On the Product Summation of Ordered Partition *

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Abstract

We define a product summation of ordered partition $f_j(n, m, r) = \sum c_1^r c_2^r \cdots c_j^r c_{j+1} \cdots c_m$, where the sum is over all positive integers $c_1, c_2, \cdots, c_m$ with $c_1 + c_2 + \ldots + c_m = n$ and $0 \leq j \leq m$. We concentrate on $f_m(n, m, r)$ in this paper. The main results are as follows:

(1) The generating function for $f_m(n, m, r)$ and the explicit formula for $f_m(n, m, 2)$, $f_m(n, m, 3)$ and $f_m(n, m, 4)$ are obtained.

(2) The relationship between $f_j(n, m, r)$ for $r = 2, 3$ and the Fibonacci and Lucas numbers is found.

Key words: product summation, recurrence, Stirling numbers of the second kind, Fibonacci numbers, Lucas numbers

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1 Introduction

In [1], Louis Comtet gives an exercise to show that $\sum_{c_1+c_2+\cdots+c_k=n} c_1c_2\cdots c_k = \binom{n+k-1}{2k-1}$. Generally, we consider

$$f_j(n,m,r) := \sum_{c_1+c_2+\cdots+c_m=n} c_1^r c_2^r \cdots c_j^r c_{j+1}\cdots c_m,$$

where all integers $c_i > 0$, and $0 \le j \le m$. Then, from Comtet's result, we have

$$f_0(n,m,r) = \sum_{c_1+c_2+\cdots+c_m=n} c_1c_2\cdots c_m = \binom{n+m-1}{2m-1},$$

and

$$f_m(n,m,r) = \sum_{c_1+c_2+\cdots+c_m=n} c_1^r c_2^r \cdots c_m^r.$$

We use $f(n,m,r)$ as an abbreviation for $f_m(n,m,r)$.

F_n is the nth *Fibonacci number* while L_n is the nth *Lucas number*. Then $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integer $n \ge 2$, while $L_0 = 2, L_1 = 1, L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for all integer $n \ge 2$. Clearly, $L_n = F_{n-1} + F_{n+1}$. For integers n and k with $n \ge k \ge 0$, let $S(n,k)$ denote the *Stirling numbers of the second kind.* The $S(n,k)$ satisfies the following recurrence:

$$S(n+1,k) = S(n,k-1) + k \cdot S(n,k). \tag{1}$$

In this paper, let m be a positive integer and we always write

$$\binom{m}{m_1\ ,\ m_2} = \frac{m!}{m_1!m_2!(m-m_1-m_2)!},$$

where the integers $m_1, m_2 \ge 0$ and $0 \le m_1 + m_2 \le m$.

2 The enumeration of $f(n,m,r)$

2.1 The generating function of $f(n,m,r)$

Let $F(t)=\sum_{n\geq 0} f(n,m,r)t^n$. Clearly, $F(t)=(\sum_{c\geq 0} c^r t^c)^m=(\sum_{c\geq 1} c^r t^c)^m$. Let $f_r(t)=\sum_{c\geq 1} c^r t^c$. Then $f_1(t)=\sum_{c\geq 1} ct^c=\frac{t}{(1-t)^2}$ and

$$f_r(t)=tf'_{r-1}(t). \qquad (2)$$

Theorem 2.1 *For all integers $r\geq 2$,*

$$f_r(t)=a_{r1}tf'_1(t)+a_{r2}t^2f''_1(t)+\cdots+a_{r(r-1)}t^{r-1}f_1^{(r-1)}(t), \qquad (3)$$

where $a_{r1}=a_{r(r-1)}=1, a_{r2}=a_{(r-1)1}+2a_{(r-1)2},\cdots,a_{ri}=a_{(r-1)(i-1)}+ia_{(r-1)i},\cdots,a_{r(r-2)}=a_{(r-1)(r-3)}+(r-1)a_{(r-1)(r-2)}$.

