Toughness of graphs and 2-factors with given properties

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Abstract

Let G be a 2-tough graph on at least five vertices and let e_1, e_2 be a pair of arbitrarily given edges of G. Then

(a) There exists a 2-factor in G containing e_1, e_2.

(b) There exists a 2-factor in G avoiding e_1, e_2.

(c) There exists a 2-factor in G containing e_1 and avoiding e_2.

All graphs considered are assumed to be simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $\deg_G(u)$ of a vertex u in G is the number of edges of G incident with u. The minimum degree of vertices in G is denoted by $\delta(G)$. If X and Y are disjoint subsets of $V(G)$, we will write $E_G(X,Y)$ and $e_G(X,Y)$ for the set and the number respectively of the edges of G joining X to Y. The number of connected components of G is denoted by $\omega(G)$. An edge e of G is said to be subdivided when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex.

For any set X of vertices in G, we define the neighbor set of X in G to be the set of all vertices adjacent to vertices in X; this set is denoted by $N_G(X)$.

Let X be a nonempty subset of $V(G)$. The subgraph of G whose vertex set is X and whose edge set is the set of those edges of G that have both ends in X is called the subgraph of G induced by X and is denoted by $G[X]$; we say that $G[X]$ is an induced subgraph of G.

A vertex cut of G is a subset X of $V(G)$ such that $G-X$ is disconnected. A k-vertex cut is a vertex cut of k elements. If $G \ncong K_n$, the connectivity $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut; otherwise, we define $\kappa(G)$ to be $n-1$. Thus $\kappa(G) = 0$ if G is either trivial or disconnected. G is said to be k-connected if $\kappa(G) \geq k$.

An edge cut of G is a subset Y of $E(G)$ of the form $e_G(X, V(G) - X)$, where X is a nonempty proper subset of $V(G)$. A k-edge cut is an edge cut of k elements. If G is nontrivial and Y is an edge cut of G, then $G - Y$ is disconnected; we then define the edge connectivity $\kappa'(G)$ of G to be the minimum k for which G has a k-edge cut. If G is trivial, $\kappa'(G)$ is defined to be zero. Thus $\kappa'(G) = 0$ if G is either trivial or disconnected. G is said to be k-edge-connected if $\kappa'(G) \geq k$.

The following theorem is a well-known result.

Theorem 1 ([7]). For every graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

The toughness of G is defined by $t(G) = \min_X \left\{ \frac{|X|}{\omega(G-X)} \right\}$, where the minimum is taken over all vertex cuts X of G. If $G \cong K_n$, then we define $t(G) = n - 1$. G is said to be t-tough if $t(G) \geq t$.

The following proposition relates the invariants of toughness, connectivity and minimum degree; we mention it because we will use it later in this paper.

Proposition 1. For every non-complete graph G,

$$t(G) \leq \frac{\kappa(G)}{2} \leq \frac{\kappa'(G)}{2} \leq \frac{\delta(G)}{2}.$$

Proof. Let X be a vertex cut of G such that $|X| = \kappa(G)$. Then clearly $t(G) \leq \frac{|X|}{\omega(G-X)} \leq \frac{\kappa(G)}{2}$, since $\omega(G - X) \geq 2$. Therefore by Theorem 1, we obtain Proposition 1. □

Let G be a graph. Given a function $f : V(G) \to \mathbb{Z}^+$, we say that G has an f-factor if there exists a spanning subgraph H of G such that $\deg_H(x) = f(x)$ for every vertex $x \in V(G)$. If f is a constant function taking the value k, then an f-factor is said to be a k-factor. Thus a k-factor of G is a k-regular spanning subgraph and a Hamilton cycle of G is clearly a special case of a 2-factor.

The following theorem is a necessary and sufficient condition for a graph to have a k-factor and it was obtained by Tutte [6].

Theorem 2 (Tutte's k-factor Theorem). A graph G has a k-factor if and only if

$$q_G(S,T;k) + \sum_{x\in T}(k - \deg_{G-S}(x)) - k|S| \le 0$$

for all pairs of disjoint subsets S and T of $V(G)$, where $q_G(S,T;k)$ denotes the number of components C of $G-(S\cup T)$ such that $k|V(C)|+e_G(V(C),T)$ is odd. (Hereafter, we refer to these components as odd components.)

In addition, Tutte proved that for any graph G and any positive integer k,

$$q_G(S,T;k) + \sum_{x\in T}(k - \deg_{G-S}(x)) - k|S| \equiv k|V(G)| \pmod 2 \quad (1)$$

The concept of toughness was first introduced by Chvátal [3]. He mainly studied relations between toughness and the existence of Hamilton cycles or k-factors, and stated several conjectures. One of the conjectures was the following: Let G be a graph and k a positive integer such that $k|V(G)|$ is even and G is k-tough. Then G has a k-factor.

