A note on B-T unitals in $PG(2, q^2)$

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Abstract

A new construction of a B-T unital using Hermitian curves and certain hypersurfaces of $PG(3, q^2)$ is presented. Some properties of an algebraic curve containing all points of a B-T unital are also examined.

Keywords: Hermitian curve; unital, hypersurface.

1 Introduction

A classical unital $\mathcal{H}$ in $PG(2, q^2)$ is the set of all $q^3 + 1$ points of a non-degenerate hermitian curve $\mathcal{H}$. A unital in $PG(2, q^2)$ is *non-classical* if it is not projectively equivalent to $\mathcal{H}$. The known non-classical unitals are those constructed by Buekenhout and Metz in 1974, see [3, 10], using the Bruck–Bose representation of $PG(2, q^2)$ in $PG(4, q)$ and some properties of spreads, ovoids (and, in particular, quadrics).

In [1], a new costruction of non-classical unitals is described; the key idea is realized within $PG(2, q^2)$, and uses hermitian curves and either quadratic transformations or certain birational transformations. The resulting non-classical unital is either a Buekenhout-Mets unital or a Buekenhout-Tits unital (a B-M unital or a B-T unital for short). For generalities on unitals in projective planes the reader is referred to [2, 5, 6]. In [1] a method is also presented to construct a non classical B-M unital starting from $PG(3, q^2)$ and using a quadratic cone.

In this paper, it is shown that above method realized in $PG(3, q^2)$, also works for B-T unitals, provided that quadratic cones are replaced by certain hypersurfaces.

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2 Construction

We recall the representation of a B-T unital in $\mathrm{PG}(2, q^2)$, where q is an odd power of 2. Fix a projective frame in $\mathrm{PG}(2, q^2)$ with homogeneous coordinates (x_0, x_1, x_2), and consider the affine plane $\mathrm{AG}(2, q^2)$ whose infinite line ℓ_∞ has equation $x_0 = 0$. Then $\mathrm{AG}(2, q^2)$ has affine coordinates (x, y) where $x = x_1/x_0$, $y = x_2/x_0$ so that $X_\infty = (0, 1, 0)$ and $Y_\infty = (0, 0, 1)$ are the infinite points of the horizontal and vertical lines, respectively.

Take $\varepsilon \in \mathrm{GF}(q^2)\backslash\mathrm{GF}(q)$ such that $\varepsilon^2+\varepsilon+\delta = 0$, for some $\delta \in \mathrm{GF}(q)\backslash\{1\}$ with $\mathrm{Tr}\,(\delta) = 1$. Here, Tr stands for the trace function $\mathrm{GF}(q) \to \mathrm{GF}(2)$. Then $\varepsilon^{2q}+\varepsilon^q+\delta = 0$. Therefore, $(\varepsilon^q+\varepsilon)^2+(\varepsilon^q+\varepsilon) = 0$, whence $\varepsilon^q+\varepsilon+1 = 0$.

Moreover, if $q = 2^e$, with e an odd integer, then

$$\sigma : x \mapsto x^{2^{(e+1)/2}}$$

is an automorphism of $\mathrm{GF}(q)$.

In the above notation, the point–set

$$U_\varepsilon = \{(1, s + t\epsilon, (s^{\sigma+2} + t^\sigma + st)\epsilon) + r | r, s, t \in GF(q)\} \cup \{Y_\infty\}, \qquad (1)$$

is a B-T unital in $\mathrm{PG}(2, q^2)$. Conversely, every B-T unital may be represented as U_ε for some choice of ε; see [4].

Now, let X_0, X_1, X_2, X_3 denote homogeneous coordinates in $\mathrm{PG}(3, q^2)$. and consider the affine space $\mathrm{AG}(3, q^2)$ whose plane at infinity has equation $X_0 = 0$. Then $\mathrm{AG}(3, q^2)$ has affine coordinates (X, Y, Z) where $X = X_1/X_0$, $Y = X_2/X_0$, $Z = X_2/X_0$.

For a given $b \in GF(q^2) \setminus GF(q)$, let $\mathcal{H}$ be the Hermitian curve

$$\mathcal{H} = \{(1, x, bx^{q+1} + r) | x \in \mathrm{GF}(q^2), r \in \mathrm{GF}(q)\} \cup \{Y_\infty\}$$

in $PG(2, q^2)$. Fix a basis $\{1, \epsilon\}$ for $GF(q^2)$ over $GF(q)$ as above and consider the map $\phi : \mathcal{H} \mapsto \mathrm{PG}(3, q^2)$ which transforms the point $P = (1, x, bx^{q+1}+r)$ into the point $\phi(P) = (1, x, [((x^q + x)\varepsilon + x)^{\sigma+2} + (x^q + x)^\sigma + ((x^q + x)\varepsilon + x)(x^q + x)]\varepsilon + bx^{q+1}, bx^{q+1} + r)$ and Y_∞ into $\phi(Y_\infty) = (0, 0, 0, 1)$.

The map ϕ is injective, thus, the set $\phi(\mathcal{H})$ consists of $q^3 + 1$ points lying on the cone $\mathfrak{C}$ of affine equation

$$Y = [((X^q+X)\varepsilon+X)^{\sigma+2}+(X^q+X)^\sigma+((X^q+X)\varepsilon+X)(X^q+X)]\varepsilon+bX^{q+1}.$$

The point $Q = (0, 0, 1, 1)$ does not lie on the cone $\mathfrak{C}$; hence, the projection ρ from Q to the plane $\pi : Y = 0$ is well defined. The point $\phi(Y_\infty)$ is on π thus we get $\rho(0, 0, 0, 1) = (0, 0, 0, 1)$.

