Up-embeddable graphs via the degree-sum of nonadjacent vertices: non-simple graphs †

Guanghua Dong[1], Yanpei Liu[2] and Ning Wang[3]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300160, P.R. China. (E-mail: gh.dong@163.com)*

[2] *Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, P.R. China.*

[3] *Department of Information Science and Technology, Tianjin University of Finance and Economics, Tianjin, 300222, P.R. China.*

Abstract

A *semi-double graph* is such a connected multi-graph that each multi-edge consists of two edges. If there is at most one loop at each vertex of a semi-double graph then this graph is called a *single-petal graph*. Via the degree-sum of nonadjacent vertices, the up-embeddability of semi-double graphs and single-petal graphs are discussed in this paper. And the results obtained in this paper can be extended to determine the up-embeddability of multi-graphs and pseudographs.

Key Words: maximum genus; up-embeddable graph; graph embedding; semi-double graph; single-petal graph

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1. Introduction

The idea of the maximum genus $\gamma_M(G)$ of a connected graph G was introduced by Nordhaus, Stewart and White [12] in 1971, and Ringeisen, who have studied the maximum genus extensively [13], [14], and [15], introduced the definition of up-embeddable graphs. From then on, many researchers have studied the up-embeddability of graphs, such as Kundu[8], Jaeger, Payan and Xuong [6], Jungerment [7], Škoviera [16], Huang and Liu[3]. Among others, recently, in terms of degree-sum of nonadjacent vertices of

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a graph, Huang and Liu[4] obtained the following result related to simple graphs:

Theorem 1.1 Let G be a 2-edge-connected (resp. 3-edge-connected) simple graph of order n, then G is up-embeddable if $d_G(u) + d_G(v) \geq \frac{2(n-2)}{3}$ (resp. $d_G(u) + d_G(v) \geq \frac{n+1}{3}$) for any two nonadjacent vertices u and v of G, furthermore the lower bound is tight.

Naturally, the problem of how to determine the up-embeddability of non-simple graphs is expected. *Even-deletion* is such an edge deleting operation on a graph G that the following requirements are satisfied: (i) the edges deleted from G may be links, multi-edges, and loops; (ii) the remainder of the graph is connected; (iii) the number of edges deleted from G should be an even number, and the subgraph induced by the deleted edges should be connected. An *even-ancestry* of a non-simple graph G is such a simple graph, or a semi-double graph, or a single-petal graph that is obtained from G by a sequence of even-deletions. For convenience, these definitions are illustrated by Fig.7, where both G_{11} and G_{12} are even-ancestries of G_{13}. It is obvious that a non-simple graph may have more than one even-ancestry. Furthermore, according to Theorem 1.2, whose proof will be given in Section 4, we can study the up-embeddability of non-simple graphs through that of simple graphs, or semi-double graphs, or single-petal graphs.

Theorem 1.2 A non-simple graph G is up-embeddable if and only if one of its even-ancestries G' is up-embeddable.

In view of the results obtained in Theorem 1.1, this paper will focus on such field as the up-embeddability of semi-double graphs and single-petal graphs.

2. Definitions and Lemmas

A graph is denoted by $G = (V(G), E(G))$, and $V(G)$, $E(G)$ denotes its vertex set and edge set respectively. Between two distinct vertices, if there is only one edge joining them, this edge is called a *link*, and if there are more than one edge joining them, these edges are called *multi-edge* of the graph. A simple graph is a graph having neither loops nor multi-edges. A multi-graph is a graph which may have multi-edges but doesn't have a loop, and a pseudograph is a graph allows loops and multi-edges. A connected multi-graph is called a *semi-double graph* if each multi-edge of this graph consists of two edges. If there is at most one loop at each vertex of a semi-double graph then this graph is called a *single-petal graph*. For example, in Figure 1, the graph G_1 is a semi-double graph, G_2 is a multi-graph but not a semi-double graph, G_3 is a single-petal graph, G_4 is a pseudograph but not a single-petal graph. The order of a graph G is the number of vertices in G. The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. A graph G is k-edge-connected (resp. k-vertex-connected) if for any $h < k$, removal of any h edges (resp. h vertices) in the graph G does not disconnect the graph. For any set X, we use $|X|$ to denote the cardinality of X. For any real number x, $\lfloor x \rfloor$ denotes the floor of x, *i.e.*, the greatest integer which is less than or equal to x, and $\lceil x \rceil$ denotes the ceiling of x, *i.e.*, the smallest integer which is greater than or equal to x. Graphs considered here are permitted to have multi-edges and loops, and are all undirected, finite and connected unless the context requires otherwise. Terminologies and notations not explained here can be seen in [17] for general graph theory. It is assumed that the reader is somewhat familiar with topological graph theory. For general background, see Liu [9], Gross and Tucker [2] or White [18].

