The Commuting Graph of the Quaternion Algebra over Residue Classes of Integers

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Abstract

The commuting graph of an arbitrary ring R, denoted by $\Gamma(R)$, is the graph whose vertices are all non-central elements of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. In this paper, we investigate the *connectivity*, the *diameter*, the *maximum degree* and the *minimum degree* of the commuting graph of the quaternion algebra $\mathbb{Z}_n[\mathbf{i}, \mathbf{j}, \mathbf{k}]$.

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1 Introduction

The commuting graph of an arbitrary ring R denoted by $\Gamma(R)$, is the graph with vertex set $R \backslash Z(R)$, where $Z(R)$ is the center of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. In this paper, we study some properties of the commuting graphs of the quaternion algebra over $\mathbb{Z}_n$, which is denoted by $\mathrm{H}_n = \mathbb{Z}_n[\mathbf{i}, \mathbf{j}, \mathbf{k}] = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} | a, b, c, d \in \mathbb{Z}_n\}$, where $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ is the ring of integers modulo n and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are formal symbols called basic units with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$.

The properties of H_n were discussed in [10] and [11], and we proved that $\mathrm{H}_n \cong M_2(\mathbb{Z}_n)$ if and only if n is odd. Moreover, a new isomorphism relation between $\mathbb{Z}_n[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ and $M_2(\mathbb{Z}_p)$ was given in [9], for all odd primes p. It is clear that the ring of Gaussian integers modulo n, $\mathbb{Z}_n[\mathbf{i}] = \{a + b\mathbf{i} \,|\, a, b \in \mathbb{Z}_n, \mathbf{i}^2 =$

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$-1\}$, is a subring of $\mathbb{Z}_n[\mathrm{i},\mathrm{j},\mathrm{k}]$. In [7] and [8], the properties of $\mathbb{Z}_n[\mathrm{i}]$ were studied. Also, the zero-divisor graph for $\mathbb{Z}_n[\mathrm{i}]$ was investigated in [2].

Let R be a ring and $R^* = R\backslash\{0\}$. We use $D(R)$, $U(R)$ to denote the set of zero-divisors of R and the group of units of R respectively. Given integers a and b, we denote by (a, b) the greatest common divisor of a and b. If p is a prime and t is a nonnegative integer, then we use the notation $p^t \| a$ to mean that $p^t | a$ and $p^{t+1} \nmid a$. The ring of n by n full matrices over a ring R is denoted by $M_n(R)$.

In this paper, all graphs are simple and undirected and $|G|$ denotes the number of vertices of the graph G. In a graph G, the degree of a vertex v is denoted by $d(v)$. And the minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We denote the vertex set of G as $V(G)$. A *path* of length r from a vertex x to another vertex y in G is a sequence of $r+1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. For a connected graph H, the diameter of H is denoted by $diam(H)$. An induced subgraph of G that is maximal, subject to being connected, is called a *connected component* of G.

In this paper, we investigate some connections between number theory, quaternion theory and graph theory motivated by the work of [1], [3], [4] and [5]. In section 2, we show that $\Gamma(\mathrm{H}_n)$ is connected if and only if $n \neq p$, $2p$, 2^2 for all odd primes p. If $\Gamma(\mathrm{H}_n)$ is connected then $diam(\Gamma(\mathrm{H}_n)) = 3$, and if $\Gamma(\mathrm{H}_n)$ is disconnected then every connected component of $\Gamma(\mathrm{H}_n)$ is a complete graph with the same size and we completely determine the vertices of every connected component. In section 3, we determine the degree of each vertex in $\Gamma(\mathrm{H}_n)$ and the maximum degree and minimum degree of $\Gamma(\mathrm{H}_n)$.

2 The connectivity and diameter of $\Gamma(\mathrm{H}_n)$

Lemma 2.1. [3, Theorem 2] *If F is a finite field, then $\Gamma(M_2(F))$ is a graph with $|F|^2+|F|+1$ connected components of size $|F|^2-|F|$ which each of them is a complete graph.*

The statements of Lemma 2.2 were proved in [10] and [11].

Lemma 2.2. *Let $\mathrm{H}_n = \mathbb{Z}_n[\mathrm{i},\mathrm{j},\mathrm{k}]$, $n \geqslant 2$.*

(1) H_2 *is commutative.*

(2) $\mathrm{H}_n \cong M_2(\mathbb{Z}_n)$ *if and only if n is odd.*

(3) *Let $n = p_1^{t_1} \cdots p_m^{t_m}$, where $m \geqslant 1$, $p_1, \ldots, p_m$ are pairwise distinct primes, $t_i \geqslant 1$, for $i \in \{1, \ldots, m\}$. Then* $\mathrm{H}_n \cong \mathrm{H}_{p_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{p_m^{t_m}}$.

Lemma 2.3. (1) *If $2 \nmid n$, then the center $Z(\mathrm{H}_n)$ of H_n is $\mathbb{Z}_n$. Therefore $|Z(\mathrm{H}_n)| = n$ and $|\Gamma(\mathrm{H}_n)| = n^4 - n$.*

(2) *If* $2|n$, *then* $Z(\mathrm{H}_n) = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_n;\ \overline{b}, \overline{c}, \overline{d} = \overline{0} \text{ or } \overline{\tfrac{1}{2}n}\right\}$. *Therefore* $|Z(\mathrm{H}_n)| = 2^3 n$ *and* $|\Gamma(\mathrm{H}_n)| = n^4 - 2^3 n$.

Proof. First of all, for $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k}$, $\beta = \overline{w} + \overline{x}\mathrm{i} + \overline{y}\mathrm{j} + \overline{z}\mathrm{k} \in \mathrm{H}_n$, we have $\alpha\beta = \beta\alpha$ if and only if the following system of congruence equations holds.

$$(*)\begin{cases} 2(cz - dy) \equiv 0 \pmod n & \text{(2-1)} \\ 2(dx - bz) \equiv 0 \pmod n & \text{(2-2)} \\ 2(by - cx) \equiv 0 \pmod n & \text{(2-3)} \end{cases}$$

(1) Assume that $2 \nmid n$. Let $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in Z(\mathrm{H}_n)$. It is clear that $\mathrm{i}\alpha = \alpha\mathrm{i}$. So by system $(*)$, we have $2c \equiv 0 \pmod n$ and $2d \equiv 0 \pmod n$. Since $2 \nmid n$, we have $n|c$ and $n|d$, i.e., $\overline{c} = \overline{d} = \overline{0}$, so $\alpha = \overline{a} + \overline{b}\mathrm{i}$. Moreover, $\mathrm{j}\alpha = \alpha\mathrm{j}$ yields that $2b \equiv 0 \pmod n$, so $\overline{b} = \overline{0}$. Thus we have $Z(\mathrm{H}_n) = \mathbb{Z}_n$ and therefore $|Z(\mathrm{H}_n)| = n$, $|\Gamma(\mathrm{H}_n)| = n^4 - n$.

