# Algebraic Properties and Panconnectivity of Folded Hypercubes*

Meijie Ma[a†] Jun-Ming Xu[b]
[a]Department of Mathematics, Zhejiang Normal University
Jinhua, 321004, China
[b]Department of Mathematics, University of Science and Technology of China
Hefei, 230026, China

**Abstract** This paper considers the folded hypercube $FQ_n$, as an enhancement on the hypercube, and obtains some algebraic properties of $FQ_n$. Using these properties the authors show that for any two vertices $x$ and $y$ in $FQ_n$ with distance $d$ and any integers $h \in \{d, n+1-d\}$ and $l$ with $h \leq l \leq 2^n - 1$, $FQ_n$ contains an $xy$-path of length $l$ and no $xy$-path of other length provided that $l$ and $h$ have the same parity.

**Keywords**: Path, Folded hypercube, Transitivity, Panconnectivity

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## 1 Introduction

It is well-known that a topological structure for an interconnection network can be modelled by a connected graph $G = (V, E)$ [14]. As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties [8, 14]. The $n$-dimensional hypercube $Q_n$ is a graph with $2^n$ vertices, each vertex with a distinct binary string $x_1x_2 \cdots x_n$ of length $n$ on the set $\{0, 1\}$, and two vertices being linked by an edge if and only if their strings differ in exactly one bit.

As a variant of the hypercube, the $n$-dimensional folded hypercube $FQ_n$, proposed first by El-Amawy and Latifi [3], is a graph obtained from

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† Corresponding author: mameij@zjnu.cn

the hypercube $Q_n$ by adding an edge between any two complementary vertices $x = (x_1x_2 \cdots x_n)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, where $\bar{x}_i = 1 - x_i$. We call these added edges complementary edges, to distinguish them from the edges, called regular edges, in $Q_n$.

From definitions, $Q_n$ is a proper spanning subgraph of $FQ_n$, and so $FQ_n$ has $2^n$ vertices. It has been shown that $FQ_n$ is $(n+1)$-regular $(n+1)$-connected, and has diameter $\lceil \frac{n}{2} \rceil$, about half the diameter of $Q_n$ [3]. Thus, the folded hypercube $FQ_n$ is an enhancement on the hypercube $Q_n$ and has recently attracted many researchers' attention [2, 4, 5, 7, 10, 12]. In this paper, we further investigate other topological properties of $FQ_n$, transitivity and panconnectivity.

A graph $G$ is called to be vertex-transitive if for any $x, y \in V(G)$ there is some $\sigma \in \mathrm{Aut}(G)$, the automorphism group of $G$, such that $\sigma(x) = y$; and edge-transitive if for any $xy, uv \in E(G)$ there is some $\phi \in \mathrm{Aut}(G)$ such that $\{\phi(x), \phi(y)\} = \{u, v\}$. It has been known that $Q_n$ is both vertex-transitive and edge-transitive [1]. However, the transitivity of $FQ_n$ has not been proved straightforwardly in the literature. In this paper, we will study some algebraic properties of $FQ_n$. Using these properties, we give another proof of a known result that $FQ_n$ is vertex and edge-transitive.

A graph $G$ is panconnected if for any two different vertices $x$ and $y$ in $G$ and any integer $l$ with $d_G(x,y) \leq l \leq |V(G)| - 1$ there exists an $xy$-path of length $l$, where $d_G(x,y)$ is the distance between $x$ and $y$ in $G$ [11]. It is easy to see that any bipartite graph with at least three vertices is not panconnected. For this reason, Li *et al* [6] suggested the concept of bipanconnected bipartite graphs. A bipartite graph $G$ is called to be bipanconnected if for any two different vertices $x$ and $y$ in $G$ and any integer $l$ with $d_G(x,y) \leq l \leq |V(G)| - 1$ such that $l$ and $d_G(x,y)$ have the same parity there exists an $xy$-path of length $l$. Li *et al* [6] have shown that $Q_n$ is bipanconnected. In this paper, we show that for any two vertices $x$ and $y$ in $FQ_n$ with distance $d$, $FQ_n$ contains an $xy$-path of length $l$ with $h \leq l \leq 2^n - 1$ such that $l$ and $h$ have the same parity, where $h \in \{d, n+1-d\}$. Hence, $FQ_n$ is bipanconnected if $n$ is odd.

