,j\geq 0} W(i,j)\frac{t^i}{i!}\frac{x^j}{j!} = e^{x(1-e^{-t})}\,. \tag{19}$$

In addition, $W(i,j)$ have an 'horizontal' generating function:

$$\sum_{j=0}^{i} W(i,j)\frac{\langle -x\rangle_j}{j!} = (-1)^i x^i\,. \tag{20}$$

Proof. For (17), we have

$$\begin{aligned}\sum_{i\geq 0} W(i,j)t^i/i! &= (-1)^j j! \sum_{i\geq 0} S(i,j)(-t)^i/i! \\ &= (-1)^j (e^{-t} - 1)^j = (1 - e^{-t})^j\,,\end{aligned}$$

and (18,19) follow directly from (17). For (20), $\sum_{j=0}^{i} W(i,j)\langle -x\rangle_j/j! = (-1)^i \sum_{j=0}^{i} (-1)^j S(i,j)\langle -x\rangle_j = (-1)^i \sum_{j=0}^{i} S(i,j)(x)_j = (-1)^i x^i$. □

Similarly, we have the generating functions for $|W(i,j)|$.

Theorem 10. *The $|W(i,j)|$ have the following generating functions:*

$$\sum_{i\geq 0} |W(i,j)|\frac{t^i}{i!} = (e^t - 1)^j\,, \quad \sum_{j=0}^{i} |W(i,j)|\frac{(x)_j}{j!} = x^i\,,$$

$$\sum_{i,j\geq 0} |W(i,j)|\frac{t^i}{i!}x^j = \frac{1}{1 - x(e^t - 1)}\,, \quad \sum_{i,j\geq 0} |W(i,j)|\frac{t^i}{i!}\frac{x^j}{j!} = e^{x(e^t-1)}\,.$$

We also get the recurrence relations satisfied by the $W(i,j)$.

Theorem 11. (see [7]) *The $W(i,j)$ satisfy the 'triangular' recurrence relation:*

$$\begin{gathered} W(i,j) = jW(i-1,j-1) - jW(i-1,j), \quad i,j \geq 1\,, \\ W(i,0) = W(0,j) = 0, \quad \text{except } W(0,0) = 1\,. \end{gathered} \tag{21}$$

For $|W(i,j)|$, this can be written

$$|W(i,j)| = j|W(i-1,j-1)| + j|W(i-1,j)|\,. \tag{22}$$

Proof. Since the Stirling numbers of the second kind $S(i,j)$ satisfy

$$\begin{gathered} S(i,j) = S(i-1,j-1) + jS(i-1,j), \quad i,j \geq 1\,, \\ S(i,0) = S(0,j) = 0, \quad \text{except } S(0,0) = 1\,, \end{gathered}$$

then (21,22) will hold in light of (16). It should be noticed that (22) can be also proved by the combinatorial meaning of $|W(i,j)|$. □

Theorem 12. *The $W(i,j)$ satisfy the 'vertical' recurrence relations:*

$$W(i,j) = \sum_{l=1}^{i-j+1} (-1)^{l-1} \binom{i}{l} W(i-l,j-1)\,, \tag{23}$$

$$W(i,j) = \sum_{l=0}^{i-j} (-1)^l j \binom{i-1}{l} W(i-l-1,j-1)\,. \tag{24}$$

Proof. Let

$$\Phi(t,x) = \sum_{i,j\geq 0} W(i,j) \frac{t^i}{i!} \frac{x^j}{j!} = e^{x(1-e^{-t})}\,.$$

For (23), equate the coefficients of $\frac{t^i}{i!} \frac{x^{j-1}}{(j-1)!}$ in $\partial\Phi/\partial x = (1-e^{-t})\Phi$. For (24), equate the coefficients of $\frac{t^{i-1}}{(i-1)!} \frac{x^j}{j!}$ in $\partial\Phi/\partial t = xe^{-t}\Phi$. □

Theorem 13. *The $W(i,j)$ satisfy the 'horizontal' recurrence relation:*

$$W(i,j) = \sum_{l=0}^{i-j} \frac{1}{j+l+1} W(i+1,j+l+1)\,. \tag{25}$$

Proof. It suffices, by (21), to replace $W(i+1,j+l+1)$ of (25) by $j+l+1(W(i,j+l) - W(i,j+l+1))$, and then to expand. □

4. Further results related to $T(i,j)$ and $W(i,j)$

Just like what we have done for the Stirling numbers of both kinds, from the fact that $T_nW_n = W_nT_n = I_n$, we can obtain an inverse relation related to $T(i,j)$ and $W(i,j)$, as the next theorem shows:

Theorem 14. *Let $\{f_n\}$ and $\{g_n\}$ be two sequences of numbers, then we have*

$$f_n = \sum_k W(n,k)g_k \tag{26}$$

if and only if

$$g_n = \sum_k T(n,k)f_k\,. \tag{27}$$

Proof. Although it's easy to prove this inverse relation by matrix representations, we will follow another way. Let $f(t) = \sum_{m\geq 0} f_m t^m/m!$ and $g(t) = \sum_{m\geq 0} g_m t^m$.