(2) Assume that $2|n$. Let $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in Z(\mathrm{H}_n)$. If $\overline{b} \neq \overline{0}, \overline{\tfrac{1}{2}n}$, then $\mathrm{j}\alpha \neq \alpha\mathrm{j}$ contradicts $\alpha \in Z(\mathrm{H}_n)$. Hence, $\overline{b} = \overline{0}$ or $\overline{\tfrac{1}{2}n}$. Similarly, we have $\overline{c}, \overline{d} = \overline{0}$ or $\overline{\tfrac{1}{2}n}$. Conversely, if $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \mathrm{H}_n$ with $\overline{a} \in \mathbb{Z}_n$ and $\overline{b}, \overline{c}, \overline{d} = \overline{0}$ or $\overline{\tfrac{1}{2}n}$, it is easy to verify that $\alpha \in Z(\mathrm{H}_n)$. Hence, the result follows. □

In the next, for $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \mathrm{H}_n$, we always suppose that a, b, c, d are nonnegative integers not greater than $n - 1$.

Theorem 2.4. *Suppose* $n = 2^t$, $t \geqslant 1$.

(1) *If* $t = 1$, *then* $|\Gamma(\mathrm{H}_n)| = 0$.

(2) *If* $t = 2$, *then* $\Gamma(\mathrm{H}_n)$ *is a graph with 7 connected components of size* 2^5 *which each of them is a complete graph.*

(3) *If* $t \geqslant 3$, *then* $\Gamma(\mathrm{H}_n)$ *is connected and* $diam(\Gamma(\mathrm{H}_n)) = 3$.

Proof. (1) By Lemma 2.2 (1), H_2 is a commutative ring, so $|\Gamma(\mathrm{H}_n)| = 0$.

(2) Clearly, $U(\mathbb{Z}_4) = \{\overline{1}, \overline{3}\}$, $D(\mathbb{Z}_4) = \{\overline{0}, \overline{2}\}$. We construct 7 subsets of $\mathrm{H}_n \setminus Z(\mathrm{H}_n)$ as follows.

$$A_1 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{b} \in U(\mathbb{Z}_4);\ \overline{c}, \overline{d} \in D(\mathbb{Z}_4)\right\}$$
$$A_2 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{c} \in U(\mathbb{Z}_4);\ \overline{b}, \overline{d} \in D(\mathbb{Z}_4)\right\}$$
$$A_3 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{d} \in U(\mathbb{Z}_4);\ \overline{b}, \overline{c} \in D(\mathbb{Z}_4)\right\}$$

$$A_4 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{c}, \overline{d} \in U(\mathbb{Z}_4);\ \overline{b} \in D(\mathbb{Z}_4)\right\}$$

$$A_5 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{b}, \overline{d} \in U(\mathbb{Z}_4);\ \overline{c} \in D(\mathbb{Z}_4)\right\}$$

$$A_6 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{b}, \overline{c} \in U(\mathbb{Z}_4);\ \overline{d} \in D(\mathbb{Z}_4)\right\}$$

$$A_7 = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \,\middle|\, \overline{a} \in \mathbb{Z}_4;\ \overline{b}, \overline{c}, \overline{d} \in U(\mathbb{Z}_4)\right\}$$

Clearly, $A_1 \cup A_2 \cup \cdots \cup A_7 = \mathrm{H}_n \setminus Z(\mathrm{H}_n)$. And $A_\lambda \cap A_s = \emptyset$, for $\lambda \neq s$. $|A_1| = |A_2| = \cdots = |A_7| = 2^5$. Moreover, it is easy to verify that for $\lambda = 1, \ldots, 7$, if $\alpha \in A_\lambda$, $\beta \in \Gamma(\mathrm{H}_n)$, then $\alpha\beta = \beta\alpha$ if and only if $\beta \in A_\lambda$. This implies that $\Gamma(\mathrm{H}_n)$ is a graph with 7 connected components of size 2^5 which each of them is a complete graph.

(3) For $\alpha, \beta \in V(\Gamma(\mathrm{H}_n))$, we put $\alpha = \overline{a}+\overline{b}\mathrm{i}+\overline{c}\mathrm{j}+\overline{d}\mathrm{k}$ and $\beta = \overline{w}+\overline{x}\mathrm{i}+\overline{y}\mathrm{j}+\overline{z}\mathrm{k}$.

Case 1. Assume that $2^\lambda|(b, c, d)$, $2^s|(x, y, z)$, for some $\lambda, s \in \{1, 2, \ldots, t-2\}$. If $\lambda + s \geqslant t-1$ then α—β is an edge of $\Gamma(\mathrm{H}_n)$. While if $\lambda + s < t-1$ then α—$2^{t-2}i$—β is a path of $\Gamma(\mathrm{H}_n)$.

Case 2. Assume that $2 \nmid (b, c, d)$, $2|(x, y, z)$, then $2^{t-2}\alpha \notin Z(\mathrm{H}_n)$, so α—$2^{t-2}\alpha$—β is a path of $\Gamma(\mathrm{H}_n)$.

Case 3. Assume that $2|(b, c, d)$, $2 \nmid (x, y, z)$, then $2^{t-2}\beta \notin Z(\mathrm{H}_n)$, so α—$2^{t-2}\beta$—β is a path of $\Gamma(\mathrm{H}_n)$.

Case 4. Assume that $2 \nmid (b, c, d)$, $2 \nmid (x, y, z)$, then $2^{t-2}\alpha$, $2^{t-2}\beta \notin Z(\mathrm{H}_n)$. So α—$2^{t-2}\alpha$—$2^{t-2}\beta$—β is a path of $\Gamma(\mathrm{H}_n)$.

Consequently, $\Gamma(\mathrm{H}_n)$ is connected and $diam(\Gamma(\mathrm{H}_n)) \leqslant 3$. Moreover, note that $\mathrm{i}, \mathrm{j} \in V(\Gamma(\mathrm{H}_n))$, suppose that $\gamma = \overline{a_0} + \overline{b_0}\mathrm{i} + \overline{c_0}\mathrm{j} + \overline{d_0}\mathrm{k}$ is adjacent to both i and j. Observe that $\mathrm{i}\gamma = \gamma\mathrm{i}$ if and only if $2c_0 \equiv 0 \pmod{2^t}$ and $2d_0 \equiv 0 \pmod{2^t}$, while $\mathrm{j}\gamma = \gamma\mathrm{j}$ if and only if $2b_0 \equiv 0 \pmod{2^t}$ and $2d_0 \equiv 0 \pmod{p^t}$. Thus we must have $b_0, c_0, d_0 \in \{0, \frac{n}{2}\}$. By Lemma 2.3, $\gamma \in Z(\mathrm{H}_n)$. Hence, there exists no vertex γ of $\Gamma(\mathrm{H}_n)$ such that i—γ—j is a path of $\Gamma(\mathrm{H}_n)$. Therefore, $diam(\Gamma(\mathrm{H}_n)) = 3$. □

By Lemma 2.2 (2), $\mathrm{H}_p \cong M_2(\mathbb{Z}_p)$ if p is an odd prime. Hence, by Lemma 2.1, $\Gamma(\mathrm{H}_p)$ is a graph with p^2+p+1 connected components of size p^2-p which each of them is a complete graph. In the following theorem, we completely determine the vertices of each connected component of $\Gamma(\mathrm{H}_p)$.