The proofs of our results are in Section 2 and Section3, respectively.

## 2 Algebraic Properties

In this section, we study some algebraic properties of $FQ_n$, and as applications, show that $FQ_n$ is vertex and edge-transitive.

The following notations will be used in the proofs of our main results. The symbol $H(x,y)$ denotes the Hamming distance between two vertices $x$ and $y$ in $Q_n$, that is, the number of different bits in the corresponding strings of both vertices. Clearly, $H(x,y) = d_{Q_n}(x,y)$. It is also clear that

$d_{FQ_n}(x,y) = i$ if and only if $H(x,y) = i$ or $n+1-i$. Let $x = 0u$ and $y = 1v$ be two vertices in $FQ_n$. It is easy to count that

$$\begin{aligned} H(0u, 1\bar{v}) = &\ H(0u, 1u) + H(1u, 1\bar{v}) \\ = &\ 1 + [(n-1) - H(1u, 1v)] \\ = &\ n + 1 - H(0u, 1v). \end{aligned} \tag{1}$$

Let $\Gamma$ be a non-trivial finite group, $S$ be a non-empty subset of $\Gamma$ without the identity of $\Gamma$ and with $S^{-1} = S$. The Cayley graph $C_\Gamma(S)$ of $\Gamma$ with respect to $S$ is defined as follows.

$$V = \Gamma; \quad (x,y) \in E \Leftrightarrow x^{-1}y \in S, \text{ for any } x, y \in \Gamma.$$

It has been proved that any Cayley graph is vertex-transitive (see, for example, Theorem 2.2.15 in [14]).

As we have known that the hypercube $Q_n$ is the Cayley graph $C_{Z_2^n}(S)$, where $Z_2$ denotes the additive group of residue classes modulo 2 on the set $\{0,1\}$, $Z_2^n = Z_2 \times Z_2 \times \cdots \times Z_2$, and $S = \{(10\cdots0), (010\cdots0), \cdots, (0\cdots 010\cdots0), \cdots, (0\cdots01)\}$ (see, for example, Example 2 in p89 in [14]). The following theorem shows that $FQ_n$ is also a Cayley graph.

**Theorem 2.1** The folded hypercube $FQ_n \cong C_{Z_2^n}(S \cup \{(11\cdots1)\})$.

**Proof** Clearly, $V(FQ_n) = Z_2^n$. Define a natural mapping

$$\begin{aligned} \varphi: \quad V(FQ_n) &\to Z_2^n \\ x &\mapsto \varphi(x) = x. \end{aligned}$$

Let $x$ and $y$ be any two vertices in $FQ_n$. Since $(x,y) \in E(FQ_n)$ if and only if $H(x,y) = 1$ or $n$. Note that $x^{-1} = x$ for any $x \in Z_2^n$. It follows that $H(x,y) = 1$ if and only if $x^{-1}y \in S$; and $H(x,y) = n$ if and only if $x^{-1}y = (11\cdots1)$, whereby $(x,y) \in E(C_{Z_2^n}(S \cup \{(11\cdots1)\}))$. Thus, $\varphi$ preserves the adjacency of vertices, which implies that $\varphi$ is an isomorphism between $FQ_n$ and $C_{Z_2^n}(S \cup \{(11\cdots1)\})$, and so $FQ_n \cong C_{Z_2^n}(S \cup \{(11\cdots1)\})$. ∎

**Corollary 2.2** The folded hypercube $FQ_n$ is vertex-transitive.

For convenience, we express $FQ_n$ as $FQ_n = L \otimes R$, where $L$ and $R$ are the two $(n-1)$-dimensional subcubes of $Q_n$ induced by the vertices with the leftmost bit is 0 and 1, respectively. A vertex in $L$ will be denoted by $0u$ and a vertex in $R$ denoted by $1v$, where $u$ and $v$ are any two vertices in $Q_{n-1}$. Between $L$ and $R$, apart from the regular edges, there exists a complementary edge joining $0u$ and $1\bar{u} \in R$ for any $0u \in L$.