Proof. The proof of the theorem is by induction on r. By (2), we have $f_2(t)=tf'_1(t)$. It is apparent that , (3) holds for $r=2$. Suppose (3) holds for $r=k$. We must show that (3) holds for $r=k+1$. By (2) and using the induction hypothesis to $f_k(t)$, we obtain

$$f_{k+1}(t)=tf'_k(t)=t\{tf'_1(t)+\sum_{i=2}^{k-2}(a_{k(i-1)}+ia_{ki})t^if'_1(t)+t^{k-1}f_1^{(k-1)}(t)\}'$$

$$=\sum_{i=1}^{k-1}[ia_{ki}t^if_1^{(i)}(t)+a_{ki}t^{i+1}f_1^{(i+1)}(t)]$$

$$=tf'_1(t)+\sum_{i=2}^{k-1}(a_{k(i-1)}+ia_{ki})t^if_1^{(i)}(t)+t^kf_1^{(k)}(t).$$

Thus the coefficient of $t^if_1^{(i)}(t)$ in $f_{k+1}(t)$ is $a_{ki}=a_{k(i-1)}+ia_{ki}$ for $2\leq i\leq k-1$ and $a_{(k+1)1}=a_{(k+1)k}=1$. Therefore, the theorem holds for all integers $r\geq 2$. ∎

Since the coefficient of $t^if_1^{(i)}(t)$ in $f_r(t)$ is $a_{ri}=a_{(r-1)(i-1)}+ia_{(r-1)i}$ for $2\leq i\leq r-2$ and $a_{r1}=a_{r(r-1)}=1$, then $a_{21}=1$. By (1) and $S(1,1)=1$, $a_{ri}=S(r-1,i)$. The following result is obtained.

Corollary 1 *For all integers $r \geq 2$*

$$f_r(t) = \sum_{i=1}^{r-1} S(r-1,i)t^i f_1^{(i)}(t).$$

Theorem 2.2

$$f_1^{(k)}(t) = \frac{k!(k+t)}{(1-t)^{k+2}}, for \quad k \geq 1. \tag{4}$$

Proof. The proof of the theorem is by the induction on k. Since $f_1(t) = \frac{t}{(1-t)^2}$ and $f_1'(t) = \frac{1}{(1-t)^2} + \frac{2t}{(1-t)^3} = \frac{1+t}{(1-t)^3}$, (4) holds for $k = 1$. Suppose (4) holds for k. We must show that (4) holds for $k+1$. Applying the induction hypothesis to $f_1^{(k)}(t)$, we obtain

$$f_1^{(k+1)}(t) = (f_1^{(k)}(t))' = (\frac{k!(k+t)}{(1-t)^{k+2}})' = \frac{k!}{(1-t)^{k+2}} + \frac{k!(k+2)(k+t)}{(1-t)^{k+3}}$$
$$= \frac{(k+1)![(k+1)+t]}{(1-t)^{k+3}}.$$

∎

By the results of Corollary 1 and Theorem 2.2, we obtain the following result.

Theorem 2.3 *$F(t) = (f_r(t))^m$, where $f_1(t) = \frac{t}{(1-t)^2}$ and for $r \geq 2$*

$$f_r(t) = \sum_{i=1}^{r-1} S(r-1,i)t^i f_1^{(i)}(t).$$

From Theorem 2.3, it is convenient to obtain a formula for $f(n,m,r)$.

