ON THE MAX-TYPE EQUATION $x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-1}\right\}$

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Abstract
It is shown that every well-defined solution to the second-order difference equation in the title, when $(A_n)_{n\in \ _0}$ is a two-periodic sequence such that $\max\{A_0, A_1\} \geq 0$, is eventually periodic with period two. In the case $\max\{A_0, A_1\} < 0$ it is shown the existence of unbounded solutions, by describing all solutions in terms of A_0, A_1, x_{-1} and x_0.

1. INTRODUCTION

Studying max-type difference equations, attracted some attention recently, see, e.g., [1]-[11], and the references therein. This type of difference equations stem from certain models in automatic control theory (see [3] and [4]).

The study of max-type equations of the following form

$$x_n = \max\left\{B_n^{(0)}, B_n^{(1)}\frac{x_{n-p_1}^{r_1}}{x_{n-q_1}^{s_1}}, B_n^{(2)}\frac{x_{n-p_2}^{r_2}}{x_{n-q_2}^{s_2}}, \ldots, B_n^{(k)}\frac{x_{n-p_k}^{r_k}}{x_{n-q_k}^{s_k}}\right\}, \quad n \in \mathbb{N}_0, \qquad (1)$$

where $k \in \mathbb{N}$, p_i, q_i are natural numbers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_k$, $r_i, s_i \in \mathbb{R}_+, i = 1, \ldots, k$ and $B_n^{(j)}$, $j = 0, 1, \ldots, k$, are real sequences, was proposed by the third author of this note in several talks (see, e.g., [5]).

Definition A sequence $(x_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there is $n_0 \in \{-k, \ldots, -1, 0, 1, \ldots\}$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$, then we say that the sequence $(x_n)_{n=-k}^{\infty}$ is periodic with period p.

One of the problems suggested by S. Stević was the following:

Research Problem. *Investigate the behavior of solutions to the difference equation*

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, x_{n-m}\right\}, \quad n \in \mathbb{N}_0, \qquad (2)$$

in terms of the sequence $(A_n)_{n\in\mathbb{N}_0}$ and $k, m \in \mathbb{N}_0$.

In [1] we showed that, when $k = 0$, $m = 2$ and $A_n = A$, $n \in \mathbb{N}_0$, every well-defined solution to Eq. (2) is eventually periodic with period three. The periodicity in this case is not so surprising. Namely, note that by using the change $x_n = \sqrt{|A|}\, y_n$, Eq. (2) in this case is reduced to the case $A = \pm 1$, which is a particular case of the following difference equation:

$$x_n = \max\{(-1)^{\gamma_1} x_{n-p_1}^{\delta_1}, \ldots, (-1)^{\gamma_k} x_{n-p_k}^{\delta_k}\}, \qquad (3)$$

where $p_i \in \mathbb{N}$, $1 \leq p_1 < \cdots < p_k$, $\gamma_i \in \{0, 1\}$ and $\delta_i \in \{-1, 1\}$, $i = 1, \ldots, k$.

This equation, although looks complicating, is a particular case of a large (folklore) class of equations whose solutions posses a simple, but interesting property mentioned in the following proposition.

Proposition 1. *Assume $f : A^k \to A$, $A \subset \mathbb{R}$, $k \in \mathbb{N}$, is such that for each vector $\vec{v} \in A^k$ the set $\{f^{[j]}(\vec{v}) : j \in \mathbb{N}\}$ is a subset of a finite set $S(\vec{v}) \subset \mathbb{R}$, where for each j, $f^{[j]}(\vec{v})$ is defined. Then every solution to the difference equation*

$$x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N},$$

with $(x_{-k}, \ldots, x_{-1}) \in A^k$ is eventually periodic.