**Theorem 2.3** Let $\sigma$ be a mapping from $V(FQ_n)$ to itself defined by

$$\begin{cases} \sigma(0u) = 0u \\ \sigma(1u) = 1\bar{u} \end{cases} \quad \text{for any } u \in V(Q_{n-1}). \tag{2}$$

Then $\sigma \in \text{Aut}(FQ_n)$. Moreover, for an edge $(x, y)$ between $L$ and $R$ in $FQ_n$, $(\sigma(x), \sigma(y))$ is complementary if and only if $(x, y)$ is regular.

**Proof** Clearly, $\sigma$ is a permutation on $V(FQ_n)$. To show $\sigma \in \text{Aut}(FQ_n)$, it is sufficient to show that $\sigma$ preserves adjacency of vertices in $FQ_n$, that is, to show that any pair of vertices $x$ and $y$ in $FQ_n$ satisfies the following condition.

$$(x, y) \in E(FQ_n) \Leftrightarrow (\sigma(x), \sigma(y)) \in E(FQ_n). \tag{3}$$

Let $FQ_n = L \otimes R$, $u$ and $v$ be any two distinct vertices in $Q_{n-1}$. Because of vertex-transitivity of $FQ_n$ by Theorem 2.1, without loss of generality, suppose $x = 0u \in L$. We consider two cases according to the location of $y$.

*Case 1* $y \in L$. In this case, let $y = 0v$. Since $\sigma$ is the identical permutation on $L \cong Q_{n-1}$, it is clear that

$$(0u, 0v) \in E(FQ_n) \Leftrightarrow (\sigma(0u), \sigma(0v)) = (0u, 0v) \in E(FQ_n).$$

*Case 2* $y \in R$. In this case, let $y = 1v$. By the definition of $FQ_n$, $(0u, 1v) \in E(FQ_n) \Leftrightarrow v = u$ or $\bar{u}$. Since $(0u, 1u), (0u, 1\bar{u}) \in E(FQ_n)$ by the definition of $FQ_n$, it follows that

$$\begin{array}{ll} (0u, 1u) \in E(FQ_n) \Leftrightarrow (\sigma(0u), \sigma(1u)) = (0u, 1\bar{u}) \in E(FQ_n) & \text{if } v = u \\ (0u, 1\bar{u}) \in E(FQ_n) \Leftrightarrow (\sigma(0u), \sigma(1\bar{u})) = (0u, 1u) \in E(FQ_n) & \text{if } v = \bar{u}. \end{array}$$

From the above arguments, we have shown $\sigma \in \text{Aut}(FQ_n)$.

We now show the remaining part of the theorem. Without loss of generality, we may suppose $x = 0u$ since $FQ_n$ is vertex-transitive. By (2), we have $\sigma(x) = \sigma(0u) = 0u$.

Suppose that $(x, y)$ is a regular edge between $L$ and $R$ in $FQ_n$. Then $y = 1u$ and $\sigma(y) = \sigma(1u) = 1\bar{u}$ by (2). By (3) $(0u, 1\bar{u}) \in E(FQ_n)$, which is a complementary edge.

Conversely, suppose that $(x, y)$ is a complementary edge in $FQ_n$. Then $y = 1\bar{u}$ and $\sigma(1\bar{u}) = 1u$ by (2), and $(0u, 1u) \in E(FQ_n)$ by (3), which is a regular edge.

The lemma follows. ∎

**Theorem 2.4** $\text{Aut}(Q_n)$ is a proper subgroup of $\text{Aut}(FQ_n)$. Moreover, for any $\sigma \in \text{Aut}(Q_n)$, $(x, y)$ is a complementary edge if and only if $(\sigma(x), \sigma(y))$ is also a complementary edge in $FQ_n$.