i) $f_2(t) = \frac{t(1+t)}{(1-t)^3}$. Then

$$\sum_{n\geq 0} f(n,m,2)t^n = \frac{t^m(1+t)^m}{(1-t)^{3m}} = t^m \sum_{i=0}^{m} \binom{m}{i} t^i \sum_{n\geq 0} \binom{-3m}{n}(-1)^n t^n$$
$$= \sum_{n\geq 0}\sum_{i=0}^{m} \binom{m}{i}\binom{n+2m-i-1}{3m-1} t^n.$$

ii) $f_3(t) = \frac{t(1+t)}{(1-t)^3} + \frac{2t^2(2+t)}{(1-t)^4} = \frac{t(t^2+4t+1)}{(1-t)^4}$. Then

$$\sum_{n\geq 0} f(n,m,3)t^n = \frac{t^m(t^2+4t+1)^m}{(1-t)^{4m}}$$

$$= t^m \sum_{m_1+m_2=0}^{m} \binom{m}{m_1\ ,\ m_2} t^{2m_1}(4t)^{m_2} \sum_{n\geq 0} \binom{-4m}{n}(-1)^n t^n$$

$$= \sum_{n\geq 0} \sum_{m_1+m_2=0}^{m} \binom{m}{m_1\ ,\ m_2} 4^{m_2} \binom{n+4m-1}{4m-1} t^{n+m+2m_1+m_2}$$

$$= \sum_{n\geq 0} \sum_{m_1+m_2=0}^{m} \binom{m}{m_1\ ,\ m_2} \binom{n+3m-2m_1-m_2-1}{4m-1} 4^{m_2} t^n.$$

iii) $f_4(t) = \frac{t(1+t)}{(1-t)^3} + 3 \times \frac{2t^2(2+t)}{(1-t)^4} + \frac{6t^3(3+t)}{(1-t)^5} = \frac{t(1+t)(t^2+10t+1)}{(1-t)^5}$. Then

$$\sum_{n\geq 0} f(n,m,4)t^n$$

$$= \frac{t^m(1+t)^m(t^2+10t+1)^m}{(1-t)^{5m}}$$

$$= t^m \sum_{i=0}^{m} \binom{m}{i} t^i \sum_{m_1+m_2=0}^{m} \binom{m}{m_1\ ;\ m_2} 10^{m_2} t^{2m_1+m_2} \sum_{n\geq 0} \binom{-5m}{n}(-1)^n t^n$$

$$= \sum_{n\geq 0} \sum_{i=0}^{m} \binom{m}{i} \sum_{m_1+m_2=0}^{m} \binom{m}{m_1\ ,\ m_2} 10^{m_2} \binom{n+5m-1}{5m-1} t^{n+m+i+2m_1+m_2}$$

$$= \sum_{n\geq 0} \sum_{i=0}^{m} \sum_{m_1+m_2=0}^{m} \binom{m}{i} \binom{m}{m_1\ ,\ m_2} 10^{m_2} \binom{n+4m-i-2m_1-m_2-1}{5m-1} t^n.$$

Comparing the coefficients of t^n in the first and last equalities of i), ii) and iii) respectively, we have the following results.

Corollary 2

$$f(n,m,2) = \sum_{i=0}^{m} \binom{m}{i} \binom{n+2m-i-1}{3m-1},$$

$$f(n,m,3) = \sum_{m_1+m_2=0}^{m} \binom{m}{m_1\ ,\ m_2} \binom{n+3m-2m_1-m_2-1}{4m-1} 4^{m_2},$$

$$f(n,m,4) = \sum_{i=0}^{m} \sum_{m_1+m_2=0}^{m} \binom{m}{i} \binom{m}{m_1\ ,\ m_2} \binom{n+4m-i-2m_1-m_2-1}{5m-1} 10^{m_2}.$$

2.2 Another formula for $f(n,m,2)$ and $f(n,m,3)$

Lemma 2.4 *Let n, t be nonnegative integers and let s be an integer. Then*

$$\sum_{c=0}^{n} \binom{c+s}{t} = \binom{n+s+1}{t+1}.$$

Proof. It is easy to show the result by induction on t. ∎

Lemma 2.5 *Let n and t be nonnegative integers. Then*

$$\sum_{c=0}^{n} \frac{(n-c)^2 c}{s+1} \binom{c+s}{2s+1} = 4\binom{n+s+2}{2s+5} + \binom{n+s+1}{2s+3}. \qquad (5)$$