Theorem 2.5. *Suppose $n = p^t$, p is an odd prime, $t \geqslant 1$.*

(1) *If $t = 1$, then $\Gamma(\mathrm{H}_n)$ is a graph with $p^2 + p + 1$ connected components of size $p^2 - p$ which each of them is a complete graph. And the following sets are the vertex sets of all different connected components of $\Gamma(\mathrm{H}_n)$:*

$$A_1 = \left\{\overline{a} + \overline{b}\mathrm{i} \,\middle|\, \overline{a} \in \mathbb{Z}_p; \overline{b} \in \mathbb{Z}_p^*\right\}$$

$$A_2 = \left\{\overline{a} + \overline{c}\mathrm{j} \mid \overline{a} \in \mathbb{Z}_p; \overline{c} \in \mathbb{Z}_p^*\right\}$$
$$A_3 = \left\{\overline{a} + \overline{d}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_p; \overline{d} \in \mathbb{Z}_p^*\right\}$$
$$B_\lambda = \left\{\overline{a} + \overline{b}\mathrm{i} + \overline{\lambda b}\mathrm{j} \mid \overline{a} \in \mathbb{Z}_p; \overline{b} \in \mathbb{Z}_p^*\right\}$$
$$C_\lambda = \left\{\overline{a} + \overline{c}\mathrm{j} + \overline{\lambda c}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_p; \overline{c} \in \mathbb{Z}_p^*\right\}$$
$$D_\lambda = \left\{\overline{a} + \overline{\lambda d}\mathrm{i} + \overline{d}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_p; \overline{d} \in \mathbb{Z}_p^*\right\}$$
$$E_{\sigma\tau} = \left\{\overline{a} + \overline{e}\mathrm{i} + \overline{\sigma e}\mathrm{j} + \overline{\sigma\tau e}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_p; \overline{e} \in \mathbb{Z}_p^*\right\}$$

Where λ, σ, $\tau = 1, 2, \ldots, p-1$.

(2) *If* $t \geqslant 2$, *then* $\Gamma(\mathrm{H}_n)$ *is connected and* $diam(\Gamma(\mathrm{H}_n)) = 3$.

Proof. (1) Clearly, the number of sets presented in (1) is equal to $3+3(p-1)+(p-1)^2 = p^2+p+1$. By an easy calculation we derive that the cardinality of each set is p^2-p, and each vertex of $\Gamma(\mathrm{H}_p)$ belongs to a unique set. Moreover, it is not difficult to verify that for α, $\beta \in \Gamma(\mathrm{H}_p)$, $\alpha\beta = \beta\alpha$ if and only if α and β belong to the same set.

(2) For α, $\beta \in V(\Gamma(\mathrm{H}_n))$, we put $\alpha = \overline{a}+\overline{b}\mathrm{i}+\overline{c}\mathrm{j}+\overline{d}\mathrm{k}$ and $\beta = \overline{w}+\overline{x}\mathrm{i}+\overline{y}\mathrm{j}+\overline{z}\mathrm{k}$.

Case 1. Assume that $p^\lambda|(b,c,d)$, $p^s|(x,y,z)$, for some $\lambda, s \in \{1, 2, \ldots, t-1\}$. If $\lambda+s \geqslant t$ then $\alpha — \beta$ is an edge of $\Gamma(\mathrm{H}_n)$. If $\lambda+s < t$ then $\alpha — p^{t-1}\mathrm{i} — \beta$ is a path of $\Gamma(\mathrm{H}_n)$.

Case 2. Assume that $p \nmid (b,c,d)$, $p|(x,y,z)$, then $p^{t-1}\alpha \notin Z(\mathrm{H}_n)$, so $\alpha — p^{t-1}\alpha — \beta$ is a path of $\Gamma(\mathrm{H}_n)$.

Case 3. Assume that $p|(b,c,d)$, $p \nmid (x,y,z)$, then $p^{t-1}\beta \notin Z(\mathrm{H}_n)$, so $\alpha — p^{t-1}\beta — \beta$ is a path of $\Gamma(\mathrm{H}_n)$.

Case 4. Assume that $p \nmid (b,c,d)$, $p \nmid (x,y,z)$, then $p^{t-1}\alpha$, $p^{t-1}\beta \notin Z(\mathrm{H}_n)$. So $\alpha — p^{t-1}\alpha — p^{t-1}\beta — \beta$ is a path of $\Gamma(\mathrm{H}_n)$.

Hence, $\Gamma(\mathrm{H}_n)$ is connected and $diam(\Gamma(\mathrm{H}_n)) \leqslant 3$. Moreover, note that $\mathrm{i}, \mathrm{j} \in V(\Gamma(\mathrm{H}_n))$, suppose that $\gamma = \overline{a_0} + \overline{b_0}\mathrm{i} + \overline{c_0}\mathrm{j} + \overline{d_0}\mathrm{k}$ is adjacent to both i and j. Since $\mathrm{i}\gamma = \gamma\mathrm{i}$ if and only if $2c_0 \equiv 0 \pmod{p^t}$ and $2d_0 \equiv 0 \pmod{p^t}$, while $\mathrm{j}\gamma = \gamma\mathrm{j}$ if and only if $2b_0 \equiv 0 \pmod{p^t}$ and $2d_0 \equiv 0 \pmod{p^t}$, we must have $b_0 = c_0 = d_0 = 0$. By Lemma 2.3, $\gamma \in Z(\mathrm{H}_n)$. Hence, there exists no vertex γ of $\Gamma(\mathrm{H}_n)$ such that $\mathrm{i} — \gamma — \mathrm{j}$ is a path of $\Gamma(\mathrm{H}_n)$. Therefore, $diam(\Gamma(\mathrm{H}_n)) = 3$. □

We next consider the commuting graph $\Gamma(\mathrm{H}_n)$ in which n has at least two prime divisors. We need the following two lemmas in the sequel.

Lemma 2.6. [6, P.161, Exercise 12] *The number of solutions of the congruence equation in* $x_1, x_2, \dots, x_k$*:*

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k \equiv b \pmod m$$

where $a_1, \dots, a_k, b$ *and* m *are integers with* $m > 1$*, is equal to*

$$m^{k-1}(a_1, a_2, \dots, a_k, m)$$

if $(a_1, \dots, a_k, m)|b$.

Lemma 2.7. *If* p *is an odd prime, then for* $\bar{b}, \bar{c} \in \mathbb{Z}_{2p} \setminus \{\bar{0}, \bar{p}\}$*, there exists a unique ordered pair* $\{\lambda, s\}$ *where* $\lambda \in \{1, 2, \dots, p-1\}$ *and* $s \in \{0, 1\}$ *such that*

$$\lambda b + sp \equiv c \pmod{2p} \tag{2-4}$$

Proof. First, since $(b, p, 2p) = 1$, by Lemma 2.6, the congruence equation (2-4) in λ, s has $2p$ solutions. Suppose $\{\lambda_0, s_0\}$ is a solution of (2-4), for some λ_0, $s_0 \in \{0, 1, 2, \dots, 2p-1\}$. Let $\lambda_0 = xp + r$ for some $x \in \{0, 1\}$ and $r \in \{1, \dots, p-1\}$. Then

$$c \equiv \lambda_0 b + s_0 p \equiv b(xp + r) + s_0 p \equiv rb + (bx + s_0)p \pmod{2p}$$

Observe that $(bx + s_0)p \equiv 0 \pmod{2p}$ if $bx + s_0$ is even, while $(bx + s_0)p \equiv p \pmod{2p}$ if $bx + s_0$ is odd. Hence exactly one of

$$\begin{cases} \lambda \equiv r \pmod{2p} \\ s \equiv 0 \pmod{2p} \end{cases} \quad \text{and} \quad \begin{cases} \lambda \equiv r \pmod{2p} \\ s \equiv 1 \pmod{2p} \end{cases}$$

is a solution of congruence equation (2-4).