Corollary 1. *Every well-defined solution of Eq.* (3) *is eventually periodic.*

Proof. Assume $x_{-p_k}, \ldots, x_{-1}$ are initial values of a well-defined solution of Eq. (3). Since the functions $g_{\pm}(x) = \pm x$ and $h_{\pm}(x) = \pm x^{-1}$ are odd involutions it is easy to see that the values of the sequence $(x_n)_{n=-p_k}^{\infty}$ belong to the set

$$S_1 = \{\pm x_{-p_k}, \ldots, \pm x_{-1}, \pm 1/x_{-p_k}, \ldots, \pm 1/x_{-1}\}.$$

Applying Proposition 1 the corollary follows. □

Here, among others, we show that every well-defined solution to the equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, x_{n-1}\right\}, \quad n \in \mathbb{N}_0, \tag{4}$$

where A_n is two-periodic with $\max\{A_0, A_1\} \geq 0$, is eventually periodic with period two. In the other cases we describe the behavior of solutions of Eq. (4).

Remark 1. Note that if $A_0 = A_1 = 0$, then Eq. (4) becomes $x_{n+1} = x_{n-1}$. Hence, in the sequel we consider the case when at least one of A_0 and A_1 is not zero.

In the rest of the paper we frequently use the following simple lemma.

Lemma 1. *Assume that $(x_n)_{n=-1}^{\infty}$ is a solution of Eq.* (4) *and there is $k_0 \in \mathbb{N}_0 \cup \{-1\}$ such that $x_{k_0} = x_{k_0+2}$ and $x_{k_0+1} = x_{k_0+3}$. Then this solution is eventually periodic with period two.*

Proof. By the method of induction we prove that

$$x_{k_0} = x_{k_0+2m} \quad \text{and} \quad x_{k_0+1} = x_{k_0+2m+1}, \quad m \in \mathbb{N}. \tag{5}$$

For $m = 1$ this is clear. Assume (5) holds for $1 \leq n \leq m_0$. We may assume k_0 is odd (case when k_0 is even, is dual). Then, by the hypothesis we have

$$x_{k_0+2m_0+2} = \max\{A_0/x_{k_0+2m_0+1}, x_{k_0+2m_0}\} = \max\{A_0/x_{k_0+1}, x_{k_0}\} = x_{k_0+2} = x_{k_0}.$$

From this and the inductive hypothesis we have

$$\begin{aligned} x_{k_0+2m_0+3} &= \max\{A_1/x_{k_0+2m_0+2}, x_{k_0+2m_0+1}\} = \max\{A_1/x_{k_0+2}, x_{k_0+1}\} \\ &= x_{k_0+3} = x_{k_0+1}, \end{aligned}$$

finishing the inductive proof of the result. □

Lemma 2. *Assume that for a solution* $(x_n)_{n=-1}^{\infty}$ *of Eq.* (4), *there is* $k_1 \in \mathbb{N}_0 \cup \{-1\}$ *such that*

$$x_{k_1} > 0 \quad \textit{and} \quad x_{k_1+1} > 0. \tag{6}$$

Then $x_n > 0$ *for* $n \geq k_1$.

Proof. The lemma is also proved by induction. For $n = k_1$ it is contained in (6). Assume we have proved that $x_n > 0$ for $k_1 \leq n \leq m_1$. Then $x_{m_1+1} = \max\{A_{m_1}/x_{m_1}, x_{m_1-1}\} \geq x_{m_1-1} > 0$, from which the lemma follows. □

Lemma 3. *Assume that a solution* $(x_n)_{n=-1}^{\infty}$ *of Eq.* (4) *is eventually positive. Then it is eventually periodic with period two.*

Proof. Assume that $k_1 \in \mathbb{N}_0 \cup \{-1\}$ is the smallest index such that (6) holds. Then from Eq. (4) we have that

$$x_{n+1}x_n = \max\{A_n, x_n x_{n-1}\}, \quad n \geq k_1 + 1. \tag{7}$$

Using (7) twice we obtain

$$\begin{aligned} x_{k_1+3}x_{k_1+2} &= \max\{A_{k_1+2}, x_{k_1+2}x_{k_1+1}\} = \max\{A_{k_1+2}, \max\{A_{k_1+1}, x_{k_1+1}x_{k_1}\}\} \\ &= \max\{A_0, A_1, x_{k_1+1}x_{k_1}\}. \end{aligned} \tag{8}$$

Now we prove by induction that

$$x_{n+1}x_n = \max\{A_0, A_1, x_{k_1+1}x_{k_1}\}, \quad n \geq k_1 + 2. \tag{9}$$

