# ON THE GLOBAL ASYMPTOTIC BEHAVIOR OF A SYSTEM OF TWO NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT

In this paper a sufficient condition is obtained for the global asmptotic stability of the following system of difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} + 1}, \; y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} + 1}, \quad n = 0, 1, 2, \ldots,$$

where the initial values $(x_k, y_k) \epsilon (0, \infty)$ (for $k = -1, 0$).

**Keywords**: Rational difference equation; System; Global asymptotic stability; Semicycle; Equilibrium point

## 1. INTRODUCTION

Difference equations are always attracting very much interest, because these equations appear in the mathematical models of some problems in biology, ecology, economy and physics, and numerical solutions of differential equations. So, recently there has been an increasing interest in the study of qualitative analyses of rational difference equations and systems of difference equations.Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the global behavior of their solutions. (see [1-13] and the references cited therein).

In [3] De Vault et al. proved that the unique equilibrium of the difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where $A \epsilon (0, \infty)$, is globally asymptotically stable and proved the oscillatory behaviour of the positive solutions of this difference equation.

From on, Papaschinopoluos and Schinas [8] extended the results obtained in [3] to the following system of difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-p}}, \; y_{n+1} = A + \frac{x_n}{x_{n-q}}, \quad n = 0, 1, 2, \ldots,$$

where $A \epsilon (0, \infty)$, $p, q$ are positive integers and

$$x_{-q}, x_{-q+1}, \ldots x_0, y_{-p}, y_{-p+1}, \ldots y_0$$

are positive initial values.

Li and Zhu [5] proved that the unique positive equilibrium of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + a}{x_n + x_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where $a \epsilon [0, \infty)$ and $x_{-1}, x_0$ are positive, is globally asymptotically stable.

From on, we [1] extended the results obtained in [5] to the following rational difference equation

$$x_{n+1} = \frac{x_n x_{n-k} + a}{x_n + x_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where $k$ is nonnegative integer, $a \epsilon [0, \infty)$ and $x_{-k}, \ldots, x_0$ are positive, is globally asymptotically stable.

Moreover, in [12] we extended the results obtained in [5] to the following system of difference equations

$$z_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \; t_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where $a \epsilon (0, \infty)$ and the initial values $(z_k, t_k) \epsilon (0, \infty)$ (for $k = -1, 0$), is globally asymptotically stable.

Also, in [13] we proved that unique positive equilibrium of the system of difference equations

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \; t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where $a \epsilon (0, \infty)$ and the initial values $(z_k, t_k) \epsilon (0, \infty)$ (for $k = -1, 0$), is globally asymptotically stable.

In this paper, we consider the following system of difference equations

$$(1) \qquad x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} + 1}, \; y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} + 1}, \quad n = 0, 1, 2, \ldots,$$

where the initial values $(x_k, y_k) \epsilon (0, \infty)$ (for $k = -1, 0$). Our main aim is to investigate the global asymptotic behavior of its solutions.

We need the following definitions and theorem [4]:

Let $I$ be some interval of real numbers and let $f, g : I \times I \to I$ be continuously differentiable functions. Then for all initial values $(x_k, y_k) \epsilon I$, $k = -1, 0$, the system of difference equations

$$(2) \qquad x_{n+1} = f(x_n, y_{n-1}),\ y_{n+1} = g(y_n, x_{n-1}), \quad n = 0, 1, 2, \ldots$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$.

**Definition 1.** *A point $(\overline{x}, \overline{y})$ called an equilibrium point of the system (2) if*

$$\overline{x} = f(\overline{x}, \overline{y}) \text{ and } \overline{y} = g(\overline{x}, \overline{y}).$$

**Definition 2.** *Let $(\overline{x}, \overline{y})$ be an equilibrium point of the system (2).*

*(a) An equilibrium point $(\overline{x}, \overline{y})$ is said to be stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for every initial points $(x_{-1}, y_{-1})$ and $(x_0, y_0)$ for which $\|(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})\| + \|(x_0, y_0) - (\overline{x}, \overline{y})\| < \delta$, the iterates $(x_n, y_n)$ of $(x_{-1}, y_{-1})$ and $(x_0, y_0)$ satisfy $\|(x_n, y_n) - (\overline{x}, \overline{y})\| < \varepsilon$ for all $n > 0$. An equilibrium point $(\overline{x}, \overline{y})$ is said to be unstable if it is not stable. (By $\|.\|$ we denote the Euclidean norm in $\mathbb{R}^2$ given by $\|(x, y)\| = \sqrt{x^2 + y^2}$)*