G_1 G_2 **Fig.1.** G_3 G_4

Recall that the maximum genus $\gamma_M(G)$ of a connected graph G is the maximum integer k such that there exists an embedding of G into the orientable surface of genus k. Since any embedding must have at least one face, the Euler characteristic for one face leads to an upper bound on the maximum genus

$$\gamma_M(G) \leq \lfloor \frac{|E(G)| - |V(G)| + 1}{2} \rfloor,$$

where the number $|E(G)| - |V(G)| + 1$ is known as the *Betti number* (or *cycle rank*) of the connected graph G and is denoted by $\beta(G)$. A graph G is said to be up-embeddable if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$.

For a subset $A \subseteq E(G)$, $c(G \backslash A)$ denotes the number of all connected components of $G \backslash A$, and $b(G \backslash A)$ denotes the number of connected components of $G \backslash A$ with odd *Betti number*, where $G \backslash A$ means the subgraph obtained from G by deleting all the edges of A from G. Let T be a spanning tree of a connected graph G. Define the *deficiency* $\xi(G,T)$ of a spanning tree T in a graph G to be the number of components of $G \backslash E(T)$ which have an odd number of edges. The deficiency $\xi(G)$ of the graph G is defined to be the minimum value of $\xi(G,T)$ over all spanning tree T of G. Note that $\xi(G) \equiv \beta(G)$ (mod 2). Let F_1, F_2, $\cdots$ F_k be k ($k \geq 2$) distinct subgraphs of a graph G, then denotes by $E_G(F_1, F_2, \cdots, F_k)$ the set of edges of $E(G)$ whose one end vertex is in $V(F_i)$ and the other in $V(F_j)$ ($1 \leq i$, $j \leq k$,

$i \neq j$), and denote by $E(F_i, G)$ the set of edges of $E(G)$ whose one end vertex is in $V(F_i)$ and the other not in $V(F_i)$ $(1 \le i \le k)$. For a vertex $v \in V(F_i)$ $(1 \le i \le k)$, we call v a non-contacting-vertex of $V(F_i)$ if v is not incident with any edge of $E(F_i, G)$, and call v a contacting-vertex of $V(F_i)$ if v is incident with at least one edge of $E(F_i, G)$, and v is called a m-contacting-vertex of $V(F_i)$ if v is incident with m $(m \geq 1)$ edge(s) of $E(F_i, G)$.

The following lemmas give some combinatorial characterizations of the maximum genus of graphs.

Lemma 2.1 (Liu ([9] and [10]), Xuong [19]) Let G be a connected graph, then

1) G is up-embeddable if and only if $\xi(G) \le 1$;

2) $\gamma_M(G) = \frac{\beta(G)-\xi(G)}{2}$.

Lemma 2.2 (Nebeský [11]) Let G be a connected graph, then

1) G is up-embeddable if and only if $c(G\backslash A) + b(G\backslash A) - 2 \le |A|$ for any subset $A \subseteq E(G)$;

2) $\xi(G) = \max\limits_{A \subseteq E(G)} \{c(G\backslash A) + b(G\backslash A) - |A| - 1\}$.

Lemma 2.3 (Huang [5]) Let G be a graph. If $\xi(G) \ge 2$, namely G is not up-embeddable, then there exists a subset $A \subseteq E(G)$ such that the following properties are satisfied:

(i) $c(G\backslash A) = b(G\backslash A) \ge 2$;

(ii) F is an vertex-induced subgraph of G for each component F of $G\backslash A$;

(iii) for any k distinct components $F_1, F_2, \cdots, F_k$ of $G\backslash A$, $|E_G(F_1, F_2, \cdots, F_k)| \le 2k - 3$. Especially $|E_G(F, H)| \le 1$ for any two distinct components F and H of $G\backslash A$;

(iv) $\xi(G) = 2c(G\backslash A) - |A| - 1$.

3. Main results

Since every 4-edge-connected graph is up-embeddable[8], we only need to discuss the graphs with edge-connectivity less than 4. We first discuss semi-double graphs.

Theorem 3.1 Let G be a connected semi-double graph of order n. For any two nonadjacent vertices u and v of G, if $d_G(u) + d_G(v) \ge 2n - 3$ then G is up-embeddable.