For $n = k_1 + 2$ this is (8). Assume the statement was proved for $k_1 + 2 \leq n \leq n_2$. Then from the inductive hypothesis and (8) we have

$$\begin{aligned} x_{n_2+2}x_{n_2+1} &= \max\{A_{n_2+1}, x_{n_2+1}x_{n_2}\} = \max\{A_{n_2+1}, \max\{A_{n_2}, x_{n_2}x_{n_2-1}\}\} \\ &= \max\{A_0, A_1, x_{n_2}x_{n_2-1}\} = \max\{A_0, A_1, x_{k_1+1}x_{k_1}\}, \end{aligned}$$

as claimed. From (9) it follows that $(x_n)_{n=-1}^{\infty}$ is an eventually periodic solution of Eq. (4) with period two, finishing the proof of the lemma. □

Lemma 4. *Assume that a solution* $(x_n)_{n=-1}^{\infty}$ *of Eq.* (4) *is eventually negative. Then it is eventually periodic with period two.*

Proof. Assume $k_2 \in \mathbb{N}_0 \cup \{-1\}$ is the smallest index such that $x_n < 0$, for $n \geq k_2$. Then by the change $x_n = -y_n$, Eq. (4) is transformed into the equation

$$y_{n+1} = \min\{A_n/y_n, y_{n-1}\},$$

where $y_n > 0$ for $n \geq k_2$. From this we have

$$\begin{aligned} x_{k_2+3}x_{k_2+2} &= \min\{A_{k_2+2}, x_{k_2+2}x_{k_2+1}\} = \min\{A_{k_2+2}, \min\{A_{k_2+1}, x_{k_2+1}x_{k_2}\}\} \\ &= \min\{A_0, A_1, x_{k_2+1}x_{k_2}\}. \end{aligned}$$

This along with the induction implies $x_{n+1}x_n = \min\{A_0, A_1, x_{k_2+1}x_{k_2}\}$, $n \geq k_2 + 2$, from which the result easily follows. □

2. Main Results

In this section we prove our main results in this paper.

Theorem 1. *Assume that one of the numbers A_0 or A_1 is equal to zero. Then every well-defined solution of Eq. (4) is eventually periodic with period two.*

Proof. First assume $A_0 = 0$. Then we have

$$x_{2n+1} = \max\{A_0/x_{2n}, x_{2n-1}\} = x_{2n-1}, \quad n \in \mathbb{N}.$$

Further we have $x_2 = \max\{A_1/x_1, x_0\}$. If $x_0 \geq \frac{A_1}{x_1}$, then $x_2 = x_0$. From this, since $x_1 = x_{-1}$ and by using Lemma 1 the result follows in this case.

If $x_0 \leq \frac{A_1}{x_1}$, then $x_2 = \frac{A_1}{x_1}$. Since $x_3 = x_1$ we have $x_4 = \max\{A_1/x_3, x_2\} = \max\{A_1/x_1, x_2\} = x_2$. By Lemma 1 the result follows in this case.

The case $A_1 = 0$ can be treated similarly, hence we omit its proof. □

Theorem 2. *Assume $A_0 \neq 0 \neq A_1$ and, $A_0 > 0$ or $A_1 > 0$. Then every well-defined solution of Eq. (4) is eventually periodic with period two.*

Proof. Since there are four independent parameters A_0, A_1, x_{-1} and x_0, there are twelve cases to be considered, with respect to their signs.

Cases (i)-(iii) Assume $x_{-1} > 0$ and $x_0 > 0$. Then by Lemma 2 we have $x_n > 0$ for $n \geq -1$, so by Lemma 3 such solutions are periodic with period two.

Case (iv) Assume $A_0 > 0$, $A_1 > 0$, $x_{-1} < 0$ and $x_0 < 0$. Then by induction we see that $x_n < 0$, $n \geq -1$. Thus the result in this case follows from Lemma 4.

Case (v) Assume $A_0 > 0$, $A_1 > 0$, $x_{-1} < 0$ and $x_0 > 0$. We have $x_1 = \max\{A_0/x_0, x_{-1}\} = A_0/x_0 > 0$. Hence this case is reduced to a case in (i)-(iii).