Furthermore, if there exist two ordered pairs $\{\lambda_1, s_1\}$ and $\{\lambda_2, s_2\}$ satisfy congruence (2-4), where $\lambda_1, \lambda_2 \in \{1, 2, \dots, p-1\}$ and $s_1, s_2 \in \{0, 1\}$. By equation (2-4), we have

$$(\lambda_1 - \lambda_2)b \equiv (s_2 - s_1)p \pmod{2p} \tag{2-5}$$

If $\lambda_1 \neq \lambda_2$, since $p \nmid b$, we have $p \nmid (\lambda_1 - \lambda_2)b$, a contradiction. So we must have $\lambda_1 = \lambda_2$. Moreover, if $s_1 \neq s_2$, then $s_2 - s_1 = -1$ *or* 1. Hence $(s_2 - s_1)p \equiv p \pmod{2p}$, which is impossible for $\lambda_1 = \lambda_2$. Therefore, $s_1 = s_2$. □

Theorem 2.8. *Suppose that* n *has at least two prime factors.*

(1) *If* $n = 2p$ *where* p *is an odd prime, then* $\Gamma(\mathrm{H}_n)$ *is a graph with* p^2+p+1 *connected components of size* $16p(p-1)$ *which each of them is a complete graph.*

(2) *If* $n \neq 2p$ *where* p *is an odd prime, then* $\Gamma(\mathrm{H}_n)$ *is a connected graph and* $diam(\Gamma(\mathrm{H}_n)) = 3$.

Proof. (1) By Lemma 2.2 (3), we have $H_n \cong H_2 \oplus H_p$. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be two vertices of $\Gamma(H_2 \oplus H_p)$. Note that H_2 is commutative, so neither α_2 nor β_2 belongs to $Z(H_p)$. Thus we have $\alpha\beta = \beta\alpha$ if and only if $\alpha_2\beta_2 = \beta_2\alpha_2$, if and only if α_2 and β_2 belong to the same connected component of $\Gamma(H_p)$. By Theorem 2.5 (1), we can construct the following subsets of $H_n \setminus Z(H_n)$.

$$A_1=\left\{\overline{a}+\overline{b}\mathrm{i}+\overline{s_1p}\mathrm{j}+\overline{s_2p}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{b} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\};\ s_1, s_2 \in \{0,1\}\right\}$$

$$A_2=\left\{\overline{a}+\overline{s_1p}\mathrm{i}+\overline{c}\mathrm{j}+\overline{s_2p}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{c} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\};\ s_1, s_2 \in \{0,1\}\right\}$$

$$A_3=\left\{\overline{a}+\overline{s_1p}\mathrm{i}+\overline{s_2p}\mathrm{j}+\overline{d}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{d} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\};\ s_1, s_2 \in \{0,1\}\right\}$$

$$B_\lambda=\left\{\overline{a}+\overline{b}\mathrm{i}+\overline{\lambda b+s_1p}\mathrm{j}+\overline{s_2p}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{b} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\};\ s_1, s_2 \in \{0,1\}\right\}$$

$$C_\lambda=\left\{\overline{a}+\overline{s_1p}\mathrm{i}+\overline{c}\mathrm{j}+\overline{\lambda c+s_2p}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{c} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\};\ s_1, s_2 \in \{0,1\}\right\}$$

$$D_\lambda=\left\{\overline{a}+\overline{\lambda d+s_1p}\mathrm{i}+\overline{s_2p}\mathrm{j}+\overline{d}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{d} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\};\ s_1, s_2 \in \{0,1\}\right\}$$

$$E_{\sigma\,\tau}=\left\{\overline{a}+\overline{e}\mathrm{i}+\overline{\sigma e+s_1p}\mathrm{j}+\overline{\sigma\tau e+s_2p}\mathrm{k} \mid \overline{a} \in \mathbb{Z}_n;\ \overline{e} \in \mathbb{Z}_n \setminus \{\overline{0},\overline{p}\}; s_1, s_2 \in \{0,1\}\right\}$$

Where λ, σ, $\tau = 1, 2, \ldots, p-1$.

By Lemma 2.7, one can show that each vertex of $\Gamma(H_n)$ belongs to exactly one of the sets above. So $\Gamma(H_n)$ is a graph with p^2+p+1 connected components of size $16p(p-1)$. It is not difficult to verify that each connected component is a complete graph.

(2) Since n has at least two prime divisors and $n \neq 2p$, we have three cases to consider.

Case 1. Suppose that $n = 2p^t$, $t > 1$ and p is an odd prime. Then by Lemma 2.2 (3), we have $H_n \cong H_2 \oplus H_{p^t}$. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be two vertices of $\Gamma(H_2 \oplus H_{p^t})$. Since H_2 is commutative, we have $\alpha_2, \beta_2 \in H_{p^t} \setminus Z(H_{p^t})$. If $\alpha_2\beta_2 = \beta_2\alpha_2$ then $\alpha — \beta$ is an edge of $\Gamma(H_2 \oplus H_{p^t})$. Otherwise, by Theorem 2.5(2), either $\alpha_2 — \xi — \beta_2$ or $\alpha_2 — \eta — \delta — \beta_2$ is a path of $\Gamma(H_{p^t})$, where ξ, η, $\delta \in H_{p^t} \setminus Z(H_{p^t})$. Put $\gamma = (\overline{0}, \xi)$, $\gamma' = (\overline{0}, \eta)$, $\gamma'' = (\overline{0}, \delta)$. Then either $\alpha — \gamma — \beta$ or $\alpha — \gamma' — \gamma'' — \beta$ is a path of $\Gamma(H_2 \oplus H_{p^t})$.

Case 2. Suppose that $n = 2p_1^{t_1} \cdots p_m^{t_m}$, where $m \geqslant 2$, $t_1, \ldots, t_m \geqslant 1$, and $p_1, \ldots, p_m$ are distinct odd primes. Then by Lemma 2.2 (3), we have $H_n \cong H_2 \oplus H_{p_1^{t_1}} \oplus \cdots \oplus H_{p_m^{t_m}}$. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$ be two vertices of $H_2 \oplus H_{p_1^{t_1}} \oplus \cdots \oplus H_{p_m^{t_m}}$. Put $\gamma = (\overline{0}, \gamma_1, \overline{0}, \ldots, \overline{0})$, $\gamma' = (\overline{0}, \overline{0}, \cdots, \gamma_m)$, where $\gamma_1 \in H_{p_1^{t_1}} \setminus Z(H_{p_1^{t_1}})$ and $\gamma_m \in H_{p_m^{t_m}} \setminus Z(H_{p_m^{t_m}})$ such that $\gamma_1\alpha_1 = \alpha_1\gamma_1$, $\gamma_m\beta_m = \beta_m\gamma_m$. Then $\alpha — \gamma — \gamma' — \beta$ is a path of $\Gamma(H_2 \oplus H_{p_1^{t_1}} \oplus \cdots \oplus H_{p_m^{t_m}})$.