Proof Assume to the contrary that G is not up-embeddable. By Lemma 2.3, there exists $A \subseteq E(G)$ such that the properties (i)-(iv) of Lemma 2.3 are satisfied. Let $\mathcal{R} = \{F_1, F_2, \cdots, F_l\}$($l = c(G\backslash A) = b(G\backslash A) \ge 2$) be all the connected components of $G\backslash A$, and x, y, and z be the number of such $F_i \in \mathcal{R}$ that $|E(F_i,G)| = 1$, 2, and 3 respectively. Counting the incidencies, it is obvious that $|A| = \frac{1}{2}\sum_{i=1}^{l}|E(F_i,G)| \geq \frac{x}{2} + y + \frac{3}{2}z + 2(l - x - y - z)$. From Lemma 2.3(iv), we have

$$\begin{aligned} 2 &\leq \xi(G) = 2l - |A| - 1 \\ &\leq 2l - (\frac{x}{2} + y + \frac{3}{2}z + 2(l - x - y - z)) - 1, \end{aligned}$$

and so $x + y + z \geq 2$. By Lemma 2.3(i), $|V(F)| \geq 2$ for each $F \in \mathcal{R}$. Noticing that for each $F \in \mathcal{R}$, if v is a non-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| - 2$; if v is a 1-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| - 1$; if v is a 2-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|$; and if v is a 3-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| + 1$. We will consider two cases in the following.

Case 1 : $l = 2$.

Let F_1 and F_2 be the two components of $G\backslash A$. From Lemma 2.3 (iii) we can get that $|E_G(F_1,F_2)| = 1$, and that the vertices in $F_i(i{=}1,2)$ are all non-contacting-vertex except one 1-contacting-vertex. By Lemma 2.3(i) we can get that $|V(F_i)| \geq 2(i = 1,2)$. It is obvious that there must be a non-contacting-vertex $v_1 \in V(F_1)$ and a non-contacting-vertex $v_2 \in V(F_2)$. Furthermore, v_1 and v_2 are two nonadjacent vertices. Thus $d_G(v_1)+d_G(v_2) \leq 2|V(F_1)|-2+2|V(F_2)|-2=2(|V(F_1)|+|V(F_2)|)-4=2n-4$. On the other hand, according to the condition required in Theorem 3.1 that $d_G(u)+d_G(v) \geq 2n-3$ for any two nonadjacent vertices u and v of G we have $d_G(v_1)+d_G(v_2) \geq 2n-3$. Thus $2n-3 \leq d_G(v_1) + d_G(v_2) \leq 2n-4$, a contradiction.

Case 2 : $l \geq 3$.

Because $x+y+z \geq 2$, without loss of generality, let F_1 and F_2 be any two components of $G\backslash A$ with the property that $1 \leq |E(F_i,G)| \leq 3(i = 1,2)$. We have the following claim.

Claim 3.1.1 There must exist two nonadjacent vertices $v_1 \in V(F_1)$ and $v_2 \in V(F_2)$ such that $d_G(v_1)+d_G(v_2) \leq 2(|V(F_1)| + |V(F_2)|) - 1$.

It is obvious that: (α) each vertex in F_i is a non-contacting-vertex, or a 1-contacting-vertex, or a 2-contacting-vertex, or a 3-contacting-vertex of $V(F_i)(i = 1,2)$; (β) if there is a non-contacting-vertex of $V(F_i)$ in F_i then this non-contacting-vertex is not adjacent to any vertex in $F_j(i,j = 1,2,i \neq j)$; (γ) if there is a 3-contacting-vertex of $V(F_i)$ in F_i then there must be a non-contacting-vertex of $V(F_i)$ in $F_i(i = 1,2)$ too. Because $|V(F_i)| \geq 2(i = 1,2)$, the vertices in $V(F_i)(i = 1,2)$ must be that one of them is a 1-contacting-vertex and the others are non-contacting-vertices; or one of them is a 2-contacting-vertex and the others are non-contacting-vertices; or one of them is a 3-contacting-vertex and the others are non-contacting-vertices; or two of them are 1-contacting-vertices and the others are non-contacting-vertices; or three of them are 1-contacting-vertices and the others are non-contacting-vertices; or one of them is a 1-contacting-vertex, another of them is a 2-contacting-vertex, and the others are non-contacting-vertices. Through an analysis we can get that among all non-adjacent vertices there must exist two nonadjacent vertices $v_1 \in V(F_1)$ and $v_2 \in V(F_2)$ such that they belong to one of the following cases: (A) one vertex is a non-contacting-vertex and the other is a non-contacting-vertex or a 1-contacting-vertex; (B) one vertex is a 1-contacting-vertex and the other is a 1-contacting-vertex or a 2-contacting-vertex. Anyway, we can always get that $d_G(v_1)+d_G(v_2) \leq 2|V(F_1)|+2|V(F_2)|-1=2(|V(F_1)|+|V(F_2)|)-1$ because for each $F \in \mathcal{R}$, if v is a non-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|-2$; if v is a 1-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|-1$; if v is a 2-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|$; and if v is a 3-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|+1$. Thus Claim 3.1.1 is obtained.

