On the Erdős-Sós Conjecture for Graphs on $n = k + 3$ Vertices

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November 17, 2009

Abstract

Erdős and Sós conjectured in 1962 that if the average degree of a graph G exceeds $k-2$, then G contains every tree on k vertices. Results from Sauer and Spencer (and independent results from Zhou) prove the special case where G has k vertices. Results from Slater, Teo and Yap prove the case where G has $k+1$ vertices. In 1996, Woźniak proved the case where G has $k+2$ vertices. We prove the conjecture for the case where G has $k+3$ vertices.

1 Terminology

We will use standard graph theory notation. We consider simple graphs (finite and undirected with no loops and no multi-edges). $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. G has order n and size $e(G)$.

Let $u, v \in V(G)$ be any two vertices in G. The degree of v, the number of edges incident to v, is denoted $deg_G(v)$. The set of neighbors of v is denoted $N(v)$, that is, $N(v) = \{w \in V(G) | vw \in E(G)\}$. Note that $deg_G(v) = |N(v)|$. If u and v are adjacent (i.e., $uv \in E(G)\}$), we say that u hits v or v hits u. If u and v are not adjacent, we say that u misses v or v misses u.

The minimum degree among all vertices in $V(G)$ is denoted $\delta(G)$. The maximum degree among all vertices in $V(G)$ is denoted $\Delta(G)$. For any graph H, we define the function $\bar{d}(H)$ to be the average degree of the vertices in H; that is $\bar{d}(H) = 2e(H)/|V(H)|$.

A leaf $w \in V(T)$ in a tree T is a vertex of degree one. If v has w (a leaf) as its neighbor, then w is referred to as a leaf neighbor of v. The set of leaf neighbors of v is denoted $L_G(v)$ (or simply $L(v)$), that is, $L(v) = \{w \in N(v) | w \text{ is a leaf}\}$.

We denote by e_{min}, the minimum number of edges in a $(k+3)$-vertex graph G that ensures that $\bar{d}(G) > k-2$. That is, $e_{min} = 1 + \lfloor \frac{1}{2}(k-2)(k+3) \rfloor$.

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Let T be a tree on k vertices and let $g : V(T) \to V(G)$ be an isomorphism from $V(T)$ to a k-subset of $V(G)$. If g preserves edges, that is, if $g(u)g(v) \in E(G)$ for every $uv \in E(T)$, then we call g an embedding of T into G. If such an embedding exists, then G contains a copy of T as a subgraph. Or, we say G contains T or simply $T \subseteq G$.

Let $T' \subseteq T$ be a proper subtree of T and let g' be an embedding of T' into G. If there exists an embedding $g : V(T) \to V(G)$ such that $g(v) = g'(v)$ for all $v \in V(T')$, we say that g' can be extended to an embedding of T or simply, that g' is T-exensible. Given any subtree $T' \subseteq T$ and any embedding $g' : V(T') \to V(G)$, to prove $T \subseteq G$, it suffices to prove that g' is T-extensible.

As an example, suppose T is a tree on k vertices that has a vertex p with 3 leaf neighbors; say $L(p) = \{l_1, l_2, l_3\}$. If f' is an embedding of $T - \{l_1, l_2, l_3\}$ into G and if $deg_G(f'(p)) \geq k-1$ (so $f'(p)$ hits at least 3 vertices in $V(G) - g'(V(T'))$), then clearly, f' is T-extensible (just map each of $\{l_1, l_2, l_3\}$ to a distinct vertex in $V(G) - g'(V(T'))$).

2 Introduction and Main Theorem

In 1959, Erdős and Gallai [2] proved the following for a fixed, positive integer k and for a graph G:

Theorem 1 *If $\bar{d}(G) > k - 2$, then G contains a path on k vertices.*

In 1962, Erdős and Sós stated the following conjecture:

Conjecture 1 *If $\bar{d}(G) > k - 2$, then G contains every tree on k vertices as a subgraph.*

Various specific cases of Conjecture 1 have already been proven. Each places limitations on the graph G or on the tree T. Woźniak [7] notes that the case where G has k vertices follows immediately from a theorem of Sauer and Spencer [5]. In 1984, B. Zhou [8] independently proved the case where G has k vertices. The case where G has $k+1$ vertices follows from a theorem on the packing of graphs by Slater, Teo and Yap [4] in 1985. In 1996, Woźniak [6] proved the following which we state as a theorem.