Case 3. Suppose that $n = q_1^{t_1} \cdots q_m^{t_m}$, $m \geqslant 2$, $2 \leqslant q_1 < \cdots < q_m$ are primes, $t_1, \ldots, t_m \geqslant 1$ and $q_1^{t_1} > 2$(this implies either $q_1 = 2$ with $t_1 > 1$ or $q_1 > 2$ with $t_1 \geqslant 1$). Then by Lemma 2.2 (3), we have $\mathrm{H}_n \cong \mathrm{H}_{q_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{q_m^{t_m}}$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two vertices of $\Gamma(\mathrm{H}_{q_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{q_m^{t_m}})$. If there exists $\sigma \in \{1, \ldots, m\}$ such that $\alpha_\sigma \in Z(\mathrm{H}_{q_\sigma^{t_\sigma}})$ or $\beta_\sigma \in Z(\mathrm{H}_{q_\sigma^{t_\sigma}})$, without loss of generality, suppose that $\alpha_\sigma \in Z(\mathrm{H}_{q_\sigma^{t_\sigma}})$. Choose $\gamma_\sigma \in \mathrm{H}_{q_\sigma^{t_\sigma}} \setminus (Z(\mathrm{H}_{q_\sigma^{t_\sigma}}) \cup \{\beta_\sigma\})$ such that $\gamma_\sigma\beta_\sigma = \beta_\sigma\gamma_\sigma$. Put $\gamma = (\overline{0}, \ldots, \overline{0}, \gamma_\sigma, \overline{0}, \ldots, \overline{0}) \in \mathrm{H}_{q_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{q_m^{t_m}}$, clearly $\gamma \notin Z(\mathrm{H}_{q_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{q_m^{t_m}})$, and $\gamma \neq \alpha, \beta$. So $\alpha - \gamma - \beta$ is a path of $\Gamma(\mathrm{H}_{q_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{q_m^{t_m}})$. Otherwise, if for $\lambda = 1, \ldots, m$, neither α_λ nor β_λ belongs to $Z(\mathrm{H}_{q_\lambda^{t_\lambda}})$, take $\gamma' = (\alpha_1, \overline{0}, \ldots, \overline{0}), \gamma'' = (\overline{0}, \overline{0}, \ldots, \beta_m)$, then $\alpha - \gamma' - \gamma'' - \beta$ is a path of $\Gamma(\mathrm{H}_{q_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{q_m^{t_m}})$.

Consequently, we have $\Gamma(\mathrm{H}_n)$ is connected and $diam(\Gamma(\mathrm{H}_n)) \leqslant 3$. Furthermore, by the similar argument of Theorem 2.5, we can conclude that there exists no vertex α of $\Gamma(\mathrm{H}_n)$ such that $\mathbf{i} - \alpha - \mathbf{j}$ is a path of $\Gamma(\mathrm{H}_n)$. Thus $diam(\Gamma(\mathrm{H}_n)) = 3$. □

By Lemma 2.2, Theorem 2.4, Theorem 2.5 and Theorem 2.8, we can get a general result.

Theorem 2.9. (1) *Let $n > 2$. Then $\Gamma(\mathrm{H}_n)$ is connected if and only if $n \neq p, 2p, 2^2$ for all odd primes p. If $\Gamma(\mathrm{H}_n)$ is connected then $diam(\Gamma(\mathrm{H}_n))$ must be 3, while if $\Gamma(\mathrm{H}_n)$ is disconnected then every connected component of $\Gamma(\mathrm{H}_n)$ must be a complete graph with the same size.*

(2) *Let $m > 1$ be an odd integer. Then $\Gamma(M_2(\mathbb{Z}_m))$ is connected if and only if m is not a prime. In this case, $diam(M_2(\mathbb{Z}_m)) = 3$.*

3 The maximum degree and minimum degree of $\Gamma(\mathrm{H}_n)$

It follows directly from Theorem 2.4 that if $n = 2^2$ then $\Delta(\Gamma(\mathrm{H}_n)) = \delta(\Gamma(\mathrm{H}_n)) = 2^5 - 1$. And by Theorem 2.5, we have $\Delta(\Gamma(\mathrm{H}_n)) = \delta(\Gamma(\mathrm{H}_n)) = n^2 - n - 1$ if n is an odd prime. By Theorem 2.8, if $n = 2p$, then $\Delta(\Gamma(\mathrm{H}_n)) = \delta(\Gamma(\mathrm{H}_n)) = 16p(p-1) - 1$.

Lemma 3.1. *Let $n = 2^t$, where $t \geqslant 2$ and $\overline{b}, \overline{c}, \overline{d} \in \mathbb{Z}_{2^t}$.*

(1) *Suppose $t \geqslant 2$ and $2 \nmid (b, c, d)$. Then the number of solutions of system $(*)$(see* Lemma 2.3*) in x, y, z is 2^{t+2}.*

(2) *Suppose $t \geqslant 3$ and $2^\tau \parallel (b, c, d)$ where $t - 2 \geqslant \tau \geqslant 1$. Then the number of solutions of system $(*)$ in x, y, z is $2^{t+2\tau+2}$.*

Proof. (1) Since $2 \nmid (b, c, d)$, without loss of generality, we can suppose $2 \nmid c$.

Case 1.1. Assume that $b, d \neq 0$. Since $(2b, 2c, 2^t) = 2$, by Lemma 2.6, the number of solutions of equation (2-3) in x, y is 2^{t+1}. Suppose

$$x \equiv x_s \pmod{2^t}, \quad y \equiv y_s \pmod{2^t}$$

are solutions of equation (2-3), $s \in \{1, 2, \ldots, 2^{t+1}\}$. Then we have

$$2(by_s - cx_s) \equiv 0 \pmod{2^t} \tag{3-6}$$

Substituting $y \equiv y_s \pmod{2^t}$ into equation (2-1), and notice that $(2c, 2^t) = 2$, thus the number of solutions of equation (2-1) in z is equal to 2, denoting them by $z \equiv z_m \pmod{2^t}$ where $m = 1, 2$. We have

$$2(cz_m - dy_s) \equiv 0 \pmod{2^t} \tag{3-7}$$

Moreover, notice that $b, d \neq 0$, so by equations (3-6) and (3-7), we have

$$2(bdy_s - cdx_s) \equiv 0 \pmod{2^t}$$
$$2(bcz_m - bdy_s) \equiv 0 \pmod{2^t}$$

From the above two equations we derive $2(cdx_s - bcz_m) \equiv 0 \pmod{2^t}$. Since $2 \nmid c$, we have $2(dx_s - bz_m) \equiv 0 \pmod{2^t}$. Hence

$$x \equiv x_s \pmod{2^t}, \quad z \equiv z_m \pmod{2^t}$$

satisfy equation (2-2). Consequently,

$$x \equiv x_s \pmod{2^t}, \; y \equiv y_s \pmod{2^t}, \; z \equiv z_m \pmod{2^t}$$

are solutions of system $(*)$. Therefore, the number of solutions of system $(*)$ is $2^{t+1} \times 2 = 2^{t+2}$.