Proof. By Lemma 2.4

$$\sum_{c=0}^{n} \frac{(n-c)^2 c}{2(s+1)} \binom{c+s}{2s+1}$$

$$= n^2 \sum_{c=0}^{n} \frac{c}{2(s+1)} \binom{c+s}{2s+1} - 2n \sum_{c=0}^{n} \frac{c^2}{2(s+1)} \binom{c+s}{2s+1} + \sum_{c=0}^{n} \frac{c^3}{2(s+1)} \binom{c+s}{2s+1}$$

$$= n^2 \binom{n+s+1}{2s+3} + \frac{n^2}{2} \binom{n+s+1}{2s+2} - 2n(2s+3) \binom{n+s+2}{2s+4}$$

$$-n(s+1) \binom{n+s+1}{2s+2} + (2s+3)(2s+4) \binom{n+s+2}{2s+5} + (s+2)$$

$$\times (2s+3) \binom{n+s+2}{2s+4} + (s+1)^2 \binom{n+s+2}{2s+3} + \frac{(s+1)^2}{2} \binom{n+s+1}{2s+2}$$

$$= 2\binom{n+s+2}{2s+5} + \frac{1}{2} \binom{n+s+1}{2s+3}.$$

Thus, we obtain (5). ∎

Lemma 2.6 *Let n and t be nonnegative integers. Then*

$$\sum_{c=0}^{n} (n-c)^2 \binom{c+s-1}{2s-1} = \frac{n}{s+1} \binom{n+s}{2s+1}.$$

Proof. By Lemma 2.4

$$\sum_{c=0}^{n}(n-c)^2\binom{c+s-1}{2s-1} = n^2\binom{n+s}{2s} - 4ns\binom{n+s}{2s+1} - 2ns\binom{n+s}{2s}$$
$$+2s(2s+1)\binom{n+s+1}{2s+2} + s^2\binom{n+s}{2s}$$
$$= \frac{n}{s+1}\binom{n+s}{2s+1}.$$

∎

Theorem 2.7 *Let n be a nonnegative integer. Then*

$$f(n,2k,2) = \sum_{i=0}^{k}\binom{k}{i}4^i f(n,2k+i,1), \quad \textit{for all integers } k \geq 1; \tag{6}$$

$$f(n,2k+1,2) = \sum_{i=0}^{k}\binom{k}{i}\frac{4^i n}{2k+i+1}f(n,2k+i+1,1), \quad \textit{for all integers } k \geq 0. \tag{7}$$

Proof. The theorem is proved by induction on k.

For $k=0$, $f(n,1,2)=n^2$. The result obviously holds . For $k=1$, we set $s=0$ in Lemma 2.5 to obtain

$$f(n,2,2) = \sum_{c=0}^{n}(n-c)^2 f(c,1,2) = 4\binom{n+2}{5} + \binom{n+1}{3}.$$

By setting $s=3$ and $s=2$ in Lemma 2.6,

$$f(n,3,2) = \sum_{c=0}^{n}(n-c)^2 f(c,2,2) = 4\sum_{c=0}^{n}(n-c)^2\binom{c+2}{5} + \sum_{c=0}^{n}(n-c)^2\binom{c+1}{3}$$
$$= \frac{4n}{8}\binom{n+3}{7} + \frac{2n}{6}\binom{n+2}{5}.$$

