Equality of Domination and Inverse Domination Numbers

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Abstract:

A set D of vertices of a graph $G = (V, E)$ is a *dominating set* if every vertex of $V-D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset of $V-D$, which is also a dominating set of G is called an *inverse dominating set* of G with respect to D. The *inverse domination number* $\gamma'(G)$ equals the minimum cardinality of an inverse dominating set D. In this paper, we study classes of graphs whose domination and inverse domination numbers are equal.

Key words: domination number, inverse domination number
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1. Introduction

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is called a *dominating set* if every vertex in $V-D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set in G. A dominating set D is called a γ*-set* if $|D| = \gamma(G)$. A subset of $V-D$ which is also a dominating set is called an *inverse dominating set* with respect to D. The inverse domination number $\gamma'(G)$ equals the minimum cardinality of an inverse dominating set with respect to a minimum dominating set D. The concept of the inverse domination number was introduced by Kulli and Sigarkanti in 1991 [3]. In that short paper they attempted to prove that for any graph G, $\gamma'(G) \leq$

$\beta_0(G)$, the vertex independence number, equals the maximum order of set of vertices, no two vertices in which are adjacent. However, the proof given is incorrect. To date, no proof of this result is known, neither are any counter examples known. Thus this has become known as the Kulli-Sigarkanti conjecture.

Since then Domke, Dunbar and Markus published a paper on the inverse domination number [2], in which they characterize the graphs for which $\gamma(G)+\gamma'(G)=n$. They also give a lower bound for the inverse domination number of a tree and give a constructive characterization of the class of trees which achieve this lower bound.

It is well known by Ore's Theorem [9] that if a graph G has no isolated vertices, then the complement of $V-D$ of every minimal dominating set contains a dominating set. Thus, every graph without isolated vertices contains an inverse dominating set with respect to a minimum dominating set, and has an inverse domination number. In this paper therefore we will assume that all graphs have no isolated vertices and we will consider classes of graphs G for which $\gamma(G)=\gamma'(G)$, or equivalently classes of graphs having two disjoint minimum dominating sets.

2. Graphs with two disjoint minimum dominating sets

Researchers have for many years studied the existence of graphs having disjoint minimum dominating sets. Most notable among this research is the study of various kinds of domatic numbers, in which one seeks to partition the vertices of G into a maximum number of various kinds of dominating sets. For a survey of this literature, the reader is referred to Zelinka [10].

Specifically the existence to two disjoint minimum dominating sets was first studied by Bange, Barkauskas and Slater in 1978 [7], who studied the existence of two disjoint minimum dominating sets in trees. In a related paper, Haynes and Henning [8] studied the existence of two disjoint minimum independent dominating sets in a tree.

When one asks the question: which classes of graphs have two disjoint minimum dominating sets?, perhaps the first three classes are the following:

1. all paths P_n where n is not congruent to $0 \bmod 3$.
2. all cycles C_n.
3. all graphs of the form $K_2 + G$ (two adjacent vertices which are both joined to all vertices of an arbitrary graph G). For these graphs the domination number equals 1, and there are at least two vertices which dominate the graphs. This class includes all complete graphs K_n.

The next observation that can be made is the well known theorem:

***Theorem** For a graph G with even order n and no isolated vertices, the domination number equals n/2 if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$, for any connected graph H.*

This theorem was proved independently by Payan and Xuong [5] and by Fink, Jacobson, Kinch and Roberts [6]. The corona $H \circ K_1$ is the graph obtained from an arbitrary graph H by attaching a leaf to each vertex in H. From this theorem it follows immediately that any corona $H \circ K_1$ has two disjoint minimum dominating sets of order $n/2$, one set consisting of the set of $n/2$ leaves, the other set consisting of the vertices in H

3. Graphs having two disjoint minimum dominating sets of size (n –1)/2

We now turn our attention to graphs for which $\gamma = \gamma' = \dfrac{n-1}{2}$.

Consider the following 5 classes of graphs. Let A be the set of all graphs given below.