Case 1.2. Assume that $b \neq 0$ and $d = 0$, by Lemma 2.6, the number of solutions of equation (2-3) in x, y is 2^{t+1}. Moreover, notice that $2 \nmid c$, so the number of solutions of equation (2-1) in z is 2, i.e., both $z \equiv 0 \pmod{2^t}$ and $z \equiv 2^{t-1} \pmod{2^t}$ satisfy equation (2-2). Hence the number of solutions of system $(*)$ is $2^{t+1} \times 2 = 2^{t+2}$. Similarly, if $d \neq 0$ and $b = 0$, we also have the same result.

Case 1.3. Assume that $b = d = 0$. Notice that $2 \nmid c$, thus

$$z \equiv 0, 2^{t-1} \pmod{2^t}, \quad y \equiv 0, 1, 2, \ldots, 2^t - 1 \pmod{2^t}$$

satisfy equation (2-1). While

$$x \equiv 0, 2^{t-1} \pmod{2^t}, \quad y \equiv 0, 1, 2, \ldots, 2^t - 1 \pmod{2^t}$$

satisfy equation (2-3). Thus

$$x \equiv 0, 2^{t-1} \pmod{2^t},\ y \equiv 0, 1, 2, \ldots, 2^t - 1 \pmod{2^t},\ z \equiv 0, 2^{t-1} \pmod{2^t}$$

satisfy system (*). Therefore the number of solutions of system (*) is equal to $2^t \times 2 \times 2 = 2^{t+2}$.

(2) We will divide our proof into two cases.

Case 2.1. Suppose $b, c, d \neq 0$. Since $2^\tau \parallel (b, c, d)$, without loss of generality, we assume that $b = 2^\lambda b_1, c = 2^\sigma c_1, d = 2^\tau d_1$, where b_1, c_1, d_1 are odd and $t - 1 \geqslant \lambda \geqslant \sigma \geqslant \tau \geqslant 1$. Since $(2c, 2d, 2^t) = (2^{\sigma+1}c_1, 2^{\tau+1}d_1, 2^t) = 2^{\tau+1}$, by Lemma 2.6, the number of solutions of equation (2-1) in y, z is $2^t \times 2^{\tau+1} = 2^{t+\tau+1}$. Suppose

$$y \equiv y_s \pmod{2^t}, \quad z \equiv z_s \pmod{2^t}$$

are solutions of equation (2-1), $s \in \{1, 2, \ldots, 2^{t+\tau+1}\}$. We have

$$2(2^\sigma c_1 z_s - 2^\tau d_1 y_s) \equiv 0 \pmod{2^t} \tag{3-8}$$

Substituting $z \equiv z_s \pmod{2^t}$ into equation (2-2) then we would derive the following equation: $2^{\tau+1}d_1 x \equiv 2^{\lambda+1}b_1 z_s \pmod{2^t}$. Since $(2^{\tau+1}d_1, 2^t) = 2^{\tau+1}$ and observe that $2^{\tau+1} | 2^{\lambda+1}$, the number of solutions of equation (2-2) in x is $2^{\tau+1}$. Denoting them by $x \equiv x_\rho \pmod{2^t}$ where $\rho = 1, 2, \ldots, 2^{\tau+1}$. Then we have

$$2(2^\tau d_1 x_\rho - 2^\lambda b_1 z_s) \equiv 0 \pmod{2^t} \tag{3-9}$$

Moreover, notice that $b_1, c_1 \neq 0$, so by congruence (3-8) and (3-9) we have

$$2(2^\tau b_1 d_1 y_s - 2^\sigma b_1 c_1 z_s) \equiv 0 \pmod{2^t} \tag{3-10}$$

$$2(2^\tau c_1 d_1 x_\rho - 2^\lambda b_1 c_1 z_s) \equiv 0 \pmod{2^t} \tag{3-11}$$

Furthermore, multiplying both sides of equation (3-10) by $2^{\lambda-\tau}$, we have:

$$2(2^\lambda b_1 d_1 y_s - 2^{\lambda+\sigma-\tau} b_1 c_1 z_s) \equiv 0 \pmod{2^t} \tag{3-12}$$

Similarly, multiplying both sides of equation (3-11) by $2^{\sigma-\tau}$, we have:

$$2(2^\sigma c_1 d_1 x_\rho - 2^{\lambda+\sigma-\tau} b_1 c_1 z_s) \equiv 0 \pmod{2^t} \tag{3-13}$$

So by equation (3-12) and (3-13), we have $2(2^\lambda b_1 d_1 y_s - 2^\sigma c_1 d_1 x_\rho) \equiv 0 \pmod{2^t}$. Since $2 \nmid d_1$, we get $2(2^\lambda b_1 y_s - 2^\sigma c_1 x_\rho) \equiv 0 \pmod{2^t}$, i.e., $2(by_s - cx_\rho) \equiv 0 \pmod{2^t}$. Hence $x \equiv x_\rho \pmod{2^t}$ and $y \equiv y_s \pmod{2^t}$ satisfy equation (2-3). Thus

$$x \equiv x_\rho \pmod{2^t},\ y \equiv y_s \pmod{2^t},\ z \equiv z_s \pmod{2^t}$$

is a solution of system (*). Therefore, the number of solutions of system (*) is $2^{t+\tau+1} \times 2^{\tau+1} = 2^{t+2\tau+2}$.

Case 2.2. Assume that at least one of b, c, d is 0. By the similar argument of Case 2.1, the result follows. □

By the similar proof of Lemma 3.1, we have the following lemma.

Lemma 3.2. *Let $n = p^t$, where p is an odd prime, $t \geqslant 1$ and $\overline{b}, \overline{c}, \overline{d} \in \mathbb{Z}_{p^t}$.*

(1) *Suppose that $t \geqslant 1$ and $p \nmid (b, c, d)$. Then the number of solutions of system $(*)$ in x, y, z is p^t.*

(2) *Suppose that $t \geqslant 2$ and $p^\tau \parallel (b, c, d)$, where $t - 1 \geqslant \tau \geqslant 1$. Then the number of solutions of system $(*)$ in x, y, z is $p^{t+2\tau}$.*

Remark 3.3. (1) *Suppose $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \mathrm{H}_{2^s}$, $s \geqslant 1$, let $A_s(\alpha) = \{\gamma \in \mathrm{H}_{2^s} \mid \alpha\gamma = \gamma\alpha\}$. By* Lemma 3.1, *we have*

$$|A_s(\alpha)| = \begin{cases} 2^{2s+2} & 2 \nmid (b, c, d) \\ 2^{2s+2\tau+2} & s \geqslant 3,\ 2^\tau \parallel (b, c, d), \text{ where } s - 2 \geqslant \tau \geqslant 1 \\ 2^{4s} & s = 1, \text{ or } 2^{s-1} | (b, c, d) \text{ with } s > 1 \end{cases}$$