By Claim 3.1.1, $l \geq 3$, and the hypothesis of Theorem 3, we obtain a contradiction.

Thus Theorem 3.1 is proved. Furthermore, the graph G_5(Fig.2.) shows that the lower bound can not be reduced to $2n-4$. So the lower bound is best possible. (Although $d(u)+d(v) \geq 4= 2n-4$ for any two nonadjacent vertices u and v of the graph G_5 depicted by Fig.2, the graph is not up-embeddable.) □

Fig.2. the graph G_5

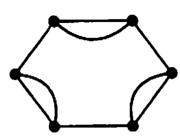


Fig.3. the graph G_6

From Theorem 3.1 we can easily get the following corollary.

Corollary 1 Let G be a connected semi-double graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{2n-3}{2} \rceil$ then G is up-embeddable.

Theorem 3.2 Let G be a 2-edge-connected semi-double graph of order n. For any two nonadjacent vertices u and v of G, if $d_G(u)+d_G(v) \geq \lceil \frac{4n-5}{3} \rceil$ then G is up-embeddable.

Proof Assume to the contrary that G is not up-embeddable. By Lemma 2.3, there exists $A \subseteq E(G)$ such that the properties (i)-(iv) of Lemma 2.3 are satisfied. Let $\mathcal{R} = \{F_1, F_2, \cdots, F_l\}$($l = c(G\backslash A) = b(G\backslash A) \geq 2$) be all the connected components of $G\backslash A$. It can be inferred from Lemma 2.3 that $l \geq 3$ and $|E(F_i, G)| \geq 2 (1 \leq i \leq l)$. Let x and y be the number of such $F_i \in \mathcal{R}$ that $|E(F_i, G)| = 2$ and 3 respectively. Counting the incidencies, it is obvious that $|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, G)| \geq x + \frac{3}{2}y + 2(l - x - y)$. Combining with Lemma 2.3(iv), we have

$$\begin{aligned} 2 &\leq \xi(G) = 2l - |A| - 1 \\ &\leq 2l - (x + \frac{3}{2}y + 2(l - x - y)) - 1, \end{aligned}$$

and so $x + y \geq 3$. By Lemma 2.3(i) we have $|V(F)| \geq 2$ for each $F \in \mathcal{R}$. Noticing that for each $F \in \mathcal{R}$, if v is a non-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| - 2$; if v is a 1-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| - 1$; if v is a 2-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|$; and if v is a 3-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| + 1$. The following two cases are considered.

Case 1 : $l = 3$.

Let F_1, F_2 and F_3 be the three components of $G \backslash A$. From Lemma 2.3 we can get that $|E_G(F_1, F_2, F_3)| = 3$ and $|E(F_i, G)| = 2$ $(i = 1, 2, 3)$. We have the following claim.

Claim 3.2.1 There must exist two nonadjacent vertices $v_i \in V(F_i)$ and $v_j \in V(F_j)$ $(1 \leq i, j \leq 3, i \neq j)$ such that $d_G(v_i)+d_G(v_j) \leq 2(|V(F_i)| + |V(F_j)|) - 2$.

It is obvious that each vertex in $F_i(i = 1, 2, 3)$ is a non-contacting-vertex, or a 1-contacting-vertex, or a 2-contacting-vertex. (I) If there exists one non-contacting-vertex v_i in one of F_i and F_j, say, F_i, then it can be deduced that there must exists a vertex $v_j \in V(F_j)$ which is not adjacent to v_i such that $d_G(v_i)+d_G(v_j) \leq 2|V(F_i)| - 2 + 2|V(F_j)|= 2(|V(F_i)| + |V(F_j)|) - 2$. (II) Neither F_i nor F_j has a non-contacting-vertex. Because $|E(F_i, G)| = |E(F_j, G)| = 2$, there must be that $|V(F_i)| = |V(F_j)| = 2$, and that all vertices in $V(F_i)$ and $V(F_j)$ are 1-contacting-vertices. Therefore there must exist two nonadjacent vertices $v_i \in V(F_i)$ and $v_j \in V(F_j)$ such that $d_G(v_i)+d_G(v_j) \leq 2|V(F_i)|-1+2|V(F_j)|-1= 2(|V(F_i)|+|V(F_j)|)-2$. By (I) and (II) Claim 3.2.1 is obtained.