**Proof** For any element $\sigma \in \text{Aut}(Q_n)$, we will prove $\sigma \in \text{Aut}(FQ_n)$.

It is clear that $\sigma$ is a permutation on $V(FQ_n)$ since $Q_n$ is a spanning subgraph of $FQ_n$. We only need to show that $\sigma$ preserves adjacency of vertices in $FQ_n$, that is, to check that (3) holds for any pair of vertices $x$ and $y$ in $FQ_n$. In fact, since

$$H(x, y) = d_{Q_n}(x, y) = d_{Q_n}(\sigma(x), \sigma(y)) = H(\sigma(x), \sigma(y))$$

and

$$(x, y) \in E(FQ_n) \Leftrightarrow H(x, y) = 1 \text{ or } n,$$

we have

$$\begin{aligned}(x, y) \in E(FQ_n) &\Leftrightarrow H(x, y) = 1 \text{ or } n \\ &\Leftrightarrow H(\sigma(x), \sigma(y)) = 1 \text{ or } n \\ &\Leftrightarrow (\sigma(x), \sigma(y)) \in E(FQ_n).\end{aligned}$$

Thus, Aut $(Q_n) \subseteq$ Aut $(FQ_n)$. It is clear that the automorphism $\sigma$ defined by (2) is not in Aut $(Q_n)$ by Theorem 2.3. Therefore, Aut $(Q_n)$ is a proper subgraph of Aut $(FQ_n)$.

By the definition of $FQ_n$, for any $\sigma \in$ Aut $(Q_n)$, it is clear that $(x, y)$ is a complementary edge in $FQ_n$ if and only if $n = H(x, y) = H(\sigma(x), \sigma(y))$, if and only if $\sigma(x, y) = (\sigma(x), \sigma(y))$ is a complementary edge in $FQ_n$.

The theorem follows. ∎

**Corollary 2.5** The folded hypercube $FQ_n$ is edge-transitive.

**Proof** For any two edges $(x, y)$ and $(x', y')$ in $FQ_n$, we will show there is an element $\sigma \in$ Aut $(FQ_n)$ such that $\{\sigma(x), \sigma(y)\} = \{x', y'\}$. Since $FQ_n$ is vertex-transitive, we may assume $x = x'$. We only need to find $\sigma \in$ Aut $(FQ_n)$ that takes $y$ to $y'$ and fixes $x$. Since for any two vertices $z$ and $t$ in $FQ_n$, $(z, t) \in E(FQ_n)$ if and only if $H(z, t) = 1$ or $n$. Without loss of generality, we may suppose that $H(x, y) = 1$, that is, $(x, y)$ is a regular edge in $FQ_n$.

If $H(x, y') = 1$, then $(x, y')$ is a regular edge. Since $Q_n$ is edge-transitive, there is an element $\sigma \in$ Aut $(Q_n)$ such that $\{\sigma(x), \sigma(y)\} = \{x, y'\}$. By Theorem 2.4, $\sigma \in$ Aut $(FQ_n)$, which satisfies our requirement.

If $H(x, y') = n$, then $y' = \bar{x}$ and $(x, y')$ is a complementary edge in $FQ_n$. Without loss of generality, we may suppose that $x = 0u$. Then $y' = 1\bar{u}$. Let $z = 1u$. Then the automorphism $\sigma$ defined in (2) can take $z$ to $y'$ and fixes $x$. If $y = z$, then the $\sigma$ satisfies our requirement. If $y \neq z$, then there is $\phi \in$ Aut $(Q_n) \subset$ Aut $(FQ_n)$ such that $\phi$ takes $y$ to $z$ and fixes $x$. Thus, $\sigma\phi(y) = \sigma(\phi(y)) = \sigma(z) = y'$ and $\sigma\phi(x) = \sigma(\phi(x)) = \sigma(x) = x$, and so $\sigma\phi$ satisfies our requirement.