Suppose the theorem is true for $m=2k$ and $m=2k+1$.Then for $m=2k+2$, we apply the induction hypothesis to $f(n,2k+1,2)$ and by Lemma 2.5,

$$f(n,2k+2,2) = \sum_{c=0}^{n}(n-c)^2 f(c,2k+1,2)$$

$$=\sum_{i=0}^{k}\binom{k}{i}4^i\sum_{c=0}^{n}\frac{(n-c)^2c}{2k+i+1}\binom{c+2k+i}{4k+2i+1}$$

$$=\sum_{i=0}^{k}\binom{k}{i}4^i\times\{4\binom{n+2k+i+2}{4k+2i+5}+\binom{n+2k+i+1}{4k+2i+3}\}$$

$$=\sum_{i=0}^{k}\binom{k}{i}4^{i+1}\binom{n+2k+i+2}{4k+2i+5}+\sum_{i=0}^{k}\binom{k}{i}4^i\binom{n+2k+i+1}{4k+2i+3}$$

$$=\sum_{i=1}^{k+1}\binom{k}{i-1}4^i\binom{n+2k+i+1}{4k+2i+3}+\sum_{i=0}^{k}\binom{k}{i}4^i\binom{n+2k+i+1}{4k+2i+3}$$

$$=4^{k+1}\binom{n+3k+2}{6k+5}+\sum_{i=1}^{k}[\binom{k}{i-1}+\binom{k}{i}]4^i\binom{n+2k+i+1}{4k+2i+3}+\binom{n+2k+1}{4k+3}$$

$$=\sum_{i=0}^{k+1}\binom{k+1}{i}\binom{n+2(k+1)+i-1}{4(k+1)+2i-1}4^i.$$

For $m=2k+3$, by Lemma 2.6,

$$f(n,2k+3,2)=\sum_{c=0}^{n}(n-c)^2f(c,2k+2,2)$$

$$=\sum_{i=0}^{k+1}\binom{k+1}{i}4^i\sum_{c=0}^{n}(n-c)^2\binom{c+2(k+1)+i-1}{4(k+1)+2i-1}$$

$$=\sum_{i=0}^{k+1}\binom{k+1}{i}\binom{n+2(k+1)+i-1}{4(k+1)+2i-1}\frac{4^in}{2(k+1)+i+1}.$$

By the principle of mathematical induction, the theorem holds. ∎

Lemma 2.8 *Let n and s be nonnegative integers. Then*

$$\sum_{c=0}^{n}(n-c)^3\binom{c+s-1}{2s-1}=6\binom{n+s+1}{2s+3}+\binom{n+s}{2s+1}.$$

Proof.

$$\sum_{c=0}^{n}(n-c)^3\binom{c+s-1}{2s-1}$$

$$= \sum_{c=0}^{n} n(n-c)^2 \binom{c+s-1}{2s-1} - 2s \sum_{c=0}^{n} \frac{(n-c)^2 c}{2s} \binom{c+s-1}{2s-1}$$

Applying Lemma 2.5 and Lemma 2.6, we have

$$\sum_{c=0}^{n} (n-c)^3 \binom{c+s-1}{2s-1}$$

$$= \frac{n^2}{s+1} \binom{n+s}{2s+1} - 4s \binom{n+s+1}{2s+3} - s \binom{n+s}{2s+1}$$

$$= (4s+6) \binom{n+s+1}{2s+3} + (s+1) \binom{n+s}{2s+1} - 4s \binom{n+s+1}{2s+3} - s \binom{n+s}{2s+1}$$

$$= 6 \binom{n+s+1}{2s+3} + \binom{n+s}{2s+1}.$$

∎

Theorem 2.9 *Let n be a nonnegative integer. Then*

$$f(n,m,3) = \sum_{i=0}^{m} \binom{m}{i} 6^i f(n, m+i, 1), \quad \textit{for all integers } m \geq 1. \tag{8}$$

Proof. The proof is by induction on m.