From Claim 3.2.1 we have that $d_G(v_i)+d_G(v_j) \leq 2(|V(F_i)|+|V(F_j)|) - 2(1 \leq i, j \leq 3, i \neq j)$. It can be deduced that

$$\begin{aligned} &2(d_G(v_1) + d_G(v_2) + d_G(v_3)) \\ \leq\ &4(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 6 = 4n - 6. \end{aligned}$$

On the other hand, the condition of Theorem 3.2 requires that $d_G(u) + d_G(v) \geq \lceil\frac{4n-5}{3}\rceil$ for any two nonadjacent vertices u and v of G. So

$$d_G(v_i) + d_G(v_j) \geq \lceil\frac{4n-5}{3}\rceil(1 \leq i, j \leq 3, i \neq j),$$

leading to a contradiction:

$$4n - 5 \leq 2(d_G(v_1) + d_G(v_2) + d_G(v_3)) \leq 4n - 6.$$

Case 2 : $l \geq 4$.

For $x+y \geq 3$, without loss of generality, let F_1, F_2 and F_3 be any three components of $G \backslash A$ with the property that $2 \leq |E(F_i, G)| \leq 3 (i = 1, 2, 3)$. It is obvious that each vertex in $F_i (i = 1, 2, 3)$ is a non-contacting-vertex, or a 1-contacting-vertex, or a 2-contacting-vertex, or a 3-contacting-vertex. We have the following claim.

Claim 3.2.2 There must exist two nonadjacent vertices $v_i \in V(F_i)$ and $v_j \in V(F_j)$ such that $d_G(v_i)+d_G(v_j) \leq 2(|V(F_i)| + |V(F_j)|) - 1$ $(1 \leq i, j \leq 3, i \neq j)$.

(I) If there exists one non-contacting-vertex v_i in one of F_i and F_j, say, F_i, then it is not a hard work to find out that there is a vertex $v_j \in V(F_j)$ which is not adjacent to v_i such that $d_G(v_i)+d_G(v_j) \leq 2|V(F_i)| - 2 + 2|V(F_j)| + 1 = 2(|V(F_i)| + |V(F_j)|) - 1$.

(II) Neither F_i nor F_j has a non-contacting-vertex. Thus every vertex in F_i and F_j is either a 1-contacting-vertex or a 2-contacting-vertex because $2 \leq |E(F_i, G)| \leq 3$ and $2 \leq |E(F_j, G)| \leq 3$. Anyway, we can always find two nonadjacent $v_i \in V(F_i)$ and $v_j \in V(F_j)$ such that one of them is a 1-contacting-vertex and the other is a 1-contacting-vertex or a 2-contacting-vertex. Therefore we have $d_G(v_i)+d_G(v_j) \leq 2|V(F_i)| - 1 + 2|V(F_j)| = 2(|V(F_i)| + |V(F_j)|) - 1$. By (I) and (II) Claim 3.2.2 is obtained.

By Claim 3.2.2 we have that

$$d_G(v_i) + d_G(v_j) \leq 2(|V(F_i)| + |V(F_j)|) - 1 (1 \leq i, j \leq 3, i \neq j).$$

Because $l \geq 4$ and $|V(F)| \geq 2$ for each $F \in \mathcal{R}$, we have that

$$\begin{aligned} & 2(d_G(v_1) + d_G(v_2) + d_G(v_3)) \\ \leq \; & 4(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 3 \\ \leq \; & 4(n-2) - 3 = 4n - 11. \end{aligned}$$

On the other hand, from the condition required in Theorem 3.2 that $d_G(u) + d_G(v) \geq \lceil \frac{4n-5}{3} \rceil$ for any two nonadjacent vertices u and v of G we have that

$$d_G(v_i) + d_G(v_j) \geq \lceil \frac{4n-5}{3} \rceil (1 \leq i, j \leq 3, i \neq j),$$

leading to a contradiction:

$$4n - 5 \leq 2(d_G(v_1) + d_G(v_2) + d_G(v_3)) \leq 4n - 11.$$

Thus Theorem 3.2 is proved. Furthermore, the graph G_6 which depicted by Fig.3. shows that the lower bound can not be reduced to $\lceil \frac{4n-5}{3} \rceil - 1$. So the lower bound is best possible. (Although $d(u) + d(v) \geq 6 = \lceil \frac{4n-5}{3} \rceil - 1$

for any two nonadjacent vertices u and v of the graph G_6, the graph is not up-embeddable.) □

The following corollary can be obtained from Theorem 3.2 and the fact that $\delta(G) \geq k'(G)$ easily, where $k'(G)$ is the edge-connectivity of G.