Since $f_n = \sum_k W(n,k)g_k$, then, using (17), we have

$$\begin{aligned} f(t) &= \sum_{m\geq 0}\frac{t^m}{m!}\sum_{k=0}^{m} W(m,k)g_k = \sum_{k\geq 0} g_k \sum_{m\geq k} W(m,k)\frac{t^m}{m!} \\ &= \sum_{k\geq 0} g_k(1-e^{-t})^k = g(1-e^{-t})\,. \end{aligned} \tag{28}$$

Let $u := 1 - e^{-t}$, then $t = -\log(1-u)$. In light of (4),

$$\begin{aligned} g(u) &= f(-\log(1-u)) = \sum_{k\geq 0} f_k \frac{(-1)^k \log^k(1-u)}{k!} \\ &= \sum_{k\geq 0} f_k \sum_{n\geq k} T(n,k)u^n = \sum_{n\geq 0} u^n \{\sum_{k=0}^{n} T(n,k)f_k\}\,, \end{aligned} \tag{29}$$

which proves (27), if we identify the coefficients of u^n of the first and the last member of (29). In a similar way, we can obtain (26) if (27) holds. □

Additionally, according to (2) and (16), some further results related to $T(i,j)$ and $W(i,j)$ can be obtained.

Theorem 15. *We have*

$$\sum_{r=k}^{n} T(n,r)S(r,k) = \frac{1}{k!}\binom{n-1}{k-1}\,, \tag{30}$$

and

$$\sum_{j=0}^{n} T(j,k)T(n-j,r) = \binom{k+r}{k} T(n,k+r)\,, \qquad (31)$$

$$\sum_{j=0}^{n} \binom{n}{j} W(j,k)W(n-j,r) = W(n,k+r)\,. \qquad (32)$$

Proof. (30) follows from the fact that ([1], p. 305) $\sum_{r=k}^{n} c(n,r)S(r,k) = \frac{n!}{k!}\binom{n-1}{k-1}$. And, by virtue of the following two equations ([1], p. 322):

$$\binom{k+r}{k} s(n,k+r) = \sum_{j=k}^{n-r} \binom{n}{j} s(j,k)s(n-j,r)\,,$$

$$\binom{k+r}{k} S(n,k+r) = \sum_{j=k}^{n-r} \binom{n}{j} S(j,k)S(n-j,r)\,,$$

(31) and (32) will be deduced. □

Theorem 16. *We have*

$$\sum_{k=0}^{n} (-1)^k T(n,k)B_k = \frac{1}{n+1}\,, \qquad (33)$$

$$\sum_{k=0}^{n} \frac{1}{k+1} W(n,k) = (-1)^n B_n\,, \qquad (34)$$

where B_n are the Bernoulli numbers.

Proof. (33) is a direct consequence of the fact that ([1], p. 328)

$$\sum_{k=0}^{n} s(n,k)B_k = \frac{(-1)^n n!}{n+1}\,,$$

and (34) follows from (33) and Theorem 14. □

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Appendix

$$T_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{11}{24} & \frac{1}{4} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{5}{12} & \frac{7}{24} & \frac{1}{12} & \frac{1}{120} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{137}{360} & \frac{5}{16} & \frac{17}{144} & \frac{1}{48} & \frac{1}{720} & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{7}{20} & \frac{29}{90} & \frac{7}{48} & \frac{5}{144} & \frac{1}{240} & \frac{1}{5040} & 0 & 0 \\ 0 & \frac{1}{8} & \frac{363}{1120} & \frac{469}{1440} & \frac{967}{5760} & \frac{7}{144} & \frac{23}{2880} & \frac{1}{1440} & \frac{1}{40320} & 0 \\ 0 & \frac{1}{9} & \frac{761}{2520} & \frac{29531}{90720} & \frac{89}{480} & \frac{1069}{17280} & \frac{1}{80} & \frac{13}{8640} & \frac{1}{10080} & \frac{1}{362880} \end{pmatrix}$$

$$W_{10} = T_{10}^{-1} =$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 14 & -36 & 24 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -30 & 150 & -240 & 120 & 0 & 0 & 0 & 0 \\
0 & -1 & 62 & -540 & 1560 & -1800 & 720 & 0 & 0 & 0 \\
0 & 1 & -126 & 1806 & -8400 & 16800 & -15120 & 5040 & 0 & 0 \\
0 & -1 & 254 & -5796 & 40824 & -126000 & 191520 & -141120 & 40320 & 0 \\
0 & 1 & -510 & 18150 & -186480 & 834120 & -1905120 & 2328480 & -1451520 & 362880
\end{pmatrix}$$