For $m = 1$, $f(n,1,3) = n^3 = n + 6\binom{n+1}{3}$. The result holds for $m = 1$. When $m > 1$, we suppose the result holds for $m = k$. Then by applying Lemma 2.8,

$$f(n, k+1, 3)$$

$$= \sum_{c=0}^{n} (n-c)^3 f(c,k,3) = \sum_{i=0}^{k} \binom{k}{i} 6^i \sum_{c=0}^{n} (n-c)^3 \binom{c+k+i-1}{2k+2i-1}$$

$$= \sum_{i=0}^{k} \binom{k}{i} 6^i \{6 \binom{n+k+i+1}{2k+2i+3} + \binom{n+k+i}{2k+2i+1}\}$$

$$= 6^{k+1} \binom{n+2k+1}{4k+3} + \sum_{i=1}^{k} [\binom{k}{i-1} + \binom{k}{i}] \binom{n+k+i}{2k+2i+1} 6^i + \binom{n+k}{2k+1}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} \binom{n+(k+1)+i-1}{2(k+1)+2i-1} 6^i.$$

Thus, the theorem holds for all integers $m \geq 1$. ∎

2.3 Relationship to Fibonacci and Lucas numbers.

In the following, we will discuss the relationship between $f(n, m, 2)$ and Fibonacci numbers and between $f(n, m, 2)$ and Lucas numbers.

Lemma 2.10

$$\sum_{m=1}^{n}\binom{n+m+i-1}{2m+2i-1}=\begin{cases} F_{2n-1}, & \text{if } i=0; \\ F_{2n-1}-\sum_{p=1}^{i}\binom{n+p-1}{2p-1}, & \text{if } i\geq 1. \end{cases} \tag{9}$$

Proof. First, we consider the case $i = 0$. .

$$\sum_{m=1}^{n}\binom{n+m-1}{2m-1}=\sum_{p\geq 0}\binom{2n-1-p}{p}=F_{2n-1}.$$

The last equality can be found in [2] (pp.201 Theorem7.1.2). For integers $i \geq 1$,

$$\sum_{m=1}^{n}\binom{n+m+i-1}{2m+2i-1}=\sum_{p=0}^{n-i-1}\binom{2n-1-p}{p}=F_{2n-1}-\sum_{p=n-i}^{n}\binom{2n-1-p}{p}$$

$$=F_{2n-1}-\sum_{p=1}^{i}\binom{n+p-1}{2p-1}.$$

∎

Lemma 2.11

$$\sum_{m=1}^{n}\frac{n}{m+i}\binom{n+m+i-1}{2m+2i-1}=\begin{cases} L_{2n-1}-2, & \text{if } i=0; \\ L_{2n-1}-2-\sum_{p=1}^{i}\frac{n}{p}\binom{n+p-1}{2p-1}, & \text{if } i\geq 1. \end{cases} \tag{10}$$

Proof.

$$\sum_{m=1}^{n}\frac{n}{m}\binom{n+m-1}{2m-1}=2\sum_{m=1}^{n}\binom{n+m}{2m}-\sum_{m=1}^{n}\binom{n+m-1}{2m-1}$$

$$=2\sum_{j=0}^{n-1}\binom{2n-j}{j}-F_{2n-1}=2(F_{2n}-1)-F_{2n-1}$$

$$= F_{2n} + F_{2n-2} - 2 = L_{2n-1} - 2.$$

For integers $i \geq 1$, we have that

$$\sum_{m=1}^{n} \frac{n}{m+i}\binom{n+m+i-1}{2m+2i-1} = 2\sum_{m=1}^{n}\binom{n+m+i}{2m+2i} - \sum_{m=1}^{n}\binom{n+m+i-1}{2m+2i-1}.$$

Proceeding in a manner similar to the proof of (9) for $i \geq 1$, we obtain (10) for $i \geq 1$. ∎

Let $\tilde{f}(n,2) = \sum_{m=1}^{n} f(n,m,2)$. Noting Eqs.(6) and (7), we have

$$\tilde{f}(n,2) = P(n,2) + Q_1(n,2),$$

where

$$P(n,2) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} f(n,2k,1) + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n}{2k+1} f(n,2k+1,1),$$

$$Q_1(n,2) = \sum_{k=1}^{n}\sum_{i=1}^{k}\binom{k}{i}4^i f(n,2k+i,1) + \sum_{k=0}^{n}\sum_{i=1}^{k}\binom{k}{i}\frac{4^i n}{2k+i+1} f(n,2k+i+1,1).$$