Theorem 2 *If a graph G has $n = k + 2$ vertices and $\bar{d}(G) > k - 2$, then G contains every tree on k vertices.*

If a longest path in a tree T has $r + 1$ vertices, then we say that T has diameter r. In 2003, McLennan [3] proved the following:

Theorem 3 *If $\bar{d}(G) > k - 2$, then every tree of order k whose diameter does not exceed 4 is contained in G as a subgraph.*

Eaton and the author of this paper [1] proved the following which we state as two theorems. They will be used in the proof of the main result (i.e., Theorem 6).

Theorem 4 *If $\bar{d}(G) > k-2$ and $\delta(G) \geq k-4$, then G contains every tree on k vertices.*

Theorem 5 *If $k \leq 8$ and $\bar{d}(G) > k-2$, then G contains every tree on k vertices.*

In this paper, we establish the following:

Theorem 6 *If a graph G has $n = k+3$ vertices and $\bar{d}(G) > k-2$, then G contains every tree on k vertices.*

3 Proof of Theorem 6

Assume that G is a connected graph (otherwise, we just apply these results to any component of G whose average degree exceeds $k-2$). We will use induction on $n = |V(G)|$. Assume the theorem holds for all graphs on fewer than n vertices and let G be a graph on n vertices. We will prove that the theorem holds for any graph on n vertices. Let $n = k+3$ and let T be a tree on k vertices.

Theorem 6 holds if $k \leq 8$, $diameter(T) \leq 4$ or $\delta(G) \geq k-4$ (by Theorems 5, 3 and 4, respectively). So, assume $k \geq 9$, $diameter(T) \geq 5$ and $\delta(G) \leq k-5$.

Assume $e(G) = e_{min} = 1 + \lfloor \frac{1}{2}(k-2)(k+3) \rfloor$. Notice if $deg_G(v) < \lfloor \frac{k}{2} \rfloor$ for some $v \in V(G)$, then $\bar{d}(G-v) > k-2$ and $G - v \subseteq G$ contains every tree on k vertices (by Theorem 2). So we assume that $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$.

Let $a_0, a_1, \ldots, a_r$ be a longest path in T (since $diameter(T) > 4$, we have $r \geq 5$). Denote the leaf neighbors of a_1 has $L_T(a_1) = \{b_1, b_2, \ldots, b_s\}$ where $s \geq 1$. Denote the leaf neighbors of a_{r-1} has $L_T(a_{r-1}) = \{c_1, c_2, \ldots, c_t\}$ where $t \geq 1$. Without loss of generality, we will assume that $s \geq t$.

We break the proof down into several cases and subcases. The cases depend on the value of $\Delta(G)$; notice $k-1 \leq \Delta(G) \leq k+2 = n-1$.

In each case, we construct subgraphs $T' \subseteq T$ and $G' \subseteq G$ that satisfy the induction assumption or that satisfy Theorem 2 (in either case, we obtain that $T' \subseteq G'$). That is, for $n' = |V(G')|$ $(< n)$ and $k' = n' - 3$, we have that T' and G' satisfy $|V(T')| \leq k'$ and $\bar{d}(G') > k' - 2$.

Since an embedding f' of T' into G exists, we will extend/modify f' to a new embedding f of T into G proving that $T \subseteq G$.

Let $z \in V(G)$ be such that $deg_G(z) = \delta(G) \leq k-5$.

Case 1 $\Delta(G) = k+2$.

Let $u \in V(G)$ be such that $deg_G(u) = k+2$. Let $G' = G - \{u, z\}$ and let $T' = T - \{a_1, b_1, \ldots, b_s\}$. So $e(G') \geq e(G) - (k+2) - (k-6) > \frac{1}{2}(k-2)(k+3) - 2k + 4 = \frac{1}{2}(k-2)(k-1)$, $\bar{d}(G') > (k-2)(k-1)/(k+1) > k-4$ and $|V(T')| \leq k-2$. By the induction assumption, $T' \subseteq G'$.

Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$. Set $f(a_1) = u$. Since $N_G(u) = V(G-u)$, u hits at least s vertices in X so f is T-extensible.

Case 2 $\Delta(G) = k+1$.