*(b) An equilibriumpoint $(\overline{x}, \overline{y})$ is said to be asymptotically stable if there exists $r > 0$ such that $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ for all $(x_{-1}, y_{-1})$ and $(x_0, y_0)$ that satisfy $\|(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})\| + \|(x_0, y_0) - (\overline{x}, \overline{y})\| < r$.*

**Definition 3.** *Let $(\overline{x}, \overline{y})$ be an equilibrium point of a map $F = (f, g)$, where $f$ and $g$ are continuously differentiable functions at $(\overline{x}, \overline{y})$. The Jacobian matrix of $F$ at $(\overline{x}, \overline{y})$ is the matrix*

$$J_F(\overline{x}, \overline{y}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{bmatrix}.$$

*The linear map $J_F(\overline{x}, \overline{y}) : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$(3) \qquad J_F(\overline{x}, \overline{y}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y})x + \frac{\partial f}{\partial y}(\overline{x}, \overline{y})y \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y})x + \frac{\partial g}{\partial y}(\overline{x}, \overline{y})y \end{bmatrix}$$

*is called the linearization of the map $F$ at $(\overline{x}, \overline{y})$.*

**Theorem 1.** *(Linearized Stability Theorem)*

*Let $F = (f, g)$ be a continuously differentiable function defined on an open set $I$ in $\mathbb{R}^2$, and let $(\overline{x}, \overline{y})$ in $I$ be an equilibrium point of the map $F = (f, g)$.*

*(a) If all the eigenvalues of the Jacobian matrix $J_F(\overline{x}, \overline{y})$ have modulus less than one, then the equilibrium point $(\overline{x}, \overline{y})$ is asymptotically stable.*

*(b) If at least one of the eigenvalues of the Jacobian matrix $J_F(\overline{x}, \overline{y})$ has modulus greater than one, then the equilibrium point $(\overline{x}, \overline{y})$ is unstable.*

*(c) An equilibrium point $(\overline{x}, \overline{y})$ of the map $F = (f, g)$ is locally asymptotically stable if and only if every solution of the characteristic equation*

$$\lambda^2 - trJ_F(\overline{x}, \overline{y})\lambda + \det J_F(\overline{x}, \overline{y}) = 0$$

*lies inside the unit circle, that is, if and only if*

$$|trJ_F(\overline{x},\overline{y})| < 1 + \det J_F(\overline{x},\overline{y}) < 2.$$

**Definition 4.** *Let $(\overline{x},\overline{y})$ be positive equilibrium point of the system (2) (see [9]).*

*A "string" of consecutive terms $\{x_s,...,x_m\}$ (resp. $\{y_s,...,y_m\}$), $s \geq -1, m \leq \infty$ is said to be a positive semicycle if $x_i \geqslant \overline{x}$ (resp. $y_i \geqslant \overline{y}$), $i\epsilon\{s,...,m\}, x_{s-1} < \overline{x}$ (resp. $y_{s-1} < \overline{y}$), and $x_{m+1} < \overline{x}$ (resp. $y_{m+1} < \overline{y}$).*

*A "string" of consecutive terms $\{x_s,...,x_m\}$ (resp. $\{y_s,...,y_m\}$), $s \geq -1, m \leq \infty$ is said to be a negative semicycle if $x_i < \overline{x}$ (resp. $y_i < \overline{y}$), $i\epsilon\{s,...,m\}, x_{s-1} \geqslant \overline{x}$ (resp. $y_{s-1} \geqslant \overline{y}$), and $x_{m+1} \geqslant \overline{x}$ (resp. $y_{m+1} \geqslant \overline{y}$).*

*A "string" of consecutive terms $\{(x_s,y_s),...,(x_m,y_m)\}$ is said to be a positive (resp. negative) semicycle if $\{x_s,...,x_m\}$, $\{y_s,...,y_m\}$ are positive (resp. negative) semicycles.*

*Finally a "string" of consecutive terms $\{(x_s,y_s),...,(x_m,y_m)\}$ is said to be a semicycle positive (resp. negative) with respect to $x_n$ and negative (resp. positive) with respect to $y_n$ if $\{x_s,...,x_m\}$ is a positive (resp. negative) semicycle and $\{y_s,...,y_m\}$ is a negative (resp. positive) semicycle.*

## 2. SOME AUXILIARY RESULTS

In this section, we give the following lemmas which show us the behaviour of semicycles of positive solutions of system (1). Proofs of Lemmas 1 and 2 are clear from (1). So, they will be omitted.