Corollary 2 Let G be a 2-edge-connected semi-double graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{4n-5}{6} \rceil$ then G is up-embeddable. In addition, any 2-edge-connected semi-double graph with order $n \leq 4$ is up-embeddable.

Theorem 3.3 Let G be a 3-edge-connected semi-double graph of order n. For any two nonadjacent vertices u and v of G, if $d_G(u)+d_G(v)$ $\geq \lceil \frac{4n-27}{3} \rceil$ then G is up-embeddable.

Proof Assume to the contrary that G is not up-embeddable. By Lemma 2.3, there exists $A \subseteq E(G)$ such that the properties (i)-(iv) of Lemma 2.3 are satisfied. Let $\mathcal{R} = \{F_1, F_2, \cdots, F_l\}(l = c(G\backslash A) = b(G\backslash A)$ $\geq 2)$ be all the connected components of $G\backslash A$. It can be inferred from Lemma 2.3 that $l \geq 4$ and $|E(F_i, G)| \geq 3(1 \leq i \leq l)$. Let x be the number of such $F_i \in \mathcal{R}$ that $|E(F_i, G)| = 3$. Counting the incidencies, it is obvious that $|A| = \frac{1}{2} \sum\limits_{i=1}^{l} |E(F_i, G)| \geq \frac{3}{2}x + 2(l - x)$. Combining with Lemma 2.3(iv), we have that

$$\begin{aligned} 2 &\leq \xi(G) = 2l - |A| - 1 \\ &\leq 2l - (\frac{3}{2}x + 2(l - x)) - 1, \end{aligned}$$

and so $x \geq 6$. Without loss of generality, let F_1, F_2 and F_3 be any three components of $G\backslash A$ such that $|E(F_i, G)| = 3(i = 1, 2, 3)$. It is obvious that $|V(F_i)| \geq 2(i = 1, 2, 3)$, and that each vertex in F_i is a non-contacting-vertex, or a 1-contacting-vertex, or a 2-contacting-vertex, or a 3-contacting-vertex of $V(F_i)$. We have the following claim.

Claim 3.3.1 There must exist an independent set Ψ of G which is composed of three vertices $v_1 \in V(F_1)$, $v_2 \in V(F_2)$, and $v_3 \in V(F_3)$ such that $d_G(v_1)+d_G(v_2)+d_G(v_3) \leq 2(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 2$.

(I) In F_1, F_2 and F_3, at least two of them contain non-contacting-vertex. Obviously these non-contacting-vertex are pairwise nonadjacent. It is not hard to find out three pairwise nonadjacent vertices $v_1 \in V(F_1)$, $v_2 \in V(F_2)$, $v_3 \in V(F_3)$ such that $d_G(v_1)+d_G(v_2)+d_G(v_3) \leq 2|V(F_1)|+2|V(F_2)|$ $+ 2|V(F_3)|-2-2 < 2(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 2$.

(II) In F_1, F_2 and F_3, only one of them contains non-contacting-vertex. Without loss of generality, let this non-contacting-vertex be $v_1 \in F_1$. It is obvious that v_1 is not adjacent to any vertex in F_2 and F_3, and that every vertex in F_2 and F_3 is either a 1-contacting-vertex or a 2-contacting-vertex. It is not a hard work to find out three pairwise nonadjacent vertices

$v_1 \in V(F_1)$, $v_2 \in V(F_2)$, $v_3 \in V(F_3)$ such that $d_G(v_1)+d_G(v_2)+d_G(v_3) \leq 2|V(F_1)| - 2+2|V(F_2)|+2|V(F_3)| \leq 2(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 2$.

(III) There is no non-contacting-vertex in F_1, F_2 and F_3. So each vertex in F_1, F_2 and F_3 is either a 1-contacting-vertex or a 2-contacting-vertex. It is not hard to find out three pairwise nonadjacent vertices $v_1 \in V(F_1)$, $v_2 \in V(F_2)$ and $v_3 \in V(F_3)$ such that at least two of them are 1-contacting-vertex. So we have $d_G(v_1)+d_G(v_2)+d_G(v_3) \leq 2|V(F_1)|+2|V(F_2)|+2|V(F_3)| -1 - 1 = 2(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 2$.

From (I), (II), (III), Claim 3.3.1 is obtained.