Let $u \in V(G)$ be such that $deg_G(u) = k+1$ and let x be the single vertex in $V(G) - \{u\}$ that is not adjacent to u. We consider two subcases: $deg_G(x) \leq k-2$ and $deg_G(x) \geq k-1$.

Case 2.1 $deg_G(x) \leq k-2$.

Let $G' = G - \{u, x\}$ and let $T' = T - \{a_1, b_1, \ldots, b_s\}$. Thus, $e(G') = e(G) - (k+1) - (k-2) > \frac{1}{2}(k-2)(k+3) - 2k + 1 = \frac{1}{2}(k-4)(k+1)$, $\bar{d}(G') > (k-4)(k+1)/(k+1) = k-4$ and $|V(T')| \leq k-2$; so $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Observe that u hits all of $f'(V(T'))$. So, set $f(a_1) = u$ and f is T-extensible.

Case 2.2 $deg_G(x) \geq k-1$.

Let $G' = G - \{u, x, z\}$ and let $T' = T - \{a_1, b_1, \ldots, b_s, a_r\}$. Thus, $e(G') \geq e(G) - (k+1) - (k+1) - (k-5) > \frac{1}{2}(k-2)(k+3) - 3k + 3 = \frac{1}{2}k(k-5)$, $\bar{d}(G') > k(k-5)/k = k-5$ and $|V(T')| = k-3$; so $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Observe that u hits every vertex in G' and x hits all but at most two.

If x hits $f'(a_2)$, then set $f(a_1) = x$, $f(a_r) = u$ and f is T-extensible. So, assume x misses $f'(a_2)$.

Suppose x hits $f'(a_{r-1})$. Set $f(a_r) = x$, $f(a_1) = u$ and f it T-extensible.

Suppose x misses $f'(a_{r-1})$. Then x hits all of $V(G') - \{f'(a_2), f'(a_{r-1})\}$. Set $f(a_{r-1}) = x$ and $f(a_1) = u$. Since $deg_G(x) \geq k-1$, x hits at least t vertices in X. And since $deg_G(u) = k+1$, f is T-extensible.

Case 3 $\Delta(G) = k$.

Let $u \in V(G)$ be such that $deg_G(u) = k$ and let x_1 and x_2 be the two vertices in $V(G-u)$ that miss u. Without loss of generality, assume $deg_G(x_1) \geq deg_G(x_2)$. We consider two subcases: $deg_T(a_1) > 2$ and $deg_T(a_1) = 2$.

Case 3.1 $deg_T(a_1) > 2$, *(i.e., $s \geq 2$)*.

Case 3.1.1 *x_1 misses x_2.*

Let $G' = G - \{u\}$ and $T' = T - \{a_0\}$. Then, $e(G') = e(G) - k > \frac{1}{2}(k-2)(k+3)-k = \frac{1}{2}(k-3)(k+2)$. Since $\bar{d}(G') > (k-3)(k+2)/(k+2) = k-3$ and $|V(T')| = k-1$, we know that $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Since $f'(a_1)f'(a_2) \in E(G')$, we know u hits at least one of $f'(a_1)$, $f'(a_2)$ (the only two vertices that u misses are x_1 and x_2 and $x_1x_2 \notin E(G')$). If u hits $f'(a_1)$, then set $f(a_0) = u$ and f is an embedding of T into G. So, assume u misses $f'(a_1)$.

Since u misses $f'(a_1)$, it must be that u hits all of $N_G(f'(a_1))$. Set $f(a_1) = u$. Since $deg_G(u) = k$, f is T-extensible.

Case 3.1.2 *x_1 hits x_2.*

Case 3.1.2.1 *$deg_G(x_1) + deg_G(x_2) \leq 2k - 2$.*

Let $G' = G - \{u, x_1, x_2\}$ and let $T' = T - \{a_1, b_1, b_2, ..., b_s\}$. So, $e(G') \geq e(G) - k - (2k-3) > \frac{1}{2}(k-2)(k+3) - 3k + 3 = \frac{1}{2}k(k-5)$, $\bar{d}(G') > k(k-5)/k = k-5$ and $|V(T')| \leq k-3$. Therefore $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Since u misses only x_1 and x_2, set $f(a_1) = u$ and f is T-extensible.