Case (vi) Assume that $A_0 > 0$, $A_1 > 0$, $x_{-1} > 0$ and $x_0 < 0$. We have $x_1 = \max\{A_0/x_0, x_{-1}\} = x_{-1} > 0$. Hence the case is reduced to Case (v).

Case (vii) Assume that $A_0 > 0$, $A_1 < 0$, $x_{-1} > 0$ and $x_0 < 0$. We have $x_1 = \max\{A_0/x_0, x_{-1}\} = x_{-1} > 0$. Further we have $x_2 = \max\{A_1/x_1, x_0\} < 0$.

If $x_0 \geq \frac{A_1}{x_1}$, then $x_2 = x_0$, and from Lemma 1 the result follows.

If $x_0 \leq \frac{A_1}{x_1}$, then $x_2 = \frac{A_1}{x_1}$. Hence $x_3 = \max\{A_0/x_2, x_1\} = x_1$ and $x_4 = \max\{A_1/x_3, x_2\} = \max\{A_1/x_1, x_2\} = x_2$, from which again by Lemma 1 the result follows in the case.

Case (viii) Assume $A_0 > 0$, $A_1 < 0$, $x_{-1} < 0$ and $x_0 > 0$. Then $x_1 = \max\{A_0/x_0, x_{-1}\} = A_0/x_0 > 0$, so this case is reduced to one in cases (i)-(iii).

Case (ix) Assume that $A_0 > 0$, $A_1 < 0$, $x_{-1} < 0$ and $x_0 < 0$. Then $x_1 = \max\{A_0/x_0, x_{-1}\} < 0$ and $x_2 = \max\{A_1/x_1, x_0\} = A_1/x_1 > 0$. By using the change $y_n = x_{n+2}$ this case is reduced to Case (viii).

Case (x) Assume that $A_0 < 0$, $A_1 > 0$, $x_{-1} < 0$ and $x_0 > 0$. Then $x_1 = \max\{A_0/x_0, x_{-1}\} < 0$.

Now if $\frac{A_0}{x_0} \leq x_{-1}$, then $x_1 = x_{-1}$. From this and since $x_2 = \max\{A_1/x_1, x_0\} = x_0$, by Lemma 1 the result follows in this case.

If $\frac{A_0}{x_0} \geq x_{-1}$, then $x_1 = \frac{A_0}{x_0}$. Of course, $x_2 = x_0$, and consequently $x_3 = \max\{A_0/x_2, x_1\} = \max\{A_0/x_0, x_1\} = x_1$, from which the result follows.

Case (xi) Assume $A_0 < 0$, $A_1 > 0$, $x_{-1} > 0$ and $x_0 < 0$. Then $x_1 = \max\{A_0/x_0, x_{-1}\} > 0$. The change $z_n = x_{n+1}$ reduces the case to Case (viii).

Case (xii) Assume that $A_0 < 0$, $A_1 > 0$, $x_{-1} < 0$ and $x_0 < 0$. Then

$$x_1 = \max\{A_0/x_0, x_{-1}\} = A_0/x_0 > 0 \text{ and } x_2 = \max\{A_1/x_1, x_0\} = A_1/x_1 > 0,$$

so that this case is reduced to one of the cases in (i)-(iii). □

From Remark 1 and Theorems 1 and 2 we obtain the following result:

Theorem 3. *Assume that $(A_n)_{n\in\mathbb{N}_0}$ is a two-periodic sequence such that A_0 and A_1 are not both negative. Then every well-defined solution of Eq. (4) is eventually periodic with period two.*

Case $A_0 < 0$ and $A_1 < 0$. Now we describe the behavior of solutions of Eq. (4) when $A_0 < 0$ and $A_1 < 0$, in terms of A_0, A_1, x_{-1} and x_0. In many cases we also obtain that solutions of Eq. (4) are eventually periodic with period two, however there are some cases in which solutions are not periodic.

Case (a) Assume $A_0 < 0$, $A_1 < 0$, $x_{-1} > 0$ and $x_0 > 0$. Then by Lemma 2, we have that $x_n > 0$ for $n \geq -1$, so by Lemma 3 such solutions are periodic with period two.