In [4], it was proved that Chvátal's conjecture is true. Furthermore it was shown that the above result is sharp in the following sense: For any positive integer k and for any positive real number ε, there exists a $(k-\varepsilon)$-tough graph G with $k|V(G)|$ even which has no k-factor.

The sharpness of the truth of Chvátal's conjecture was also proved independently by Tsikopoulos [5].

Chen [2] obtained the following two theorems, which strengthen the truth of Chvátal's conjecture.

Theorem 3. Let G be a graph and k a positive integer, where $k \ge 2$, such that $k|V(G)|$ is even and G is k-tough. Then for every edge $e \in E(G)$, G has a k-factor containing e.

Theorem 4. Let G be a graph and k a positive integer, such that $k|V(G)|$ is even, $|V(G)| \ge k+2$ and G is k-tough. Then for every $e \in E(G)$, the graph $G - \{e\}$ has a k-factor.

The main purpose of this paper is to present the following three theorems which fit into the above-mentioned literature.

Theorem 5. Let G be a 2-tough graph. Then for every pair of edges e_1, e_2 of G, there exists a 2-factor in G containing them.

Theorem 6. Let G be a 2-tough graph with $|V(G)| \geq 5$. Then for every pair of edges e_1, e_2 of G, the graph $G - \{e_1, e_2\}$ has a 2-factor.

Theorem 7. Let G be a 2-tough graph with $|V(G)| \geq 5$. Then for every pair of edges e_1, e_2 of G, the graph $G - \{e_1\}$ has a 2-factor containing e_2.

Theorems 5 and 6 are stronger than Theorems 3 and 4 respectively, for the special case when $k = 2$. Furthermore Theorem 7 is an improvement of both Theorems 3 and 4, again when $k = 2$.

For the proof of Theorem 5, we shall need the following lemmas.

Lemma 1. Let G be a graph. Suppose that there exists a pair of disjoint subsets S and T of $V(G)$, such that

$$q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) - 2|S| \geq 2 \tag{2}$$

furthermore, S is minimal with respect to (2). Then for any vertex $v \in S$, $\deg_G(v) \geq 4$.

Proof. Since S is minimal with respect to (2), for any vertex $v \in S$,

$$q_G(S - \{v\}, T; 2) + \sum_{x \in T} (2 - \deg_{G-(S-\{v\})}(x)) - 2|S - \{v\}| \leq 0. \tag{3}$$

Combining (2) and (3), we have

$$q_G(S,T;2) - q_G(S - \{v\}, T; 2) + e_G(\{v\}, T) - 2 \geq 2.$$

Note that $q_G(S - \{v\}, T; 2) \geq q_G(S,T;2) - \deg_{G-S-T}(v)$, thus

$$\begin{aligned} \deg_G(v) &\geq \deg_{G-S-T}(v) + e_G(\{v\}, T) \\ &\geq q_G(S,T;2) - q_G(S - \{v\}, T; 2) + e_G(\{v\}, T) \geq 4. \end{aligned}$$ □

Lemma 2. Let G be a 2-tough graph with $|V(G)| \geq 5$ and let S, T be a pair of disjoint subsets of $V(G)$. If $|S| \geq 2$, then

$$q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) - 2|S| \leq -4.$$

Proof. Suppose that there exists a pair of disjoint subsets S, T of $V(G)$ with $|S| \geq 2$, such that

$$q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) - 2|S| > -4.$$

But from (1),

$$q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) - 2|S| \geq -2. \qquad (4)$$

Claim: $|T| \geq 2$.

If $T = \emptyset$, then $q_G(S,\emptyset;2) \geq 2|S| - 2 \geq 2$ by (4), and hence S is a vertex cut of G. Since G is 2-tough, $|S| \geq 2\omega(G-S) \geq 2q_G(S,\emptyset;2) \geq 4|S| - 4$, $|S| \leq 4/3$, a contradiction. If $|T| = 1$, say $T = \{u\}$, then by (4),

$$q_G(S,\{u\};2) \geq |S| + \big(|S| + \deg_{G-S}(u)\big) - 4 \geq |S| + \deg_G(u) - 4 \geq |S|,$$

since by the definition of toughness and Proposition 1, $\delta(G) \geq 4$; hence $S \cup \{u\}$ is a vertex cut of G. Therefore, $|S| + 1 = |S \cup \{u\}| \geq 2\omega(G - (S \cup \{u\})) \geq 2q_G(S,\{u\};2) \geq 2|S|$, a contradiction. Thus we have proved the claim.