(2) *Suppose $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \mathrm{H}_{p^t}$, where $t \geqslant 1$ and p is an odd prime, let $B_{p^t}(\alpha) = \{\gamma \in \mathrm{H}_{p^t} \mid \alpha\gamma = \gamma\alpha\}$. By* Lemma 3.2 *we have*

$$|B_{p^t}(\alpha)| = \begin{cases} p^{2t} & p \nmid (b, c, d) \\ p^{2t+2\tau} & t \geqslant 2,\ p^\tau \parallel (b, c, d), \text{ where } t - 1 \geqslant \tau \geqslant 1 \\ p^{4t} & p^t | (b, c, d) \end{cases}$$

Theorem 3.4. *Suppose $n = 2^t$ where $t \geqslant 3$, $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \Gamma(\mathrm{H}_n)$.*

(1) *If $2 \nmid (b, c, d)$, then $d(\alpha) = 2^{2t+2} - 2^{t+3} - 1$.*

(2) *If $2^\tau \parallel (b, c, d)$, then $d(\alpha) = 2^{2t+2\tau+2} - 2^{t+3} - 1$, where $t - 2 \geqslant \tau \geqslant 1$.*

(3) *The minimum degree $\delta(\Gamma(\mathrm{H}_n)) = 2^{2t+2} - 2^{t+3} - 1$, while $d(\alpha) = \delta(\Gamma(\mathrm{H}_n))$ if and only if $2 \nmid (b, c, d)$.*

(4) *The maximum degree $\Delta(\Gamma(\mathrm{H}_n)) = 2^{4t-2} - 2^{t+3} - 1$, while $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ if and only if $2^{t-2} \parallel (b, c, d)$.*

Proof. (1) By Remark 3.3, we obtain that $|A_t(\alpha)| = 2^{2t+2}$. Moreover, by Lemma 2.3, we have $|Z(\mathrm{H}_n)| = 2^3 n = 2^{t+3}$. Hence

$$d(\alpha) = |A_t(\alpha)| - |Z(\mathrm{H}_n)| - 1 = 2^{2t+2} - 2^{t+3} - 1.$$

(2) By Remark 3.3, we have $|A_t(\alpha)| = 2^{2t+2\tau+2}$. Hence

$$d(\alpha) = |A_t(\alpha)| - |Z(\mathrm{H}_n)| - 1 = 2^{2t+2\tau+2} - 2^{t+3} - 1.$$

(3) and (4) follows directly by (1) and (2). □

Theorem 3.5. *Suppose $n = p^t$ where p is an odd prime and $t \geqslant 2$, $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \Gamma(\mathrm{H}_n)$.*

(1) *If $p \nmid (b, c, d)$, then $d(\alpha) = p^{2t} - p^t - 1$.*

(2) *If $p^\tau \parallel (b, c, d)$, where $t - 1 \geqslant \tau \geqslant 1$, then $d(\alpha) = p^{2t+2\tau} - p^t - 1$.*

(3) *The minimum degree $\delta(\Gamma(\mathrm{H}_n)) = p^{2t} - p^t - 1$, while $d(\alpha) = \delta(\Gamma(\mathrm{H}_n))$ if and only if $p \nmid (b, c, d)$.*

(4) *The maximum degree $\Delta(\Gamma(\mathrm{H}_n)) = p^{4t-2} - p^t - 1$, while $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ if and only if $p^{t-1} \parallel (b, c, d)$.*

Proof. By Lemma 3.2, and by the similar proof of Theorem 3.4, the result follows. □

Now, it remains to calculate the degree of vertices in $\Gamma(\mathrm{H}_n)$ where n has at least two prime divisors and $n \neq 2p$ for all odd primes p.

Theorem 3.6. *Suppose that $n = 2^{t_0} p_1^{t_1} \cdots p_m^{t_m}$ (where $t_0 \geqslant 0$, $m, t_1, \ldots, t_m \geqslant 1$ and $p_1 < \cdots < p_m$ are odd primes) and $n \neq 2^\mu, p^\mu, 2p$ (where p is an odd prime and $\mu \geqslant 1$). For $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in \mathrm{H}_n$, we define two subsets $I, J \subseteq M = \{1, 2, \ldots, m\}$ as follows:*

$$\begin{aligned} I &= \left\{\sigma \in M \;\middle|\; p_\sigma^{\tau_\sigma} \parallel (b, c, d), \text{ for some } 1 \leqslant \tau_\sigma \leqslant t_\sigma - 1\right\} \\ J &= \left\{\lambda \in M \;\middle|\; p_\lambda^{t_\lambda} | (b, c, d)\right\}. \end{aligned}$$

(1) *Assume that $t_0 = 0$ or 1.*

(i) *The degree of α is $d(\alpha) = 2^{2t_0} n^2 \prod\limits_{\sigma \in I} p_\sigma^{2\tau_\sigma} \prod\limits_{\lambda \in J} p_\lambda^{2t_\lambda} - 2^{3t_0} n - 1$.*

(ii) *The minimum degree $\delta(\Gamma(\mathrm{H}_n)) = 2^{2t_0} n^2 - 2^{3t_0} n - 1$, while $d(\alpha) = \delta(\Gamma(\mathrm{H}_n))$ if and only if $p_\lambda \nmid (b, c, d)$, for $\lambda = 1, 2, \ldots, m$.*

(iii) *The maximum degree $\Delta(\Gamma(\mathrm{H}_n)) = \frac{n^4}{p_1^2} - 2^{3t_0} n - 1$, while $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ if and only if $p_1^{t_1 - 1} \parallel (b, c, d)$ and $p_s^{t_s} | (b, c, d)$ for $s = 2, 3, \ldots, m$.*

(2) *Assume that $t_0 \geqslant 2$.*

(i) *Let $g = (b, c, d)$, then*

$$d(\alpha) = \begin{cases} 2^2 n^2 \prod\limits_{\sigma \in I} p_\sigma^{2\tau_\sigma} \prod\limits_{\lambda \in J} p_\lambda^{2t_\lambda} - 2^3 n - 1, & 2 \nmid g \\ 2^{2e+2} n^2 \prod\limits_{\sigma \in I} p_\sigma^{2\tau_\sigma} \prod\limits_{\lambda \in J} p_\lambda^{2t_\lambda} - 2^3 n - 1, & t_0 > 2,\ 2^e \parallel g,\ t_0 - 2 \geqslant e \geqslant 1 \\ 2^{2t_0} n^2 \prod\limits_{\sigma \in I} p_\sigma^{2\tau_\sigma} \prod\limits_{\lambda \in J} p_\lambda^{2t_\lambda} - 2^3 n - 1, & 2^{t_0 - 1} | g \end{cases}$$

(ii) *The minimum degree* $\delta(\Gamma(\mathrm{H}_n)) = 2^2n^2 - 2^3n - 1$, *while* $d(\alpha) = \delta(\Gamma(\mathrm{H}_n))$ *if and only if* $2 \nmid (b,c,d)$ *and for* $\lambda = 1,2,\ldots,m$, $p_\lambda \nmid (b,c,d)$.