The corollary follows. ∎

# 3 Panconnectivity

In this section, we investigate the panconnectivity of $FQ_n$. The proof of the main theorem in this section is strongly dependent on the following lemmas.

**Lemma 3.1** [6] If $n \geq 2$, then $Q_n$ is bipanconnected, that is, for any two vertices $x$ and $y$ in $Q_n$ there exists an $xy$-path of length $l$ with $H(x, y) \leq l \leq 2^n - 1$ such that $l$ and $H(x, y)$ have the same parity.

**Lemma 3.2** [13] $FQ_n$ is a bipartite graph if and only if $n$ is odd. Moreover, if $n$ is even, then the length of the shortest odd cycle in $FQ_n$ is $n + 1$.

**Theorem 3.3** For any two distinct vertices $x$ and $y$ in $FQ_n$ with distance $d$, $FQ_n$ contains an $xy$-path of length $l$ with $h \leq l \leq 2^n - 1$ such that $l$ and $h$ have the same parity, where $h \in \{d, n + 1 - d\}$.

**Proof** If $n = 1$, the theorem is true clearly since $FQ_1 = K_2$. Assume $n \geq 2$ below. Without loss of generality, we may assume $x = 0u, y = 1v$ since $d \geq 1$ and $FQ_n$ is vertex-transitive by Corollary 2.2. We first deduce two conclusions from Lemma 3.2 and Theorem 2.3.

(a) By Lemma 3.1, $Q_n$ contains an $xy$-path $P$ of length $l$ with $H(x, y) \leq l \leq 2^n - 1$ such that $l$ and $H(x, y)$ have the same parity. Since $Q_n$ is a spanning subgraph of $FQ_n$, $P$ is an $xy$-path of length $l$ in $FQ_n$.

(b) Consider the vertex $z = 1\bar{v}$. By Lemma 3.1, $Q_n$ contains an $xz$-path $R$ of length $l'$ with $H(x, z) \leq l' \leq 2^n - 1$ such that $l'$ and $H(x, z)$ have the same parity. Since $H(x, z) = n + 1 - H(x, y)$ by (1), $l'$ and $n + 1 - H(x, y)$ have the same parity. Let $\sigma \in \mathrm{Aut}\,(FQ_n)$ defined in (2). Then $P' = \sigma(R)$ is an $xy$-path of length $l'$ with $n + 1 - H(x, y) \leq l' \leq 2^n - 1$ such that $l'$ and $n + 1 - H(x, y)$ have the same parity.

To prove the theorem, it is sufficient to check that $H(x, y) = d$ or $n + 1 - d$. In fact, it is clear that if $H(x, y) \leq \lceil \frac{n}{2} \rceil$ then $d = H(x, y)$; if $H(x, y) > \lceil \frac{n}{2} \rceil$ then $H(x, y) = n - d + 1$. The theorem is proved. ∎

**Corollary 3.4** If $n$ is odd, then $FQ_n$ is bipanconncted.

**Proof** If $n$ is odd, then $FQ_n$ is a bipartite graph by Lemma 3.2. Let $x$ and $y$ be any two vertices in $FQ_n$ with distance $d$. Since $n$ is odd, the condition that $l$ and $n + 1 - d$ have the same parity implies that $l$ and $d$ have the same parity. Note that $d \leq n + 1 - d$ since $d \leq \lceil \frac{n}{2} \rceil$. By Theorem 3.3, $FQ_n$ contains an $xy$-path of length $l$ with $d \leq l \leq 2^n - 1$ such that $l$ and $d$ have the same parity, and so $FQ_n$ is bipanconncted. ∎

**Corollary 3.5** If $n$ is even then for any two different vertices $x$ and $y$ with $d_{FQ_n}(x, y) = d$ in $FQ_n$, there is an $xy$-path of length $l$ for each $l$ satisfying $n - d + 1 \leq l \leq 2^n - 1$ and there is also an $xy$-path of length $l'$ for each $l'$ satisfying $d \leq l' \leq n - d$ such that $l'$ and $d$ have the same parity; there is no $xy$-path of other length.