We may assume further that T is minimal with respect to (4). By the minimality of T, for an arbitrary vertex $u \in T$, we have

$$q_G(S, T - \{u\}; 2) + \sum_{x \in T - \{u\}} (2 - \deg_{G-S}(x)) - 2|S| \leq -4. \qquad (5)$$

Combining (4) and (5), $q_G(S,T;2) - q_G(S, T-\{u\};2) + (2 - \deg_{G-S}(u)) \geq 2$, that is,

$$\begin{aligned} \deg_{G-S}(u) &\leq q_G(S,T;2) - q_G(S,T-\{u\};2) && (6)\\ &\leq \deg_{G-S-T}(u). && (7) \end{aligned}$$

Therefore, $\deg_{G-S}(u) = \deg_{G-S-T}(u)$, that is, $|N_G(u) \cap T| = 0$, and thus T is an independent set of G. Moreover, the inequalities (6) and (7) become equalities. So we have that

if $e_G(V(C), \{u\}) \neq 0$, then C is an odd component of $(G - S) - T$ and $e_G(V(C), \{u\}) = 1$. Hence, if C is an even component, then $e_G(V(C), T) = 0$. $(*)$

Assume that $C_1, C_2, \dots, C_t$ are the components of $G-(S \cup T)$. Let $H = G-(S\cup T)$. Let $V_1 = \{v \in V(H) : |N_G(v) \cap T| = 1\}$ and $V_2 = \{v \in V(H) : |N_G(v) \cap T| \geq 2\}$. Let $C_1, C_2, \dots, C_l$ be the components containing at least one element of V_1. Choose arbitrary $u_i \in V(C_i) \cap V_1$ for $i = 1, \dots, l$. Define $X = \{u_1, u_2, \dots, u_l\}$ and $Y = (N_G(T) \cap V(H)) - X$.

By the definition of V_1 and V_2, we have

$$|V_1| + 2|V_2| \leq e_G(T, V(H)). \tag{8}$$

Thus $|V_1| + |V_2| \leq e_G(T, V(H)) - |V_2|$, and $|Y| = |N_G(T) \cap V(H)| - |X| = |V_1| + |V_2| - |X| \leq e_G(T, V(H)) - |V_2| - |X|$.

Obviously,

$$|V_2| + |X| \geq r, \tag{9}$$

where r is the number of components of $G-(S \cup T)$ which is joined to T. So $|Y| \leq e_G(T, V(H)) - r$ and $|S| + |Y| \leq |S| + e_G(T, V(H)) - r$.

By the choice of Y, $\omega(G-(S\cup Y)) \geq |T| + \omega(G-(S \cup T)) - r \geq 2$ and so $S \cup Y$ is a vertex cut of G. As G is 2-tough, we have

$$\begin{aligned} |S| + |Y| &\geq 2\omega(G-(S\cup Y)) && (10)\\ &\geq 2|T| + 2\omega(G-(S\cup T)) - 2r. && (11) \end{aligned}$$

Thus $|S| + e_G(T, V(H)) - r \geq |S| + |Y| \geq 2|T| + 2\omega(G-(S \cup T)) - 2r$ and so

$$2|T| - e_G(T, V(H)) \leq |S| + r - 2\omega(G-(S\cup T)).$$

Then we have

$$\begin{aligned} &q_G(S,T;2) + \sum_{x\in T}(2 - \deg_{G-S}(x)) - 2|S| \\ =&q_G(S,T;2) + 2|T| - e_G(V(H), T) - 2|S| \\ \leq&q_G(S,T;2) - |S| + r - 2\omega(G-(S\cup T)) \\ \leq& -|S| && (12)\\ \leq& -2. && (13) \end{aligned}$$

By (4), $|S| = 2$, and all the labeled inequalities become equalities.

Since $|S| = 2$, for any vertex $x \in T$, $\deg_{G-S}(x) = \deg_H(x) \geq \deg_G(x) - |S| \geq 2$ as $\deg_G(x) \geq 4$. But equality holds in (12), so it yields

$$q_G(S,T;2) = \omega(G-(S\cup T)) = r. \tag{14}$$

Hence $(G-S)-T$ has no even component. Moreover for every component C_i the equality in (9) yields

$$|V_2 \cap V(C_i)| = 0, \text{ when } 1 \le i \le l \\ \text{and } |V_2 \cap V(C_i)| = 1, \text{ when } l+1 \le i \le t. \quad (**)$$

The equalities in (10) and (11) together with (14) imply that

$$|S| + |Y| = 2\omega(G - (S \cup Y)) = 2|T|. \tag{15}$$

Now if C_i contains a vertex in V_2, then $|V(C_i)| = 1$ since $\omega(G-(S \cup Y)) = |T|$ and by (**).

The equality in (8) also implies that $|N_G(x) \cap T| = 2$, for every $x \in V_2$.

Therefore, if $x \in V_2$, then $V(C_i) = \{x\}$ with $|N_G(x) \cap T| = 2$ and so C_i is an even component. Thus we have $V_2 = \emptyset$ by using (*).