**Lemma 1.** *A positive solution $\{(x_n,y_n)\}_{n=-1}^{\infty}$ of the system (1) is eventually equal to (1,1) if and only if*

$$(x_{-1}-1)(x_0-1)(y_{-1}-1)(y_0-1) = 0.$$

**Lemma 2.** *Assume that $\{(x_n,y_n)\}_{n=-1}^{\infty}$ is a positive solution of the system (1) which is not eventually equal to (1,1). Then the following statements are true:*

*(i) $(x_{n+1}-x_n)(1-x_n) > 0$ and $(y_{n+1}-y_n)(1-y_n) > 0$ for all $n \geq 0$,*

*(ii) $(x_{n+1}-1)(x_n-1)(1-y_{n-1}) > 0$ and $(y_{n+1}-1)(y_n-1)(1-x_{n-1}) > 0$ for all $n \geq 0$.*

**Lemma 3.** *Assume that $\{(x_n,y_n)\}_{n=-1}^{\infty}$ is a positive solution of the system (1) and suppose that the case, Case1: $x_k, y_k < 1$ (for $k = -1, 0$), holds. Then $(x_n,y_n)$ is a negative semicycle of system (1) with an infinite number of terms and it monotonically tends to the positive equilibrium $(\overline{x},\overline{y}) = (1,1)$.*

*Proof.* If $x_k, y_k < 1$ (for $k = -1, 0$), then by Lemma 2 (ii), it follows that $x_n, y_n < 1$ for $n \geq -1$, i.e., this negative semicycle has an infinite number of terms. Furthermore, according to Lemma 2 (i), we know that $(x_n, y_n)$ is strictly increasing for all $n \geq 0$. So, the limits

$$\lim_{n\to\infty} x_n = l_1 \text{ and } \lim_{n\to\infty} y_n = l_2$$

exist and are finite. Taking limits on both sides of the system (1), we have $l_1 = l_2 = 1$. □

**Lemma 4.** *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1), *and consider the cases;*

*Case2:* $x_{-1}, x_0, y_{-1}, y_0 > 1$

*Case3:* $x_{-1}, y_{-1} > 1$ *and* $x_0, y_0 < 1$,

*Case4:* $x_{-1}, y_{-1} < 1$ *and* $x_0, y_0 > 1$.

*If one of the above cases occurs, then*

*(i) Every positive semicycle consists of two terms;*

*(ii) Every negative semicycle consists of one term;*

*(iii) Every positive semicycle of length two is followed by a negative semicycle of length one;*

*(iv) Every negative semicycle of length one is followed by a positive semicycle of length two.*

*Proof.* If Case 2 occurs, then in view of Lemma 2(ii) we have: $x_1, y_1 < 1$ and $x_{3n+2}, y_{3n+2} > 1$; $x_{3n+3}, y_{3n+3} > 1$ and $x_{3n+4}, y_{3n+4} < 1$ for all $n \geq 0$ which imply that every positive semicycle of system (1) of length two is followed by a negative semicycle of length one which in turn is followed by a positive semicycle of length two.

If Case 3 occurs, then in view of Lemma 2(ii) we have: $x_{3n+1}, y_{3n+1} > 1$; $x_{3n+2}, y_{3n+2} > 1$ and $x_{3n+3}, y_{3n+3} < 1$ for all $n \geq 0$ which imply that every positive semicycle of system (1) of length two is followed by a negative semicycle of length one which in turn is followed by a positive semicycle of length two.

Similarly, if Case 4 occurs, then in view of Lemma 2(ii) we have: $x_1, y_1 > 1$; $x_2, y_2 < 1$ and $x_{3n+3}, y_{3n+3} > 1$; $x_{3n+4}, y_{3n+4} > 1$ and $x_{3n+5}, y_{3n+5} < 1$ for all $n \geq 0$. Therefore, the proof is complete. □

We omit the proofs of the following two results since they can easily be obtained in a way similar to the proof of Lemma 4.