Also,

$$P(n,2) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n+2k-1}{4k-1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}\frac{n}{2k+1}\binom{n+2k}{4k+1}$$

$$= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n+2k-1}{4k-1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}\binom{n+2k}{4k+1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}\frac{n-(2k+1)}{2k+1}\binom{2n+2k}{4k+1}$$

$$= \sum_{i=1}^{n}\binom{n+i-1}{2i-1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}\frac{n-(2k+1)}{2k+1}\binom{2n+2k}{4k+1}.$$

By Lemma 2.10,

$$P(n,2) = F_{2n-1} + Q_2(n,2),$$

where

$$Q_2(n,2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}\frac{n-(2k+1)}{2k+1}\binom{2n+2k}{4k+1}.$$

Let

$$Q(n,2) = Q_1(n,2) + Q_2(n,2). \tag{11}$$

Then

$$\tilde{f}(n,2) = F_{2n-1} + Q(n,2). \tag{12}$$

In another way, we can obtain the relationship between $f(n,m,2)$ and Lucas numbers. Note that

$$\begin{aligned}
&P(n,2)\\
&= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{2k}\binom{n+2k-1}{4k-1} - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n-2k}{2k}\binom{n+2k-1}{4k-1} + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} \frac{n}{2k+1}\binom{n+2k}{4k+1}\\
&= (\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{2k}\binom{n+2k-1}{4k-1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} \frac{n}{2k+1}\binom{n+2k}{4k+1}) - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n-2k}{2k}\binom{n+2k-1}{4k-1}\\
&= \sum_{i=1}^{n} \frac{n}{i}\binom{n+i-1}{2i-1} - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n-2k}{2k}\binom{n+2k-1}{4k-1}.
\end{aligned}$$

By Lemma 2.11

$$P(n,2) = L_{2n-1} - 2 + Q_3(n,2),$$

where

$$Q_3(n,2) = -\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n-2k}{2k}\binom{n+2k-1}{4k-1}.$$

Let

$$Q'(n,2) = Q_1(n,2) + Q_3(n,2). \tag{13}$$

Then

$$\tilde{f}(n,2) = L_{2n-1} + Q'(n,2). \tag{14}$$

Theorem 2.12

$$\sum_{m=1}^{n} f(n,m,2) = F_{2n-1} + Q(n,2) = L_{2n-1} - 2 + Q'(n,2)$$

where $Q(n,2)$ and $Q'(n,2)$ are defined in Eqs.(11) *and* (13), *respectively.*

Applying Theorem 2.9, it easy to deduce the following result.

Theorem 2.13

$$\sum_{m=1}^{n} f(n,m,3) = F_{2n-1} + W(n,3),$$

where

$$W(n,3) = \sum_{m=1}^{n}\sum_{i=1}^{m}\binom{m}{i}6^i f(n,m+i,1).$$

Proof.

$$\sum_{m=1}^{n} f(n,m,3) = \sum_{m=1}^{n} f(n,m,1) + \sum_{m=1}^{n}\sum_{i=1}^{m}\binom{m}{i}6^i f(n,m+i,1)$$
$$= F_{2n-1} + W(n,3)$$

∎

2.4 Some values of $f(n,i,r)$

Clearly, for all integers $r \geq 1$, $f(n,1,r) = n^r$, $f(n,n,r) = 1$ and $f(n,i,r) = 0$, if $i > n$. We also have the following results for $f(n,i,r)$.

i)$f(n,n-1,r) = 2^r(n-1)$, for all integers $r \geq 1$;