Assume that there exists a vertex $u \in T$ having two neighbors $u_1 \in V(H)$ and $u_2 \in V(H)$, such that $|N_G(u_i) \cap T| = 1$. By (*), u_1 and u_2 are in distinct odd components. We may assume that $u_1, u_2 \in X$. Then $\omega(G-(S \cup Y \cup \{u\})) \ge \omega(G-(S \cup Y))+1 = |T|+1 \ge 3$, and thus $S \cup Y \cup \{u\}$ is a vertex cut of G. As G is 2-tough, then $2|T|+1 = |S \cup Y \cup \{u\}| \ge 2\omega(G-(S \cup Y \cup \{u\})) \ge 2|T|+2$, a contradiction.

Hence for any vertex $u \in T$, u has at most one neighbor in H, and thus $\deg_G(u) \le |S|+1 = 3$, a contradiction, since $\delta \ge 4$.

Therefore, (4) is false and this completes the proof of the lemma. □

Lemma 3. Let G be a graph, and $e = ab$ be an edge of G. Let G' be the graph obtained from G by inserting a new vertex u into the edge e. Then for any pair of disjoint subsets S', T' of $V(G')$,

$$q_{G'}(S',T';2) + \sum_{x \in T'} (2-\deg_{G'-S'}(x)) = q_G(S,T;2) + \sum_{x \in T} (2-\deg_{G-S}(x)) + 2\varepsilon, \tag{16}$$

where $S = S' - \{u\}$, $T = T' - \{u\}$ and $\varepsilon = 0, 1$.

Proof. From the parity equality in (1), (16) holds so we only need to prove that $\varepsilon = 0, 1$.

By the construction of G', G' is obtained from G by deleting one edge ab and adding two adjacent edges ua, ub. Thus we have

$$-1 \le e_{G'}(S',T') - e_G(S,T) \le 2.$$

We consider the following four cases:

Case 1: $e_{G'}(S',T') = e_G(S,T) + 2$.

Then $\{ua, ub\} \subseteq E_{G'}(S',T')$, and thus $q_{G'}(S',T';2) = q_G(S,T;2)$. Clearly, $\varepsilon = 1$.

Case 2: $e_{G'}(S',T') = e_G(S,T) + 1$.

Then exactly one of $\{ua, ub\}$ is in $E_{G'}(S',T')$. Hence one vertex of $\{a,b\}$, say b, is in a component of $(G' - S') - T'$, and then $q_{G'}(S',T';2) = q_G(S,T;2) - 1$ or $q_{G'}(S',T';2) = q_G(S,T;2) + 1$. Hence $\varepsilon = 0, 1$.

Case 3: $e_{G'}(S',T') = e_G(S,T)$.

Then $\varepsilon = 0, 1$, as $0 \le q_{G'}(S',T';2) - q_G(S,T;2) \le 2$.

Case 4: $e_{G'}(S',T') = e_G(S,T) - 1$.

Then edge $ab \in E_G(S,T)$ and vertex $u \notin S' \cup T'$. Since u has only two neighbors a and b, $\{u\}$ is a component of $(G' - S') - T'$. Moreover, $\{u\}$ is an odd component of $(G' - S') - T'$. Then $q_{G'}(S',T';2) = q_G(S,T;2) + 1$, and hence $\varepsilon = 0$. □

Proof of Theorem 5. We may assume that $|V(G)| \ge 5$ since for all 2-tough graphs having less than five vertices the theorem clearly holds.

Let G_1 be the graph obtained from G after the subdivision of e_1(insert a new vertex u_1 on e_1), and let G_2 be the graph obtained from G_1 after the subdivision of e_2(insert a new vertex u_2 on e_2). Then G has a 2-factor containing e_1 and e_2 if and only if G_2 contains a 2-factor.

Suppose that the graph G_2 contains no 2-factors. Then by Tutte's k-factor theorem, there exists a pair of disjoint subsets S_2, T_2 of $V(G_2)$, such that

$$q_{G_2}(S_2,T_2;2) + \sum_{x \in T_2} (2 - \deg_{G_2 - S_2}(x)) - 2|S_2| \ge 2. \qquad (17)$$

We may assume that S_2 is minimal with respect to (17). Then by Lemma 1, $S_2 \cap \{u_1, u_2\} = \emptyset$. Let $S = S_1 = S_2$, $T_1 = T_2 - \{u_2\}$ and $T = T_1 - \{u_1\}$. By Lemma 3,

$$\begin{aligned} & q_{G_2}(S_2,T_2;2) + \sum_{x \in T_2} (2 - \deg_{G_2 - S_2}(x)) \\ \le & q_{G_1}(S_1,T_1;2) + \sum_{x \in T_1} (2 - \deg_{G_1 - S_1}(x)) + 2, \end{aligned} \qquad (18)$$

and

$$q_{G_1}(S_1,T_1;2)+\sum_{x\in T_1}(2-\deg_{G_1-S_1}(x)) \leq q_G(S,T;2)+\sum_{x\in T}(2-\deg_{G-S}(x))+2. \quad (19)$$

Combining (18) and (19), we have

$$q_{G_2}(S_2,T_2;2)+\sum_{x\in T_2}(2-\deg_{G_2-S_2}(x)) \leq q_G(S,T;2)+\sum_{x\in T}(2-\deg_{G-S}(x))+4.$$

Thus (17) yields

$$q_G(S,T;2)+\sum_{x\in T}(2-\deg_{G-S}(x))-2|S| \geq -2$$

and so by Lemma 2, $|S| \leq 1$.