**Proof** If $n$ is even, then $d$ and $n - d + 1$ have different parity. Thus, for any integer $l$, either $l$ and $d$ have the same parity, or $l$ and $n - d + 1$

have the same parity. Since $d \le \frac{n}{2}$, $d < n - d + 1$. By Theorem 3.3, there is an $xy$-path of length $l$ with $n - d + 1 \le l \le 2^n - 1$ in $FQ_n$.

Since the length of the shortest odd cycle in $FQ_n$ is $n+1$ by Lemma 3.2, $FQ_n$ contains no $xy$-path of length $l$ with $d < l \le n - d$ if $l$ and $d$ have different parity. In other words, the length $l$ of the second shortest path between $x$ and $y$ with distance $d$ is certainly $n - d + 1$ if $l$ and $d$ have different parity. It follows from Theorem 3.3 that there is an $xy$-path of length $l'$ with $d \le l' \le n - d$ provided $l'$ and $d$ have the same parity.

The corollary is proved. ∎

A graph is called to be hamiltonian connected if there is a hamiltonian path between any two vertices. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected. For this reason, Simmons [9] introduces the concept of hamiltonian laceable for hamiltonian bipartite graphs. A hamiltonian bipartite graph is hamiltonian laceable if there is a hamiltonian path between any two vertices in different bipartite sets. It is clear that if a bipartite graph is bipanconnected then it is certainly hamiltonian laceable. It follows from Corollary 3.4 and Corollary 3.5 that the following result is true clearly.

**Corollary 3.6** *$FQ_n$ is hamiltonian laceable if $n$ is odd, and hamiltonian connected if $n$ is even.* ∎

# References


[1] A.E. Brower, A.M. Cohen and A. Neumaier, Distance Regular Graphs, Springer, Berlin, 1989.

[2] D. R. Duh, G. H. Chen and J. F. Fang, Algorithms and properties of a new two-level network with folded hypercubes as basic modules, *IEEE Trans. Parallel and Distributed Systems*, **6**(7) (1995), 714-723.

[3] A. El-Amawy and S. Latifi, Properties and performance of folded hypercubes, *IEEE Trans. Parallel and Distrib. Syst.*, **2** (1991), 31-42.

[4] X. Hou, M. Xu and J.-M. Xu, Forwarding indices of folded $n$-cubes, *Discrete Applied Mathematics*, **145** (2005), 490-492.

[5] C. N. Lai, G. H. Chen and D. R. Duh, Constructing one-to-many disjoint paths in folded hypercubes, *IEEE Trans. Comput.*, **51** (1) (2002), 33-45.

[6] L. K. Li, C. H. Tsai, J. M. Tan and L. H. Hsu, Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes, *Information Processing Letters*, **87** (2003), 107-110.

[7] S. C. Liaw and G. J. Chang, Generalized diameters and Rabin numbers of networks, *Journal of Combinatorial Optimization*, **2** (1998), 371-384.

[8] Y. Saad and M. H. Schultz, Topological properties of hypercubes, *IEEE Trans. Comput.*, **37**(7) (1988), 867-872.

[9] G. Simmons, Almost all $n$-dimensional rectangular lattices are hamilton laceable. *Congr. Numer.*, **21** (1978), 103-108.

[10] E. Simó and J. L. A. Yebra, The vulnerability of the diameter of folded $n$-cubes, *Discrete Math.*, **174** (1997), 317-322.

[11] Z. M. Song and Y. S. Qin, A new sufficient condition for panconnected graphs, *Ars Combin.*, **34** (1992), 161-166.

[12] D. Wang, Embedding hamiltonian cycles into folded hypercubes with faulty links, *J. Parallel and Distrib. Comput.*, **61** (2001), 545-564.

[13] J. M. Xu and M. Ma, Cycles in folded hypercubes, *Applied Mathematics Letters*, **19** (2006), 140-145.

[14] Junming Xu, *Topological Structure and Analysis of Interconnection Networks*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.