By Claim 3.3.1, $l \geq x \geq 6$, and $|V(F_i)| \geq 2$ for each $F_i \in \mathcal{R}(i = 1, 2, 3)$, we have

$$\begin{aligned} & 2(d_G(v_1) + d_G(v_2) + d_G(v_3)) \\ \leq & \ 2(2(|V(F_1)| + |V(F_2)| + |V(F_3)|) - 2) \\ \leq & \ 4(n - 2 \times 3) - 4 = 4n - 28. \end{aligned}$$

Combining with the hypothesis of Theorem 3.3 we can get a contradiction: $4n - 27 \leq 2(d_G(v_1) + d_G(v_2) + d_G(v_3)) \leq 4n - 28$. Thus Theorem 3.3 is obtained. Furthermore, the graph G_7 depicted by Fig.4. shows that the lower bound can not ba reduced to $\lceil \frac{4n-27}{3} \rceil - 1$. So the lower bound is best possible. (Although $d(u) + d(v) \geq 6= \lceil \frac{4n-27}{3} \rceil - 1$ for any two nonadjacent vertices u and v of the graph G_7, the graph is not up-embeddable.) □

The following corollary can be obtained from Theorem 3.3 and the fact that $\delta(G) \geq k'(G)$ easily, where $k'(G)$ is the edge-connectivity of G.

Corollary 3 Let G be a 3-edge-connected semi-double graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{4n-27}{6} \rceil$ then G is up-embeddable. In addition, any 3-edge-connected semi-double graph with order $n \leq 11$ is up-embeddable.

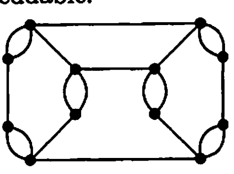


Fig.4. the graph G_7

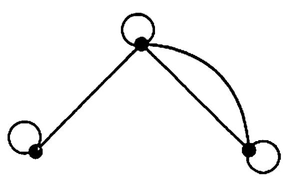


Fig.5. G_8: connected

As for single-petal graphs we have the following results.

Theorem 3.4 Let G be a connected single-petal graph of order n. For any two nonadjacent vertices u and v of G, if $d_G(u) + d_G(v) \geq 2n + 2$ then G is up-embeddable.

Theorem 3.5 Let G be a 2-edge-connected single-petal graph of order n. For any two nonadjacent vertices u and v of G, if $d_G(u) + d_G(v) \geq 2n+2$ then G is up-embeddable.

Theorem 3.6 Let G be a 3-edge-connected single-petal graph of order n. For any two nonadjacent vertices u and v of G, if $d_G(u) + d_G(v) \geq 2n-1$ then G is up-embeddable.

Proof (of Theorem 3.4, 3.5, 3.6) From a deduction similar to that of Theorem 3.1, 3.2, and 3.3 respectively, Theorems 3.4, 3.5, and 3.6 can be obtained noticing that if v is a non-contacting-vertex of $F_i(\in \mathcal{R})$ then $d_G(v) \leq 2|V(F_i)|$; if v is a 1-contacting-vertex then $d_G(v) \leq 2|V(F_i)| + 1$; if v is a 2-contacting-vertex then $d_G(v) \leq 2|V(F_i)| + 2$; and if v is a 3-contacting-vertex then $d_G(v) \leq 2|V(F_i)| + 3$. Furthermore, the graph G_8(Fig.5), G_9(Fig.6), and G_{10}(Fig.6) shows that the lower bound $2n+2, 2n+2$, and $2n-1$ can not be reduced to $2n+1, 2n+1$, and $2n-2$ respectively. □

G_9: 2-edge-connected G_{10}: 3-edge-connected

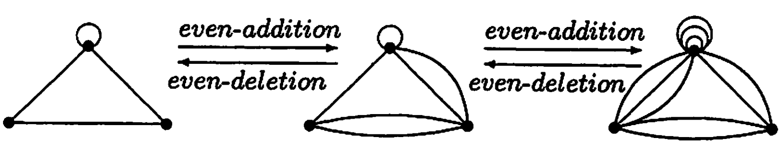
Fig.6.

The following corollary can be obtained easily.

Corollary 4 Let G be a connected(resp. 2-edge-connected, 3-edge-connected) single-petal graph of order n, if the minimum degree $\delta(G) \geq n+1$(resp. $n+1, \lceil \frac{2n-1}{2} \rceil$) then G is up-embeddable.

4. Conclusions

Let G be a simple graph, or a semi-double graph, or a single-petal graph. *Even-addition* on G is such an edge-adding operation on G which meets the following requirements: (i) the edges added to G may be links, multi-edges and loops; (ii) the number of edges added to G should be an even number; (iii) the subgraph induced by the edges added to G should be connected. The graph G^* obtained from G by a sequence of even-additions is called an *even-posterity* of G. For convenience, these definitions are illustrated by Fig.7, where the graph G_{11} is a single-petal graph, both G_{12} and G_{13} are even-posterities of G_{11}.



the graph G_{11} the graph G_{12} the graph G_{13}

Fig.7.