Case 1: $S = S_1 = S_2 = \emptyset$.

If $x \in T_2 - \{u_1, u_2\}$, $\deg_{G_2}(x) = \deg_G(x) \geq 4$ and if $x \in T_2 \cap \{u_1, u_2\}$, $\deg_{G_2}(x) = 2$. Thus (17) implies

$$q_{G_2}(\emptyset,T_2;2) \geq \sum_{x\in T_2}(\deg_{G_2}(x)-2)+2 \geq 2|T_2-\{u_1,u_2\}|+2 = 2|T|+2. \quad (20)$$

Since $\omega(G-T) \geq \omega(G_2-T_2)-2$, $\omega(G-T) \geq q_{G_2}(\emptyset,T_2;2)-2 \geq 2|T|$.

If $T \neq \emptyset$, then T is a vertex cut of G, and $|T| \geq 2\omega(G-T) \geq 4|T|$, a contradiction. So we assume that $T = \emptyset$. But if this is the case then (20) implies that $q_{G_2}(\emptyset,T_2;2) \geq 2$ and so T_2 is a vertex cut of G_2. Since $T = T_2 - \{u_1,u_2\} = \emptyset$, $T_2 \subseteq \{u_1,u_2\}$. Therefore, the edge set $\{e_1,e_2\}$ is a 2-edge cut of G, a contradiction to the fact that G is 4-connected.

Case 2: $|S| = 1$.

Let m be the number of elements u_i of $T_2 \cap \{u_1,u_2\}$ such that $|N_{G_2}(u_i) \cap S| = 1$. Clearly $0 \leq m \leq 2$.

For every $x \in T$ we have $|S_2|+\deg_{G_2-S_2}(x) = |S|+\deg_{G-S}(x) \geq \deg_G(x) \geq 4$ and so

$\sum_{x\in T}\deg_{G_2-S_2}(x) \geq 3|T|$ since $|S_2| = |S| = 1$.

Moreover $\sum_{x\in T_2\cap\{u_1,u_2\}} \deg_{G_2-S_2}(x) = 2|T_2\cap\{u_1,u_2\}|-m$. Thus (17) yields

$$\begin{aligned}
&q_{G_2}(S_2,T_2;2)\\
\geq&2|S_2| + \sum_{x\in T_2}\big(\deg_{G_2-S_2}(x)-2\big)+2\\
=&2+\sum_{x\in T}\big(\deg_{G_2-S_2}(x)-2\big)+\sum_{x\in T_2\cap\{u_1,u_2\}}\big(\deg_{G_2-S_2}(x)-2\big)+2\\
\geq&4+|T|-m.
\end{aligned}$$

But we have

$$\begin{aligned}
\omega(G-(S\cup T)) &\geq q_{G_2}(S_2,T_2;2)-(2-m)\\
&\geq 4+|T|-m-(2-m)\\
&=|T|+2.
\end{aligned}$$

Thus $S\cup T$ is a vertex cut of G and so $|T|+1=|S\cup T|\geq 2\omega(G-(S\cup T))\geq 2|T|+4$, which is a contradiction. □

For the proof of Theorems 6 and 7, we will also use the following lemma.

Lemma 4. Let G be a graph, $e=ab$ be an edge of G and let $G'=G-\{e\}$. Then for any pair of disjoint subsets S,T of $V(G)$,

$$q_{G'}(S,T;2)+\sum_{x\in T}(2-\deg_{G'-S}(x)) = q_G(S,T;2)+\sum_{x\in T}(2-\deg_{G-S}(x))+2\varepsilon \tag{21}$$

where $\varepsilon=0,1$.

Proof. Since $G'=G-\{e\}$, so $\deg_{G'}(a)=\deg_G(a)-1$ and $\deg_{G'}(b)=\deg_G(b)-1$.

From (1), clearly (21) holds, so we only need to prove that $\varepsilon=0,1$. According to the location of a,b, there are three cases to consider:

Case 1: $a,b\in T$.

Then $q_{G'}(S,T;2)=q_G(S,T;2)$ and $\sum_{x\in T}\deg_{G'-S}(x)=\sum_{x\in T}\deg_{G-S}(x)-2$. Therefore, $\varepsilon=1$.

Case 2: Assume that exactly one of $\{a,b\}$, say a, belongs to T.

If $b \in S$, then $\sum_{x \in T} \deg_{G'-S}(x) = \sum_{x \in T} \deg_{G-S}(x)$ and $q_G(S,T;2) = q_{G'}(S,T;2)$; and if $b \notin S \cup T$, then $\sum_{x \in T} \deg_{G'-S}(x) = \sum_{x \in T} \deg_{G-S}(x) - 1$ and $q_{G'}(S,T;2) - q_G(S,T;2) = 1$ or -1. Therefore $\varepsilon = 0, 1$.