For any $(x, r) \in \mathrm{GF}(q^2) \times \mathrm{GF}(q)$, set

$$\alpha_x = [((x^q + x)\varepsilon + x)^{\sigma+2} + (x^q + x)^\sigma + ((x^q + x)\varepsilon + x)(x^q + x)]\varepsilon + bx^{q+1}$$

and

$$P_{x,r} = (1, x, \alpha_x, bx^{q+1} + r).$$

The line $P_{x,r}Q$ has point set

$$\{(1, x, \alpha_x + \lambda, bx^{q+1} + r + \lambda) | \lambda \in \mathrm{GF}(q^2)\} \cup \{(0,0,1,1)\}$$

and intersects the plane π at $\rho(P_{x,r}) = (1, x, 0, bx^{q+1} + r + \alpha_x)$. We are going to show that no 2-secant lines of $\phi(\mathcal{H})$ pass through Q. Let $P_{x_1,r_1}(1, x_1, \alpha_{x_1}, bx_1^{q+1} + r_1)$ and $P_{x_2,r_2}(1, x_2, \alpha_{x_2}, bx_2^{q+1} + r_2)$ be two distinct points of $\phi(\mathcal{H})$. The line $P_{x_1,r_1}P_{x_2,r_2}$ is the point set

$$\{(\lambda+1, x_1+\lambda x_2, \alpha_{x_1}+\lambda\alpha_{x_2}, b(x_1^{q+1}+\lambda x_2^{q+1})+r_1+\lambda r_2) | \lambda \in \mathrm{GF}(q^2)\} \cup \{P_{x_2,r_2}\}.$$

The point at infinity of the line $P_{x_1,r_1}P_{x_2,r_2}$ is the point

$$P_\infty = (0, x_1 + x_2, \alpha_{x_1} + \alpha_{x_2}, b(x_1^{q+1} + x_2^{q+1}) + r_1 + r_2)$$

If $P_\infty = Q$ then $x_1 = x_2$ and $\alpha_{x_1} + \alpha_{x_2} \neq 0$, which is impossible. Therefore, $|\rho(\phi(\mathcal{H}))| = q^3 + 1$. Writing $x = s + t\varepsilon$, where $s, t \in GF(q)$, we have

$$t = x^q + x, \tag{2}$$

$$s = x + (x^q + x)\epsilon \tag{3}$$

and hence

$$\rho(P_{x,r}) = (1, s + t\epsilon, 0, (s^{\sigma+2} + t^\sigma + st)\epsilon + r).$$

Therefore it is possible to choose homogeneous coordinates for the plane π in such a way as $\rho(\phi(\mathcal{H}))$ turns out to be a B-T unital in π.

3 Algebraic curves containing all points of a B-T unital in $PG(2, q^2)$

Let U_ε be the Buekenhout-Tits unital (1) of $PG(2, q^2)$. Setting $x = s + t\epsilon$ and $y = (s^{\sigma+2} + t^\sigma + st)\epsilon + r$ we have (2), (3) and

$$s^{\sigma+2} + t^\sigma + st = y^q + y. \tag{4}$$

Substituting (2) and (3) in (4) gives

$$y^q + y = [x + (x^q + x)\epsilon]^{\sigma+2} + (x^q + x)^\sigma + (x^{2q} + x^2)\epsilon + x^{q+1} + x^2. \tag{5}$$

Hence a point $P(1, x, y)$ is on U if and only if its coordinates satisfy (5). Let C_ε denote the algebraic plane curve with affine equation (5) and consider the birational transform of $AG(2, q^2)$ into itself

$$\gamma : (x, y) \mapsto (x, y + (\epsilon^\sigma + \epsilon^{\sigma+2})x^{q\sigma+2} + \epsilon^{\sigma+2}x^{q(\sigma+2)} + x^\sigma + (1+\epsilon)x^2).$$

Basic facts on rational transformations of projective planes are found in [9, Section 3.3]. The curve C_ε is transformed by γ into the non–degenerate Hermitian curve of equation

$$\mathcal{H} : y^q + y + x^{q+1} = 0.$$

Thus we have the following

Theorem 3.1. *The algebraic curve C_ε is birationally equivalent over $GF(q^2)$ to a non–degenerate Hermitian curve.*

Remark 3.2. Algebraic curves over finite fields may be used to obtain good families of codes with prescribed parameters. The general construction technique for linear codes from algebraic curves was introduced by Goppa in [7]. These codes are called *algebraic-geometry.*

Amongst all algebraic-geometry codes, of particular interest are those associated with maximal curves, since they may have length and minimum distance as large as possible for a given genus g. The parameters of linear codes arising from a Hermitian curve by Goppa's method were computed in [11].

It is interesting to see if an algebraic curve of low degree whose rational points are the same as those of a non–classical unitals also provides good codes. In [1] it is noted that the algebraic-geometry codes arising from a Hermitian curve $\mathcal{H}$ of $PG(2, q^2)$, and those arising from an irreducible algebraic curve of degree $2q$ containing all the points of a non–classical B-M unital are the same.

Since the algebraic-geometric codes are determined by the function fields of the corresponding algebraic curves and the function fields of two birationally equivalent plane curves are isomorphic, Theorem 3.1 implies that also algebraic-geometry codes arising from a Hermitian curve $\mathcal{H}$ of $PG(2, q^2)$ and those arinsing from the irreducible curve C_ε are the same.

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