(iii) *The maximum degree* $\Delta(\Gamma(\mathrm{H}_n)) = \frac{n^4}{2^2} - 2^3n - 1$, *while* $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ *if and only if* $2^{t_0-2} \parallel (b,c,d)$ *and* $p_\lambda^{t_\lambda} | (b,c,d)$ *for* $\lambda = 1,2,\ldots,m$.

Proof. (1) (i) First suppose $t_0 = 0$. By Lemma 2.2 (3), we have $\mathrm{H}_n \cong \mathrm{H}_{p_1^{t_1}} \oplus \cdots \oplus \mathrm{H}_{p_m^{t_m}}$. Let $\alpha = (\alpha_1,\ldots,\alpha_m)$ and $\beta = (\beta_1,\ldots,\beta_m)$ be two vertices of $\Gamma(\mathrm{H}_n)$. Then $\alpha\beta{=}\beta\alpha$ if and only if $\alpha_\lambda\beta_\lambda = \beta_\lambda\alpha_\lambda$ for $\lambda = 1,2,\ldots,m$. Hence, for $\alpha = \overline{a} + \overline{b}\mathrm{i} + \overline{c}\mathrm{j} + \overline{d}\mathrm{k} \in V(\Gamma(\mathrm{H}_n))$, by Remark 3.3 and note that $|Z(\mathrm{H}_n)| = n$, we have

$$\begin{aligned} d(\alpha) &= |B_{p_1^{t_1}}(\alpha)|\cdots|B_{p_m^{t_m}}(\alpha)| - |Z(\mathrm{H}_n)| - 1 \\ &= \prod_{\sigma\in I} p_\sigma^{2t_\sigma+2\tau_\sigma} \prod_{\lambda\in J} p_\lambda^{4t_\lambda} \prod_{s\notin I,\,J} p_s^{2t_s} - n - 1 \\ &= n^2 \prod_{\sigma\in I} p_\sigma^{2\tau_\sigma} \prod_{\lambda\in J} p_\lambda^{2t_\lambda} - n - 1. \end{aligned}$$

We next suppose $t_0 = 1$. Since H_2 is commutative, clearly, for $\gamma \in \mathrm{H}_2$, $|A_{t_0}(\gamma)| = 2^4$. And note that $|Z(\mathrm{H}_n)| = 2^3n$, similarly, we have

$$\begin{aligned} d(\alpha) &= |A_{t_0}(\alpha)||B_{p_1^{t_1}}(\alpha)|\cdots|B_{p_m^{t_m}}(\alpha)| - |Z(\mathrm{H}_n)| - 1 \\ &= 2^4 \prod_{\sigma\in I} p_\sigma^{2t_\sigma+2\tau_\sigma} \prod_{\lambda\in J} p_\lambda^{4t_\lambda} \prod_{s\notin I,\,J} p_s^{2t_s} - 2^3n - 1 \\ &= 2^2n^2 \prod_{\sigma\in I} p_\sigma^{2\tau_\sigma} \prod_{\lambda\in J} p_\lambda^{2t_\lambda} - 2^3n - 1. \end{aligned}$$

(ii) Since $\prod_{\sigma\in I} p_\sigma^{2\tau_\sigma} \prod_{\lambda\in J} p_\lambda^{2t_\lambda} = 1$ if and only if $p_\lambda \nmid (b,c,d)$ for $\lambda = 1,\ldots,m$, we have $\delta(\Gamma(\mathrm{H}_n)) = 2^{2t_0}n^2 - 2^3n - 1$, as desired.

(iii) By (1)(i), we can write $d(\alpha)$ as

$$d(\alpha) = \frac{n^4}{\prod_{\sigma\in I} p_\sigma^{2t_\sigma-2\tau_\sigma} \prod_{s\notin I,J} p_s^{2t_s}} - 2^{3t_0}n - 1.$$

Since $p_1^2 \leqslant \prod_{\sigma\in I} p_\sigma^{2t_\sigma-2\tau_\sigma} \prod_{s\notin I,J} p_s^{2t_s}$, we obtain

$$\Delta(\Gamma(\mathrm{H}_n)) = \frac{n^4}{p_1^2} - 2^{3t_0}n - 1.$$

Hence, if $t_1 = 1$, then $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ if and only if $p_1 \nmid (b,c,d)$ and for $s = 2,\ldots,m$, $p_s^{t_s}|(b,c,d)$. While if $t_1 > 1$, then $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ if and only if $p_1^{t_1-1} \parallel (b,c,d)$ and for $s = 2,\ldots,m$, $p_s^{t_s}|(b,c,d)$.

(2)(i) By the similar argument of (1)(i) and by Remark 3.3, the result follows.

(ii) Clearly, $\prod_{\sigma\in I} p_\sigma^{2\tau_\sigma} \prod_{\lambda\in J} p_\lambda^{2t_\lambda} = 1$ if and only if $p_\lambda \nmid (b,c,d)$ for $\lambda = 1,\dots,m$. And note that $2^2 < 2^{2e+2} < 2^{2t_0}$, we derive that $\delta(\Gamma(\mathrm{H}_n)) = 2^2n^2 - 2^3n - 1$. Therefore, $d(\alpha) = \delta(\Gamma(\mathrm{H}_n))$ if and only if $2 \nmid (b,c,d)$ and $p_\lambda \nmid (b,c,d)$ for $\lambda = 1,2,\dots,m$.

(iii) Suppose that $t_0 = 2$. If $2 \nmid (b,c,d)$, then by (2)(i), we can write $d(\alpha)$ as:

$$d(\alpha) = \frac{n^4}{2^2 \prod\limits_{\sigma\in I} p_\sigma^{2t_\sigma - 2\tau_\sigma} \prod\limits_{s\notin I,J} p_s^{2t_s}} - 2^3n - 1.$$

If $2|(b,c,d)$, then

$$d(\alpha) = \frac{n^4}{\prod\limits_{\sigma\in I} p_\sigma^{2t_\sigma - 2\tau_\sigma} \prod\limits_{s\notin I,J} p_s^{2t_s}} - 2^3n - 1.$$

Since $2^2 \leqslant 2^2 \prod\limits_{\sigma\in I} p_\sigma^{2t_\sigma-2\tau_\sigma} \prod\limits_{s\notin I,J} p_s^{2t_s}$ and $2^2 \leqslant \prod\limits_{\sigma\in I} p_\sigma^{2t_\sigma-2\tau_\sigma} \prod\limits_{s\notin I,J} p_s^{2t_s}$, we have $\Delta(\Gamma(\mathrm{H}_n)) = \frac{n^4}{2^2} - 2^3n - 1$. Clearly, $d(\alpha) = \Delta(\Gamma(\mathrm{H}_n))$ if and only if $2 \nmid (b,c,d)$ and $p_\lambda^{t_\lambda}|(b,c,d)$ for $\lambda = 1,2,\dots,m$.

Now suppose $t_0 > 2$, by the similar argument of the case $t_0 = 2$, the result follows. □

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