Case 3: $\{a, b\} \cap T = \emptyset$.

Then $\sum_{x \in T} \deg_{G'-S}(x) = \sum_{x \in T} \deg_{G-S}(x)$ and $q_{G'}(S,T;2) - q_G(S,T;2) = 0$ or 2. Therefore, $\varepsilon = 0, 1$. □

Proof of Theorem 6. Let G_1 be the graph obtained from G by deleting the edge e_1, and let G_2 be the graph obtained from G_1 by deleting the edge e_2. Then the graph G has a 2-factor excluding $\{e_1, e_2\}$ if and only if G_2 contains a 2-factor.

Suppose that the graph G_2 contains no 2-factors. Then by Tutte's k-factor theorem, there exists a pair of disjoint subsets S and T of $V(G_2)$, such that

$$q_{G_2}(S,T;2) + \sum_{x \in T} (2 - \deg_{G_2-S}(x)) - 2|S| \geq 2. \tag{22}$$

Let $H = G - (S \cup T)$, and let m_1 be the number of elements of $\{e_1, e_2\}$ having both ends in T; let m_2 be the number of elements of $\{e_1, e_2\}$ having one end in T and the other in $V(H)$; let m_3 be the number of elements of $\{e_1, e_2\}$ having both ends in H. Clearly,

$$m_1 + m_2 + m_3 \leq 2 \tag{23}$$

and

$$\sum_{x \in T} \deg_{G_2-S}(x) = \sum_{x \in T} \deg_{G-S}(x) - 2m_1 - m_2. \tag{24}$$

By Lemma 4,

$$q_{G_2}(S,T;2) + \sum_{x \in T} (2 - \deg_{G_2-S}(x)) \leq q_{G_1}(S,T;2) + \sum_{x \in T} (2 - \deg_{G_1-S}(x)) + 2$$

and

$$q_{G_1}(S,T;2) + \sum_{x \in T} (2 - \deg_{G_1-S}(x)) \leq q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) + 2.$$

Then

$$q_{G_2}(S,T;2) + \sum_{x \in T} (2 - \deg_{G_2-S}(x)) \leq q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) + 4.$$

Thus (22) yields

$$q_G(S,T;2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) \geq 2|S| - 2$$

and so Lemma 2 implies that $|S| \leq 1$.

We also have

$$\begin{aligned}
&\omega(G - (S \cup T)) \\
\geq & q_{G_2}(S,T;2) - m_3 \\
\geq & \sum_{x \in T} (\deg_{G_2 - S}(x) - 2) + 2|S| + 2 - m_3 && \text{by (22)} \\
= & \sum_{x \in T} (\deg_{G-S}(x) - 2) - 2m_1 - m_2 + 2|S| + 2 - m_3 && \text{by (24)} \\
\geq & \sum_{x \in T} (\deg_{G-S}(x) - 2) - m_1 + 2|S| && \text{by (23)} \\
\geq & \sum_{x \in T} (4 - |S| - 2) - m_1 + 2|S| && \text{since } \delta(G) \geq 4 \\
= & |T|(2 - |S|) - m_1 + 2|S|. && (25)
\end{aligned}$$

Case 1: $|S| = 1$.

Then by (25)

$$\omega(G - (S \cup T)) \geq |T| - m_1 + 2 \geq 2,$$

since $|T| \geq m_1$. Hence $S \cup T$ is a vertex cut of G. Since G is 2-tough, $|T| + 1 = |S \cup T| \geq 2\omega(G - (S \cup T)) \geq 2|T| - 2m_1 + 4$. So $|T| - 2m_1 + 3 \leq 0$ and since $|T| \geq m_1$, we obtain $m_1 \geq 3$ contradicting (23).

Case 2: $|S| = 0$.

Then $T \neq \emptyset$. Otherwise, by (22) $q_{G_2}(\emptyset, \emptyset; 2) \geq 2$, which yields that $\{e_1, e_2\}$ is a 2-edge cut of G, a contradiction to the fact that G is 4-connected. By the definition of m_1, if $m_1 = 1$, then $|T| \geq 2$; if $m_1 = 2$, then $|T| \geq 3$. Hence $|T| \geq m_1 + 1$. Thus by (25)

$$\omega(G - T) \geq 2|T| - m_1 \geq |T| + 1 \geq 2,$$

and hence T is a vertex cut of G. Since G is 2-tough, $|T| \geq 2\omega(G - T) \geq 4|T| - 2m_1 \geq 2|T| + 2$, a contradiction. □

Proof of Theorem 7. Let G_1 be the graph obtained from G by deleting the edge e_1. Let G_2 be the graph obtained from G_1 after the subdivision of the edge $e_2 = u_1u_2$ (insert a new vertex u on e_2). Then the graph G contains a 2-factor containing e_2 and avoiding e_1 if and only if G_2 contains a 2-factor.