Case 3.1.2.2 *$deg_G(x_1) + deg_G(x_2) \geq 2k - 1$.*

It must be that $2k - 1 \leq deg_G(x_1) + deg_G(x_2) \leq 2k$, $deg_G(x_1) = k$ and $k-1 \leq deg_G(x_2) = k$ (since $deg_G(x_1) \geq deg_G(x_2)$. And, notice that $x_1 \neq z \neq x_2$ (since $deg_G(z) \leq k-5$) and that u hits z.

Let $G' = G - \{u, z, x_1, x_2\}$ and let $T' = T - \{a_1, b_1, \ldots, b_s, a_r\}$. So, $e(G') \geq e(G)-k-(2k-1)-(k-6) > \frac{1}{2}(k-2)(k+3)-4k+7 = \frac{1}{2}(k^2-7k+8)$. Since $\bar{d}(G') > (k^2-7k+8)/(k-1) > k-6$ and $|V(T')| \leq k-4$, we have that $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Notice that u hits all of $V(G')$ and x_1 misses at most one vertex in $V(G')$. In particular, x_1 hits either $f'(a_{r-1})$ or $f'(a_{r-2})$.

If x_1 hits $f'(a_{r-1})$, then set $f(a_r) = x_1$. If x_1 misses $f'(a_{r-1})$, then x_1 hits all of $G' - f'(a_{r-1})$ so set $f(a_{r-1}) = x_1$ and $f(a_r) = x_2$.

Set $f(a_1) = u$ and f is T-extensible.

Case 3.2 *$deg_T(a_1) = 2$, (i.e., $s = 1$).*

Assume first that there is a $w \in V(G) - \{u, x_1, x_2\}$ such that w hits both x_1 and x_2. Let $G' = G - \{u, w\}$ and let $T' = T - \{a_0, a_1\}$.

Then, $e(G') \geq e(G)-k-(k-1) > \frac{1}{2}(k-2)(k+3)-2k+1 = \frac{1}{2}(k-4)(k+1)$, $\bar{d}(G') > (k-4)(k+1)/(k+1) = k-4$ and $|V(T')| = k-2$; so $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Note that u hits w and that at least one of them hits $f'(a_2)$; without loss of generality, assume w hits $f'(a_2)$. Set $f(a_1) = w$, $f(a_0) = u$ and f is an embedding of T into G.

Now, assume there is no such $w \in V(G) - \{u, x_1, x_2\}$ that hits both x_1 and x_2. Then, each vertex in $V(G) - \{u, x_1, x_2\}$ sends at most one edge to $\{x_1, x_2\}$. Thus, $deg_G(x_1) + deg_G(x_2) \leq k+2$.

Let $G' = G - \{u, x_1, x_2\}$ and $T' = T - \{a_0, a_1\}$. Then, $e(G') \geq e(G) - k - (k+1) > \frac{1}{2}(k-2)(k+3) - 2k - 1 = \frac{1}{2}(k^2 - 3k - 8)$. Since $\bar{d}(G') > (k^2-3k-8)/k \geq k-4$ and $|V(T')| \leq k-2$, $T' \subseteq G'$. Let f' be an embedding of T' into G', let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Since u hits all of $V(G')$ (and at least one additional vertex), set $f(a_2) = u$ and f is T-extensible.

Case 4 $\Delta(G) = k-1$.

Let $\{u_1, u_2, u_3, u_4\} \subset V(G)$ be four vertices of degree $k-1$. Such four vertices exist as otherwise,
$e(G) \leq \lfloor \frac{1}{2}[(k-5) + 3(k-1) + (k-1)(k-2)] \rfloor = \lfloor \frac{1}{2}(k^2 + k - 6) \rfloor < e_{min}$; a contradiction.

Each one of $\{u_1, u_2, u_3, u_4\}$ misses exactly 3 vertices in $V(G)$. Let $\{x_1, x_2, x_3\} \subset V(G)$ be the three vertices in $V(G) - \{u_1\}$ that miss u_1.

