A Class of Optimal Quaternary Codes

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Abstract

A construction of optimal quaternary codes from symmetrical BIB design $(4t-1, 2t-1, t-1)$ is described.

1 Introduction

Binary and q-ary codes have been constructed from symmetrical BIB designs (BIBDs) and nested BIBDs respectively see, Stinson and van Rees (1984), Sinha (1994), and Sinha and Mitra (1999). It is known (see Tonchev, 1988, 1998) that, for any equidistant (n, M, d, q) code, $d \leq \frac{M(q-1)}{(M-1)q}$ ($= d_{opt}$, say) where the equality is achieved if and only if M is a multiple of q and each of the symbols $0, 1, \ldots, q-1$ occurs exactly M/q times in each colum of the $M \times n$ matrix formed by the codewords.

Furthermore, some of the designs constructed by Sinha (1994), and Sinha and Mitra (1999) are nearly optimal. These optimal q-ary codes were obtained from resolvable BIB designs, see Semakov and Zinoviev (1968) op.cit. Tonchev (1988, 1998). Here, we describe a new construction of optimal quaternary codes from symmetrical BIBD $(4t-1, 2t-1, t-1)$.

2 Construction

Theorem 2.1 The existence of a symmetrical BIBD $(4t-1, 2t-1, t-1)$ implies the existence of optimal quaternary codes with the parameters

(i) $n = (4t-1)(2t-1)$, $M = 4t$, $d = 3t(2t-1)$, $t \geq 4$ when there exists a Hadamard matrix of order $4t$.

(ii) $n = (4t-1)s$, $M = 4t$, $d = 3ts$, $s \geq 1$, when $4t-1$ is prime.

Proof. (i) It is well known that a symmetrical BIBD $(4t-1, 2t-1, t-1)$ exists, whenever there exists Hadamard matrix of order $4t$. Let us form ${}^{4t-1}C_2$ all possible combinations of pairs of blocks of the symmetrical BIBD. Here, each pair of blocks give rise to a subset of $(4t-1)$ elements having three sub-subsets, consisting of (a) the $t-1$ common elements, (b) the remaining t elements of the first block, and (c) the remaining t elements of the second block.

Now, corresponding to each subset, we form a column where the elements of first sub-subset are denoted by 1, second sub-subset by 2, third sub-subset by 3 and the rest of the $(4t-1)$ elements by 0. Thus we get an arrangement of $(4t-1)(2t-1)$ columns and $(4t-1)$ rows. Now, by adding a row of 1's and identifying the rows as codewords, we get the optimal code (i). Here, $d ={}^{4t-1} C_2 - \{{}^{t-1}C_2 + (t-1)t + {}^{t}C_2\} = 3t(2t-1)$. For an optimal code, $d_{opt} = \frac{nM(q-1)}{(M-1)q} = 3t(2t-1)$, which shows that the code is optimal.

Example 1. Let us consider the symmetric BIBD (7, 3, 1) obtained by developing (1, 2, 4) mod 7. Now, we form pairwise intersection of blocks to obtain the 21 subsets each divided into 3 sub-subsets as

(2 14 35)	(4 12 36)	(4 12 57)	(1 24 56)
(2 14 67)	(1 24 37)	(3 25 46)	(5 23 47)
(5 23 16)	(2 35 67)	(3 25 17)	(4 36 57)
(6 34 15)	(6 34 27)	(3 46 17)	(5 47 16)
(7 45 26)	(7 54 13)	(6 15 27)	(1 56 37)
(7 26 13)			

Further, we form 7×21 matrix, which alongwith a row of ones, yields an optimal quaternary code with $n = 21$, $M = 8$, $d = 18$ as given below.

(2 2 2 1 2 1 0 0 3 0 0 0 3 0 3 3 0 3 2 1 3)
(1 2 2 2 1 2 2 2 2 1 2 0 0 3 0 0 3 0 3 0 2)
(3 3 0 0 0 3 1 2 2 2 1 2 2 2 1 0 0 3 0 3 3)
(2 1 1 2 2 2 3 3 0 0 3 1 2 2 2 2 2 2 0 0 0)
(3 0 3 3 0 0 2 1 1 2 2 3 3 0 0 1 2 2 2 2 0)
(0 3 0 3 3 0 3 0 3 3 3 2 1 1 2 3 3 0 1 2 2)
(0 0 3 0 3 3 0 3 0 3 0 3 0 3 3 2 1 1 3 3 1)
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)

(ii) When $(4t-1)$ is a prime, BIBD $(4t-1, 2t-1, t-1)$ is cyclic. We develop it once to form a difference set $(a_1 a_2 \dots a_{t-1}; b_1 b_2 \dots b_t; c_1 c_2 \dots c_t)$ where $a_1 a_2 \dots a_{t-1}$ are common between the first and second blocks $b_1 b_2 \dots b_t$ and $c_1 c_2 \dots c_t$ are remaining elements from first and second blocks, respectively. Now corresponding to each subset, we form a column as indicated above to get a matrix of order $(4t-1) \times (4t-1)$. Further, a row of 1s is added to this matrix and by identifying the rows as codewords, we get an optimal code as $(4t-1, 4t, 3t, 4)$. This is then repeated s times column-wise to get an optimal code as $((4t-1)s, 4t-1, 3ts, 4)$.

Remark 1. If $s = 2t-1$ we get the optimal code as obtained above in (i).

Example 2. By combining first two blocks of BIBD (1, 2, 4) mod 7, we get BIBD (2, 14, 35) mod 7, optimal code as (7, 8, 6, 4) given below.

(2 0 0 3 2 3 1)
(1 2 0 0 3 2 3)
(3 1 2 0 0 3 2)
(2 3 1 2 0 0 3)
(3 2 3 1 2 0 0)
(0 3 2 3 1 2 0)
(0 0 3 2 3 1 2)
(1 1 1 1 1 1 1)

Remark 2. An intersting feature of the above structure is that, if we add a column of 1s instead of adding a row of 1s, we still get an optimal code column-wise as (7, 8, 6, 4).

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