Theorem 4.1 Let G be a simple graph, or a semi-double graph, or a single-petal graph, and G^* be an even-posterity of G. If G is up-embeddable then G^* is up-embeddable.

Proof According to the definition of the even-posterity of G, the edges added to G each time are an even number of edges, and the subgraph induced by the edges added to G each time is a connected subgraph of G^*, so the deficiency of G^* is no more than that of G. By Lemma 2.1 we can get that if G is up-embeddable then G^* is up-embeddable. □

(The proof of Theorem 1.2) According to Lemma 2.1, if the non-simple graph G is up-embeddable, then there must exist a spanning tree T of G such that the deficiency $\xi(G)$ of G is at most one. Performing some times of even-deletions on G with respect to T, G', which is up-embeddable and an even-ancestry of G, would be obtained.

Conversely, if one of the even-ancestries G' of G is up-embeddable, then G is up-embeddable according to Theorem 4.1. □

Remark 1 Since k-vertex-connectivity implies k-edge-connectivity, the condition required in the theorems obtained in this paper that G is a k-edge-connected graph can be replaced by that G is a k-vertex-connected graph ($k = 1, 2, 3$).

Remark 2 Theorem 4.1 provides a sufficient condition but not a sufficient and necessary condition, *i.e.*, if the even-posterity G^* of G is up-embeddable, G may be not up-embeddable. For example, in Fig.8, the graph G_{15}, which is an even-posterity of G_{14}, is up-embeddable, but the graph G_{14} is not up-embeddable.

Remark 3 According to Theorem 1.2, we can study the up-embeddability of non-simple graphs through that of simple graphs, or semi-double graphs, or single-petal graphs. In our another paper [1], we have studied the up-embeddability of semi-double graphs and single-petal graphs via the degree-sum of adjacent vertices. Whether there are other ways to study the up-embeddability of semi-double graphs and single-petal graphs is a question we are interested in.

the graph G_{14} the graph G_{15}

Fig.8.

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References

[1] G. Dong, Y. Liu, Up-embeddable graphs via the degree-sum of adjacent vertices, accepted to Ars Combin..

[2] J.L. Gross, T.W. Tucker, Topological Graph Theory. Wiley-Interscience, New York, 1987.

[3] Y. Huang, Y. Liu, Face size and the maximum genus of a graph, J Combin. Theory Ser B. 80 (2000) 356–370.

[4] Y. Huang, Y. Liu, The degree sum of nonadjacent vertices and up-embeddability of graphs, Chin Ann of Math. 5 (19A) (1998) 651–656 (in Chinese).

[5] Y. Huang, Maximum genus and chromatic number of graphs, Discrete Math. 271 (2003) 117–127.

[6] F. Jaeger, C. Payan, N.H. Xuong, A classes of upper embeddable graphs, J. Graph Theory. 3 (1979) 387–391.

[7] M. Jungerment, A characterization of upper embeddable graphs, Trans. Math. Soc. 241 (1979) 401–406.

[8] S. Kundu, Bounds on number of disjoint spanning trees, J. Combin. Theory Ser. B. 17 (1974) 199–203.

[9] Y. Liu, Embeddability in Graphs. Kluwer Academic, Dordrecht, Boston, London, 1995. 255–265.

[10] Y. Liu, The maximum orientable genus of a graph. *Scientia Sinical (Special Issue)*, (II): 41–55 (1979)

[11] L. Nebeský, A new characterization of the maximum genus of a graph. Czechoslova Math. J. 31(106) (1981) 604–613.

[12] E.A. Nordhause, B.M. Stewart, A.T. White, On the maximum genus of a graph, J. Combin. Theory. 11 (1971) 258–267.

[13] R. Ringeisen, The maximum genus of a graph, Ph. D. Thesis. Michigan State Univ., 1970.

[14] R. Ringeisen, Upper and lower embeddable graphs, Graph Theory and Applications. Lecture Notes in Math., Springer-Verlag, Berlin and New York, 303 (1972).

[15] R. Ringeisen, Determining all compact orientable 2-manifolds upon which $K_{m,n}$ has 2-cell imbeddings, J. Combin. Theory Ser. B. 12 (1974) 101–104.

[16] M. Škoviera, The maximum genus of graphs diameter two, Discrete Math. 87 (1991) 175–180.

[17] D.B. West, Introduction to Graph Theory (Second Edition), China Machine Press, Beijing, 2004.

[18] A.T. White, Graphs, Groups, and Surfaces. Amsterdam, North-Holland, 1984.

[19] N.H. Xuong, How to determine the maximum genus of a graph, J. Combin. Theory Ser. B. 26 (1979) 217–225.