Case 4.1 *$\{x_1, x_2, x_3\}$ share at most one edge.*

Without loss of generality, assume x_1 hits x_2. Let $G' = (G - u_1) - x_1x_2$ (so $\{x_1, x_2, x_3\}$ share no edges in G'). Then, $e(G') \geq e(G) - (k-1) - 1 > \frac{1}{2}(k-2)(k+3) - k = \frac{1}{2}(k-3)(k+2)$. So $\bar{d}(G') > (k-3)(k+2)/(k+2) = k-3$. Let $T' = T - a_0$. Since $|V(T')| \leq k-1$, $T' \subseteq G'$. Let f' be an embedding of T' into G and let $X = V(G) - f'(V(T'))$.

If u_1 hits $f'(a_1)$, set $f(a_0) = u_1$ and f is an embedding of T into G. So, assume u_1 misses $f'(a_1)$.

It must be that u_1 hits all of $N_G(f'(a_1))$ so set $f(a_1) = u_1$. Since u_1 hits $k-1$ vertices in G (and therefore, at least 1 vertex in X), f is T-extensible.

Case 4.2 *$\{x_1, x_2, x_3\}$ share at least two edges.*

Case 4.2.1 *$deg_T(a_1) = 2$, (i.e., $s = 1$).*

Without loss of generality, assume x_1 hits both of $\{x_2, x_3\}$. Let $G' = (G - \{u_1, x_1\}) - x_2x_3$. Then, $e(G') \geq e(G) - (k-1) - (k-1) - 1 > \frac{1}{2}(k-2)(k+3) - 2k + 1 = \frac{1}{2}(k-4)(k+1)$. So $\bar{d}(G') > (k-4)(k+1)/(k+1) = k-4$. Let

$T' = T - \{a_0, a_1\}$. Since $|V(T')| = k-2$, $T' \subseteq G'$. Let f' be an embedding of T' into G' and let $f = f'$.

Suppose u_1 hits $f'(a_2)$. Set $f(a_1) = u_1$. Since u_2 hits $k-1$ vertices in G (and therefore, at least 1 vertex in $V(G) - f'(V(T'))$), f is T-extensible.

Suppose u_1 misses $f'(a_2)$. Then $f'(a_2) = x_2$ or x_3; without loss of generality, assume $f'(a_2) = x_2$. Set $f(a_1) = x_1$. If $x_3 \notin f'(V(T'))$, then set $f(a_0) = x_3$ and f is an embedding of T into G. So, assume $x_3 \in f'(V(T'))$; and let $w \in V(T')$ be such that $x_3 = f'(w)$. Note that $f'(w) = x_3$ does not hit $f'(a_2) = x_2$ so it must be that u_1 hits all of $f'(N_T(f'(w)))$. Set $f(w) = u_1$, $f(a_0) = x_3$ and f is an embedding of T into G.

Case 4.2.2 *$deg_T(a_1) > 2$, (i.e., $s \geq 2$).*

If $\{u_1, u_2, u_3, u_4\}$ share no edges, then $\{x_1, x_2, x_3\}(= \{u_2, u_3, u_4\})$ share no edges and $T \subseteq G$ by Case 4.1. So, assume $\{u_1, u_2, u_3, u_4\}$ share at least one edge; without loss of generality, assume u_1 hits u_2.

Let $G' = (G - \{u_1, u_2, z\}) - \{x_1x_2, x_1x_3, x_2x_3\}$. Then, $e(G') \geq e(G) - (k-1) - (k-2) - (k-5) - 3 > \frac{1}{2}(k-2)(k+3) - 3k + 5 = \frac{1}{2}(k-4)(k-1)$. So $\bar{d}(G') > (k-4)(k-1)/k > k-5$. Let $T' = T - \{a_1, b_1, b_2, \ldots, b_s\}$. Since $|V(T')| \leq k-3$, $T' \subseteq G'$. Let f' be an embedding of T' into G'. Let $X = V(G) - f'(V(T'))$ and let $f = f'$.

Suppose u_1 hits $f'(a_2)$. Set $f(a_1) = u_1$. Since u_1 hits $k-1$ vertices in G, f is T-extensible.

Suppose u_1 misses $f'(a_2)$. Then u_1 hits all of $N(f'(a_2))$. So, set $f(a_2) = u_1$ and $f(a_1) = u_2$. Since u_2 hits $k-1$ vertices in G, f is T-extensible. □

4 Acknowledgments

I am grateful to Nancy Eaton for enlightening me to the Erdős-Sós Conjecture. I am also grateful to Mariusz Woźniak for writing [6] which was helpful in my approach to this paper.

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