Suppose that G_2 contains no 2-factor. Then by Tutte's k-factor theorem, there exists a pair of disjoint subsets S_2 and T_2 of $V(G_2)$, such that

$$q_{G_2}(S_2, T_2; 2) + \sum_{x \in T_2} (2 - \deg_{G_2 - S_2}(x)) - 2|S_2| \geq 2. \tag{26}$$

Without loss of generality, we assume that S_2 is minimal with respect to (26). By Lemma 1, $u \notin S_2$. Let $S = S_1 = S_2$, $T_1 = T_2 - \{u\}$ and $T = T_1$. Let m_1 be the number of ends of e_1 belonging to T_2 and $m_2 = |\{u_1u, u_2u\} \cap E_{G_2}(S_2, T_2)|$.

By Lemma 3,

$$\begin{aligned} & q_{G_2}(S_2, T_2; 2) + \sum_{x \in T_2} (2 - \deg_{G_2 - S_2}(x)) \\ \leq & q_{G_1}(S_1, T_1; 2) + \sum_{x \in T_1} (2 - \deg_{G_1 - S_1}(x)) + 2 \end{aligned}$$

and by Lemma 4,

$$q_{G_1}(S_1, T_1; 2) + \sum_{x \in T_1} (2 - \deg_{G_1 - S_1}(x)) \leq q_G(S, T; 2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) + 2.$$

Thus

$$q_{G_2}(S_2, T_2; 2) + \sum_{x \in T_2} (2 - \deg_{G_2 - S_2}(x)) \leq q_G(S, T; 2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) + 4.$$

Hence (26) yields

$$q_G(S, T; 2) + \sum_{x \in T} (2 - \deg_{G-S}(x)) \geq 2|S| - 2$$

and so by Lemma 2, $|S| \leq 1$.

We also have,

$$\begin{aligned}\sum_{x\in T_2}(2-\deg_{G_2-S_2}(x)) &= \sum_{x\in T_2}(2-\deg_{G_2}(x))+e_{G_2}(S_2,T_2)\\ &= \sum_{x\in T_1}(2-\deg_{G_1}(x))+e_{G_2}(S_2,T_2)\\ &= \sum_{x\in T}(2-\deg_{G}(x))+m_1+e_{G_2}(S_2,T_2)\\ &\le \sum_{x\in T}(2-\deg_{G}(x))+m_1+e_G(S,T)+m_2\\ &= \sum_{x\in T}(2-\deg_{G-S}(x))+m_1+m_2. \end{aligned}\tag{27}$$

At this point we must mention that $m_2 \le 1$, since $|S| \le 1$ and $u \notin S$.

Case 1: $m_1 = 0$ or 1.

Then $\omega(G-(S\cup T)) \ge \omega(G_2-(S_2\cup T_2)) - (1-m_1) - (1-m_2)$.

Therefore,

$$\begin{aligned}&\omega(G-(S\cup T))\\ \ge& q_{G_2}(S_2,T_2;2)-(1-m_1)-(1-m_2)\\ \ge& 2|S_2|+2-\sum_{x\in T_2}(2-\deg_{G_2-S_2}(x))-(1-m_1)-(1-m_2)\\ \ge& 2|S_2|-\sum_{x\in T}(2-\deg_{G-S}(x)) \qquad \text{by (27)}\\ \ge& 2|S|+|T|(4-|S|-2)\\ =& 2|S|+|T|(2-|S|).\end{aligned}\tag{28}$$

If $|S|=1$, then by (28), $\omega(G-(S\cup T)) \ge 2+|T|$. Hence $S\cup T$ is a vertex cut of G, $1+|T| = |S\cup T| \ge 2\omega(G-(S\cup T)) \ge 2(2+|T|)$, a contradiction to the fact that G is 2-tough.

Thus we may assume that $|S|=0$. Then by (28), $\omega(G-T)\ge 2|T|$. If $|T|\ge 1$, then T is a vertex cut of G, and $|T| \ge 2\omega(G-T) \ge 4|T|$, a contradiction. Hence we assume that $T=\emptyset$. By (26), $\omega(G_2) \ge q_{G_2}(\emptyset,\emptyset;2) \ge 2$ and so by the construction of G_2, the edge e_1 is a cut edge of G, contradicting the fact that G is 4-connected.

Case 2: $m_1 = 2$.