Case (b) Assume that $A_0 < 0$, $A_1 < 0$, $x_{-1} < 0$ and $x_0 > 0$. Then $x_1 = \max\{A_0/x_0, x_{-1}\} < 0$. There are two cases to be considered.

If $\frac{A_0}{x_0} \leq x_{-1}$, then $x_1 = x_{-1}$. We have $x_2 = \max\{A_1/x_1, x_0\} > 0$.

Now, there are two subcases. If $x_0 \geq \frac{A_1}{x_1}$, then $x_2 = x_0$ and by Lemma 1 it follows that these solutions are periodic with period two.

If $x_0 \leq \frac{A_1}{x_1}$, then $x_2 = \frac{A_1}{x_1}$. Further we have $x_3 = \max\{A_0/x_2, x_1\} = \max\{A_0/A_1x_1, x_1\}$.

If $x_3 = x_1$, which is equivalent with $A_0 \leq A_1 < 0$, then

$$x_4 = \max\{A_1/x_3, x_2\} = \max\{A_1/x_1, x_2\} = x_2,$$

so that these solutions are also eventually periodic with period two.

If $x_3 = \frac{A_0}{x_2}$, which is equivalent to $A_1 \leq A_0 < 0$, we obtain

$$x_4 = \max\{A_1/x_3, x_2\} = \max\{(A_1x_2)/A_0, x_2\} = (A_1x_2)/A_0 = A_1^2/(A_0x_{-1})$$

and

$$x_5 = \max\{A_0/x_4, x_3\} = \max\{(A_0x_3)/A_1, x_3\} = (A_0x_3)/A_1 = (A_0^2x_{-1})/A_1^2.$$

By induction we obtain that

$$x_{2n} = \left(\frac{A_1}{A_0}\right)^n \frac{A_0}{x_{-1}} \quad \text{and} \quad x_{2n+1} = \left(\frac{A_0}{A_1}\right)^n x_{-1}, n \in \mathbb{N}_0. \tag{10}$$

If $\frac{A_0}{x_0} \geq x_{-1}$, then $x_1 = \frac{A_0}{x_0}$. If $x_2 = x_0$ then $x_3 = \max\{A_0/x_2, x_1\} = \max\{A_0/x_0, x_1\} = x_1$. So this solution is eventually periodic with period two.

If $x_2 = \frac{A_1}{x_1}$, then $x_3 = \max\{A_0/x_2, x_1\} = \max\{A_0/A_1x_1, x_1\}$.

If $x_3 = x_1$, which is equivalent with $A_0 \leq A_1 < 0$, then

$$x_4 = \max\{A_1/x_3, x_2\} = \max\{A_1/x_1, x_2\} = x_2,$$

from which two periodicity follows.

If $x_3 = \frac{A_0}{x_2}$, which is equivalent with $A_1 \leq A_0 < 0$, then

$$x_4 = \max\{A_1/x_3, x_2\} = \max\{(A_1x_2)/A_0, x_2\} = (A_1x_2)/A_0$$

and

$$x_5 = \max\{A_0/x_4, x_3\} = \max\{(A_0x_3)/A_1, x_3\} = (A_0x_3)/A_1.$$

By induction we obtain

$$x_{2n} = (A_1/A_0)^n x_0 \quad \text{and} \quad x_{2n+1} = (A_0/A_1)^n A_0/x_0, n \in \mathbb{N}_0. \tag{11}$$

Case (c) Assume $A_0 < 0$, $A_1 < 0$ and $x_0 < 0$. Then $x_1 = \max\{A_0/x_0, x_{-1}\} > 0$. Hence, no matter which sign has x_{-1}, this case is reduced to Case (b), more precisely, if we use the change $z_n = x_{n+1}$ all the results in Case (b) hold for z_n.

Remark 2. Note that in the cases when all the solutions of Eq. (4) are not periodic (see Case (b)), we have explicit formulae (10) and (11) from which the behavior of these solutions easily follows. For example, if $A_0 \neq A_1$ from (10) and (11) the existence of unbounded solutions follows.

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