**Lemma 5.** *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1) *and consider the cases;*

*Case5:* $x_{-1}, x_0 < 1$ *and* $y_{-1}, y_0 > 1$,

*Case6:* $x_0, y_{-1} > 1$ *and* $x_{-1}, y_0 < 1$,

*Case7:* $x_{-1}, y_{-1}, y_0 > 1$ *and* $x_0 < 1$,

*Case8:* $x_0, y_{-1}, y_0 > 1$ *and* $x_{-1} < 1$,

*Case9:* $x_{-1}, x_0, y_{-1} < 1$ *and* $y_0 > 1$,

*Case10:* $x_0, y_{-1}, y_0 < 1$ *and* $x_{-1} > 1$.

*If one of the above cases occurs, then*

*(i) Every positive semicycle associated with* $\{x_n\}$ *of system* (1) *consists of one term;*

*(ii) Every negative semicycle associated with* $\{x_n\}$ *of system* (1) *consists of one or three terms;*

*(iii) Every positive semicycle associated with* $\{x_n\}$ *of length one is followed by a negative semicycle of one or three terms;*

*(iv) Every negative semicycle associated with* $\{x_n\}$ *of length one or three terms is followed by a positive semicycle of length one;*

*(v) Every positive semicycle associated with* $\{y_n\}$ *of system* (1) *consists of four terms;*

*(vi) Every negative semicycle associated with* $\{y_n\}$ *of system* (1) *consists of two terms;*

*(vii) Every positive semicycle associated with* $\{y_n\}$ *of length four is followed by a negative semicycle of length two;*

*(viii) Every negative semicycle associated with* $\{y_n\}$ *of length two is followed by a positive semicycle of length four.*

**Lemma 6.** *Assume that* $\{(x_n, y_n)\}_{n=-1}^{\infty}$ *is a solution of system* (1) *and consider the cases;*

*Case11:* $x_{-1}, y_0 > 1$ *and* $x_0, y_{-1} < 1$,

*Case12:* $x_{-1}, x_0 > 1$ *and* $y_{-1}, y_0 < 1$,

*Case13:* $x_{-1}, x_0, y_{-1} > 1$ *and* $y_0 < 1$,

*Case14:* $x_{-1}, x_0, y_0 > 1$ *and* $y_{-1} < 1$,

*Case15:* $x_{-1}, x_0, y_0 < 1$ *and* $y_{-1} > 1$,

*Case16:* $x_{-1}, y_{-1}, y_0 < 1$ *and* $x_0 > 1$.

*If one of the above cases occurs, then*

*(i) Every positive semicycle associated with* $\{x_n\}$ *of system* (1) *consists of four terms;*

*(ii) Every negative semicycle associated with* $\{x_n\}$ *of system* (1) *consists of two terms;*

*(iii) Every positive semicycle associated with* $\{x_n\}$ *of length four is followed by a negative semicycle of length two;*

*(iv) Every negative semicycle associated with* $\{x_n\}$ *of length two is followed by a positive semicycle of length four.*

*(v) Every positive semicycle associated with* $\{y_n\}$ *of system* (1) *consists of one term;*

*(vi) Every negative semicycle associated with* $\{y_n\}$ *of system* (1) *consists of one or three terms;*

*(vii) Every positive semicycle associated with* $\{y_n\}$ *of length one is followed by a negative semicycle of one or three terms;*

*(viii) Every negative semicycle associated with* $\{y_n\}$ *of length one or three terms is followed by a positive semicycle of length one.*

## 3. Main result

**Theorem 2.** *The positive equilibrium point* $(\overline{x},\overline{y}) = (1,1)$ *of the system* (1) *is globally asymptotically stable.*

*Proof.* We must show that the positive equilibrium point $(\overline{x},\overline{y}) = (1,1)$ of the system (1) is both locally asymptotically stable and $(x_n, y_n) \to (\overline{x},\overline{y})$ as $n \to \infty$. The characteristic equation of the system (1) about the positive equilibrium point $(\overline{x},\overline{y}) = (1,1)$ is

$$\lambda^2 - 0.\lambda + 0 = 0$$

and so it is clear from Theorem 1 that positive equilibrium point $(\overline{x},\overline{y}) = (1,1)$ of the system (1) is locally asymptotically stable. It remains to verify that every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (1) converges to $(\overline{x},\overline{y}) = (1,1)$ as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \overline{x} = 1 \text{ and } \lim_{n\to\infty} y_n = \overline{y} = 1. \tag{4}$$