By the definition of m_1, $|T| \geq 2$. We also have,

$$\begin{aligned}
\omega(G-(S\cup T)) &\geq q_{G_2}(S_2,T_2;2)-(1-m_2)\\
&\geq 2|S_2|+2-\sum_{x\in T_2}(2-\deg_{G_2-S_2}(x))-(1-m_2)\\
&\geq 2|S_2|-\sum_{x\in T}(2-\deg_{G-S}(x))-1\\
&\geq 2|S|+|T|(4-|S|-2)-1\\
&= 2|S|+|T|(2-|S|)-1. \qquad (29)
\end{aligned}$$

Thus $S\cup T$ is a vertex cut of G. But G is 2-tough, so $|S\cup T| \geq 2\omega(G-(S\cup T)) \geq 4|S|+2|T|(2-|S|)-2$ by (29); and hence $3|T|-2 \leq (2|T|-3)|S|$. As $|T| \geq 2$ and $|S| = 0$ or 1, $3|T|-2 \leq 2|T|-3$, a contradiction to $|T| \geq 2$.

This completes the proof of Theorem 7. □

We will next show that Theorems 5, 6 and 7 are in some sense best possible.

We will first prove that the number of edges in Theorem 5 cannot be increased. For this purpose we will describe a family of graphs G, which are 2-tough, having edges e_1, e_2, e_3 such that there is not a 2-factor in G containing all of them. We construct such a family of graphs as follows: We start from a complete graph K_n, where n is a positive integer ($n \geq 4$) and a copy of K_1. Let $\{v_1,v_2,v_3\} \subset V(K_n)$ and $V(K_1) = \{v\}$. Choose a vertex $w \in V(K_n)-\{v_1,v_2,v_3\}$. Join v to v_1,v_2,v_3,w. The resulting graph G is clearly 2-tough. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$ and $e_3 = v_3v_1$. We claim that G does not possess a 2-factor containing e_1, e_2, e_3. For the proof of the above statement we work as follows. If there exists a 2-factor of G containing e_1,e_2,e_3, then $G-\{v_1,v_2,v_3\}$ has a 2-factor. But this does not hold since the degree of v in $G-\{v_1,v_2,v_3\}$ is 1.

We will next show that the number of deleted edges in Theorem 6 cannot be increased. For this purpose we will describe a family of graphs G, which are 2-tough, having edges e_1, e_2, e_3 such that $G-\{e_1,e_2,e_3\}$ does not have a 2-factor. We construct such a family of graphs as follows: We start from a complete graph H_0 having n vertices, where $n \geq 4$; and three copies of K_2's, say H_1, H_2, H_3. Let $V(H_2) = \{v_1,v_2\}$, $V(H_3) = \{v_3,v_4\}$, $E(H_2) = \{e_2\}$, $E(H_3) = \{e_3\}$ and let $\{u_1,u_2,u_3,u_4\} \subseteq V(H_0)$. We join v_1 to u_1, v_2 to u_2, v_3 to u_3 and v_4 to u_4. Finally we join every vertex of H_1 to all the vertices of H_0, H_2, H_3. The resulting graph G is clearly 2-tough. Furthermore, if we let e_1 to be the edge joining v_1 to u_1, the graph $G_1 = G-\{e_1,e_2,e_3\}$ does not have a 2-factor. Let $S = V(H_1)$ and

$T = \{v_1, v_2, v_3, v_4\}$. Then

$$q_{G_1}(S,T;2) + \sum_{x \in T}(2 - \deg_{G_1 - S}(x)) > 2|S|$$

since $q_{G_1}(S,T;2) = 1$, $\sum_{x \in T}(2 - \deg_{G_1-S}(x)) = 5$ and $|S| = 2$.

Finally we will show by constructing again a family of graphs G, that the number of involved edges in Theorem 7 cannot be increased. We construct such a family of graphs G as follows: We start from two copies of K_2's, say H_1, H_2; and a copy of K_n, say H_3, where $n \geq 2$. Let $V(H_1) = \{u_1, u_2\}$, $E(H_1) = \{e_1\}$, $V(H_2) = \{v_1, v_2\}$, $E(H_2) = \{e_2\}$ and let $\{w_1, w_2\} \subseteq V(H_3)$. We join w_1 to v_1, w_2 to v_2 and every vertex of H_1 to all the vertices of H_2 and H_3. The resulting graph G is clearly 2-tough and furthermore if we let e_3 to be the edge joining w_2 to v_2, the graph $G - \{e_2, e_3\}$ will not have a 2-factor containing e_1. For the proof of our last statement, we work as follows: We subdivide the edge e_1 of the graph $G - \{e_2, e_3\}$. Let G^* be the resulting graph. Then $G - \{e_2, e_3\}$ has a 2-factor containing e_1 if and only if G^* has a 2-factor. But we will prove G^* does not possess a 2-factor. Let $S = V(H_1)$, $T = \{v_1, v_2, v\}$ where $V(G^*) - V(G) = \{v\}$. Then

$$q_{G^*}(S,T;2) + \sum_{x \in T}(2 - \deg_{G^*-S}(x)) > 2|S|$$

since $q_{G^*}(S,T;2) = 1$, $\sum_{x \in T}(2 - \deg_{G^*-S}(x)) = 5$ and $|S| = 2$.

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