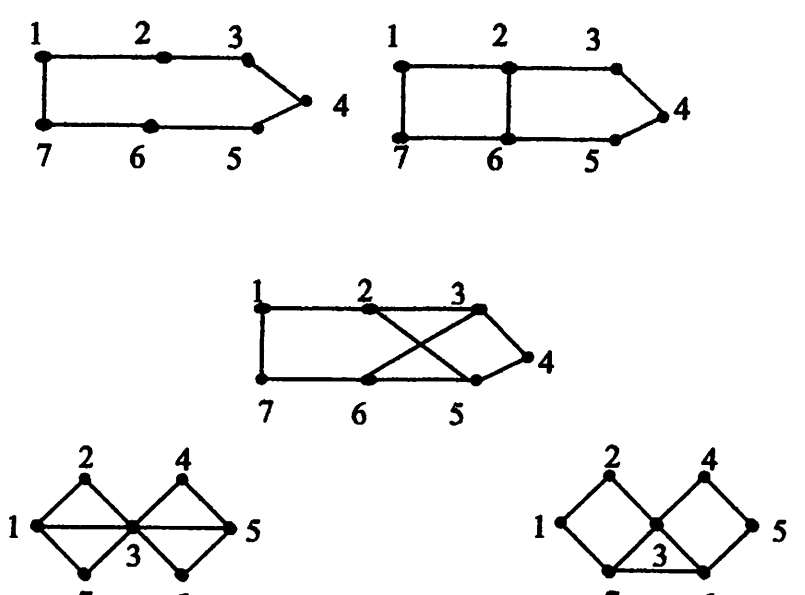

Fig 1

Let B be the set of all graphs given below

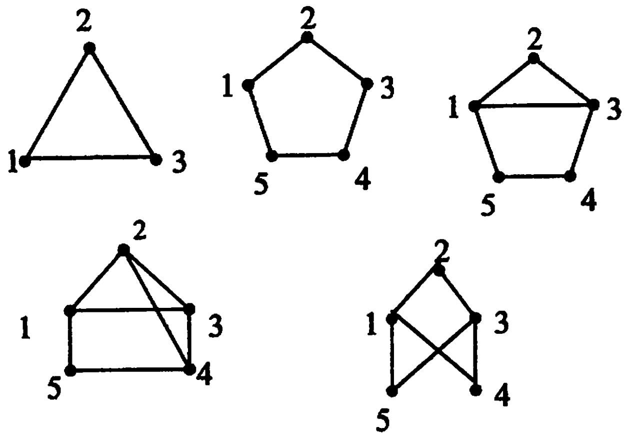

Fig 2

Let $Q_1 = A \bigcup B$. For any graph H, let $S(H)$ denote the set of connected graphs, each of which can be formed from $H \circ K_1$ by adding a new vertex x and edges joining x to two or more vertices of H. Then define $Q_2 = \bigcup_H S(H)$, where the union is taken over all graphs H. Let y be a vertex of a copy of C_4 and for a graph $G \in Q_2$, let $\theta(G)$ be the graph obtained by joining G to C_4 with the single edge xy where x is the new vertex added in forming G. Then define $Q_3 = \{ \theta(G) / G \in Q_2\}$. Let u, v, *and* w be a vertex sequence of the path P_3. For any graph H, let $P(H)$ be the set of connected graphs, which may be formed from $H \circ K_1$ by joining each of u and w to one or more vertices of H. Then define $Q_4 = \bigcup_H P(H)$.

Let H be a graph and $X \in B$. Let $R(H, X)$ be the set of connected graphs, which may be formed from $H \circ K_1$ by joining each vertex of $U \subseteq V(X)$ to one or more vertices of H, such that no set with fewer than $\gamma(X)$ vertices of X dominates $V(X) - U$. Then define $Q_5 = \bigcup_{H,X} R(H, X)$.

Lemma 3.1. *For all graphs in Q_1, Q_2, Q_3, Q_4, Q_5, we have* $\gamma = \gamma' = \frac{n-1}{2}$.

Proof. $D = \{1, 3, 6\}$ and $D' = \{2, 5, 7\}$ form the γ-set and γ'-set respectively for all the graphs in A with $\frac{n-1}{2}$ vertices each. For C_3 in B, $\{1\}$ and $\{2\}$ are the γ and γ'-sets. $D = \{1, 4\}$ and $D' = \{3, 5\}$ are the γ and γ'-sets for all the other graphs in B. Thus $\gamma = \gamma' = \frac{n-1}{2}$ for all the graphs in Q_1.

Let x be adjacent to y and z in H. Now, y and z are adjacent to the pendent vertices y_1 and z_1 respectively. Since there are $\frac{n-1}{2}$ pendent vertices, which are adjacent to distinct $\frac{n-1}{2}$ vertices of H, any γ-set of G contains at least $\frac{n-1}{2}$ vertices. Further, the set $D = V(H) \cup \{y_1\} - \{y\}$ is a dominating set containing $\frac{n-1}{2}$ elements and hence it is a γ-set. $D' = V(G) - D \cup \{x\}$ is a dominating set in V–D, containing $\frac{n-1}{2}$ vertices. Hence D' is a γ'-set and thus $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ $\forall\ G \in Q_2$.

The graphs in Q_2 are of the form given below:

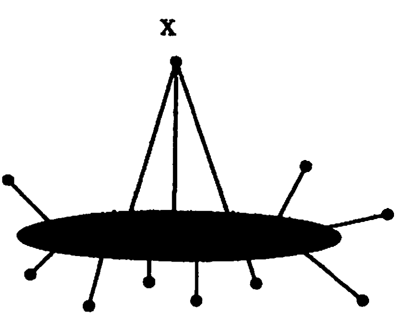

Fig 3

The graphs in Q_3 are of the form given below:

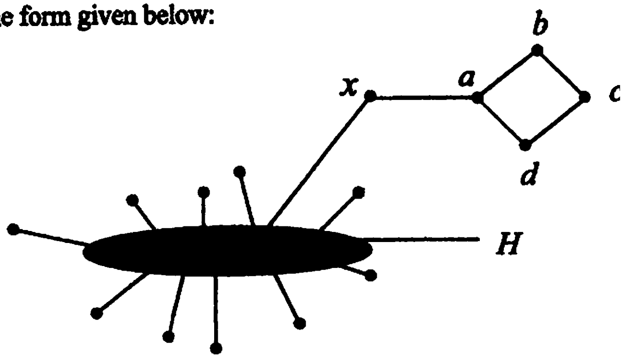

Fig 4

$D = V(H) \cup \{b, d\}$ is a γ-set and $D' = V(G) - D \cup \{x\}$ is a γ'-set for $G \in Q_3$. Thus, $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ for all $G \in Q_3$. Any graph in Q_4 can be visualized as given in the following figure

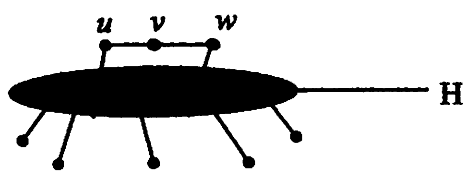

Fig 5

$D = V(H) \cup \{u\}$ is a γ-set with $\frac{n-1}{2}$ vertices and $D' = V(G) - D \cup \{w\}$ is a γ'-set containing $\frac{n-1}{2}$ vertices. Thus $\gamma = \gamma' = \frac{n-1}{2}$ for all $G \in Q_4$.

Any graph G in $R(H, X)$ is of the form

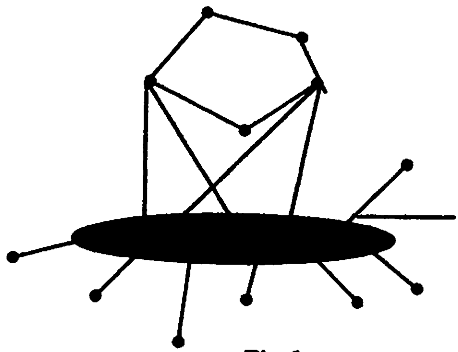

Fig 6

Let $|V(H)| = m$ and $|V(X)| = p$. Then $|V(R(H,X))| = 2m + p = n$. Let D_X and D'_X be the γ -set and γ'- set of the graph X respectively. Then $D = V(H) \cup D_X$ is a dominating set and no set with fewer vertices dominates G and hence D is a γ-set. $D' = D'_X$ together with all the pendent vertices forms a γ'-set of $R(H,X)$. Hence $|D| = m + \frac{p-1}{2} = \frac{2m+p-1}{2} = \frac{n-1}{2}$ and similarly $|D'| = \frac{n-1}{2}$. Hence $\gamma = \gamma' = \frac{n-1}{2}$. Thus $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ for all graphs in $G = \bigcup_{i=1}^{5} Q_i$.

Remark 3.2. Note that the classes of graphs Q_1, Q_3, Q_4 and Q_5 are the same as the classes of graphs G_2, G_4, G_5 and G_6 respectively and Q_2 is a subclass of the class G_3 given in Theorem 2.6 [1].

Theorem 3.3. *A connected graph G satisfies* $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$ *if and only if* $G \in Q = \bigcup_{i=1}^{5} Q_i$.

Proof. Let $G \in Q = \bigcup_{i=1}^{5} Q_i$. By Lemma 3.1 $\gamma(G) = \gamma'(G) = \frac{n-1}{2}$. Conversely, suppose that $\gamma(G)=\gamma'(G)=\frac{n-1}{2}$. Now by Theorem 2.6 [1], $G \in G = \bigcup_{i=1}^{6} G_i$ since n is odd, G cannot be in G_1. In view of Remark 3.2, $G \in Q_1$ or Q_3 or Q_4 or Q_5 and $G \in G_3$. It is enough to prove that whenever $G \in G_3$, we get that $G \in Q_2$. If $G \in G_3$, let S be the set of end vertices in G and T be the set of neighbours of vertices in S. If $|T|=t$, then by Lemma 2.5 [1], $|S|=t$ or $|S|=t+1$. Hence there is a γ-set of G containing T. Let $G'=G-(S\cup T)$.

Case (i). If $|S|=t+1$. One can show that $G'=\phi$ as in the course of proof of Theorem 2.6 [1] for this case. Hence we get $\gamma = t$ and $\gamma'=t+1$ a contradiction. Therefore $|S|\neq t+1$.

Case (ii). If $|S|=t$, then either G' contains isolated vertices or G' contains no isolated vertices.

Sub Case (i). If G' contains isolated vertices. Let y be an isolated vertex of G'. As in the proof of Theorem 2.6 [1], one can prove that $G'-y$ is empty. Since y is not a pendent vertex of G, y is adjacent to two or more vertices of T. Hence $G \in Q_2$.

Sub Case (ii). If G' contains no isolated vertices as in the above-mentioned proof $G \notin Q_3$. If $G \in G_4$ then $G \in Q_3$. If $G \in G_5$ then $G \in Q_4$. If $G \in G_6$ then $G \in Q_5$. Thus $G \in \bigcup_{i=1}^{5} Q_i$.

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