If the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (1) is nonoscillatory about the positive equilibrium point $(\overline{x},\overline{y}) = (1,1)$ of the system (1), then according to Lemmas 1 and 3, respectively, we know that the solution is either eventually equal to $(1,1)$ or an eventually negative one which has an infinite number of terms and monotonically tends the positive equilibrium point $(\overline{x},\overline{y}) = (1,1)$ of the system (1) and so equation (4) holds. Therefore, it suffices to prove that equation (4) holds for strictly oscillatory solutions. Now let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be strictly oscillatory about the positive equilibrium point $(\overline{x},\overline{y}) = (1,1)$ of the system (1). By virtue of Lemmas 2 (ii) and 4 one can see that every positive semicycle of this solution has two terms and every negative semicycle has one term. Every positive semicycle of length two is followed by a negative semicycle of length one. For the convenience of statement, without loss of generality, we use the following notation. We denote by $x_p, x_{p+1}$ (resp. $y_p, y_{p+1}$) the terms of a positive semicycle of length two, followed by $x_{p+2}$ (resp. $y_{p+2}$) which is the term of a negative semicycle of length one. Afterwards, there is the positive semicycle $x_{p+3}, x_{p+4}$ (resp. $y_{p+3}, y_{p+4}$) in turn followed by the negative semicycle $x_{p+5}$ (resp. $y_{p+5}$) so on.

Therefore, we have the following sequences consisting of positve and negative semicycles (for $n = 0, 1, \ldots$):

$$\{x_{p+3n}, x_{p+3n+1}\}_{n=0}^{\infty} ; \{x_{p+3n+2}\}_{n=0}^{\infty}$$

and

$$\{y_{p+3n}, y_{p+3n+1}\}_{n=0}^{\infty} ; \{y_{p+3n+2}\}_{n=0}^{\infty},$$

We get the following assertions:

(i) $x_{p+3n} > x_{p+3n+1}$ (resp. $y_{p+3n} > y_{p+3n+1}$);

(ii) $x_{p+3n+1}x_{p+3n+2} > 1$ (resp. $y_{p+3n+1}y_{p+3n+2} > 1$);

(iii) $x_{p+3n+2}x_{p+3n+3} < 1$ (resp. $y_{p+3n+2}y_{p+3n+3} < 1$).

Combining the above inequalities, we derive

$$(5) \qquad \begin{aligned} \frac{1}{x_{p+3n}} < \frac{1}{x_{p+3n+1}} < x_{p+3n+2} < \frac{1}{x_{p+3n+3}} \\ \frac{1}{y_{p+3n}} < \frac{1}{y_{p+3n+1}} < y_{p+3n+2} < \frac{1}{y_{p+3n+3}} \end{aligned}$$

From equation (5), one can see that $\{x_{p+3n+2}\}_{n=0}^{\infty}$ (resp. $\{y_{p+3n+2}\}_{n=0}^{\infty}$) is increasing with upper bound 1. So the limit $\lim_{n\to\infty} x_{p+3n+2} = l_3$ (resp. $\lim_{n\to\infty} y_{p+3n+2} = l_4$) exists and is finite. Accordingly, in view of equation (5), we obtain (for $m = 0, 1, 3$)

$$\lim_{n\to\infty} x_{p+3n+m} = \frac{1}{l_3} \quad (\text{resp.} \lim_{n\to\infty} y_{p+3n+m} = \frac{1}{l_4}) .$$

It suffices to verify that $l_3 = l_4 = 1$. To this end, note that

$$x_{p+3n+3} = \frac{x_{p+3n+2} + y_{p+3n+1}}{x_{p+3n+2}y_{p+3n+1} + 1} \text{ and } y_{p+3n+3} = \frac{y_{p+3n+2} + x_{p+3n+1}}{y_{p+3n+2}x_{p+3n+1} + 1}$$

Take the limits on both sides of the above equality and obtain

$$\frac{1}{l_3} = \frac{l_3 + 1/l_4}{l_3.1/l_4 + 1} \text{ and } \frac{1}{l_4} = \frac{l_4 + 1/l_3}{l_4.1/l_3 + 1}$$

which imply that $l_3 = l_4 = 1$. So, we have shown that

$$\lim_{n\to\infty} x_{p+3n+m_1} = \lim_{n\to\infty} y_{p+3n+m_1} = 1 \quad \text{for } m_1 \epsilon \{0, 1, 2, 3\} .$$

Similarly, by virtue of Lemmas 2 (ii) and 5 (resp. 6) one can see that equation (4) holds. Therefore, the proof is complete. □

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