Since the ordered partition of $c_1 + c_2 + \cdots + c_{n-1} = n$ for all integers $c_i \geq 1$ means that the partition should have $n-2$ c_i such that $c_i = 1$ and have 1 $c_i = 2$, then

$$f(n,n-1,r) = \sum_{c_1+c_2+\cdots+c_{n-1}=n} c_1^r c_2^r \cdots c_n^r = 2^r\binom{n-1}{1} = 2^r(n-1).$$

Similarly, for all integers $r \geq 1$, we have the following formulas.

ii)$f(n,n-2,r) = 3^r(n-2) + 4^r\binom{n-2}{2}$;

iii)$f(n,n-3,r) = 4^r(n-3) + 2\times 6^r\binom{n-3}{2} + 8^r\binom{n-3}{3}$;

iv)$f(n,n-4,r) = 5^r(n-4) + (2\times 8^r + 9^r)\binom{n-4}{2} + 3\times 12^r\binom{n-4}{3} + 16^r\binom{n-4}{4}$.

3 The enumeration of $f_j(n,m,r)$

In this section we will go back to discussing $f_j(n,m,r)$ and give some results for $r=2$ and $r=3$.

3.1 The enumeration of $f_j(n,m,2)$

Theorem 3.1 *Let n be a nonnegative integer and let integers $m \geq 1$. Then*

$$f_{2l}(n,m,2) = \sum_{i=0}^{l} \binom{l}{i} 4^i f(n,m+i,1), \quad \textit{for all integers} \quad 0 \leq l \leq \lfloor \frac{m}{2} \rfloor; \tag{15}$$

$$f_{2l+1}(n,m,2) = \sum_{i=0}^{l} \binom{l}{i} \frac{4^i n}{m+i} f(n,m+i,1), \quad \textit{for all integers} \quad 0 \leq l \leq \lfloor \frac{m}{2} \rfloor. \tag{16}$$

The proof of this theorem is by induction on l. The proof is similar to that of Theorem 2.7 and we can omit it here.

Set $\tilde{f}_j(n,m,2) = \sum_{m=1}^{n} f_j(n,m,2)$. We have the following theorem.

Theorem 3.2

$$\tilde{f}_{2l}(n,m,2) = \begin{cases} F_{2n-1}, & \text{if } l=0; \\ 5^l F_{2n-1} - \sum_{i=1}^{l}\sum_{p=1}^{i} \binom{l}{i}\binom{n+p-1}{2p-1} 4^i, & \text{if } l \geq 1; \end{cases}$$

$$\tilde{f}_{2l+1}(n,m,2) = \begin{cases} L_{2n-1} - 2, & \text{if } l=0; \\ 5^l (L_{2n-1} - 2) - \sum_{i=1}^{l}\sum_{p=1}^{i} \binom{l}{i} \frac{4^i n}{p} \binom{n+p-1}{2p-1}, & \text{if } l \geq 1. \end{cases}$$

Proof. From Theorem 3.1, $f_0(n,m,2) = \binom{n+m-1}{2m-1}$. And $f_1(n,m,2) = \frac{n}{m}\binom{n+m-1}{2m-1}$. Thus from Lemma 2.10 and Lemma 2.11, $\tilde{f}_0(n,m,2) = F_{2n-1}$, $\tilde{f}_1(n,m,2) = L_{2n-1} - 2$. When $l \geq 1$, by Theorem 3.1 and Lemma 2.10

$$\begin{aligned} \tilde{f}_{2l}(n,m,2) &= \sum_{m=1}^{n}\sum_{i=0}^{l} \binom{l}{i} 4^i \binom{n+m+i-1}{2m+2i-1} \\ &= \sum_{i=0}^{l} \binom{l}{i} 4^i \sum_{m=1}^{n} \binom{n+m+i-1}{2m+2i-1} \\ &= \sum_{i=0}^{l} \binom{l}{i} 4^i \{F_{2n-1} - \sum_{p=1}^{i} \binom{n+p-1}{2p-1}\} \end{aligned}$$

$$= F_{2n-1}\sum_{i=0}^{l}\binom{l}{i}4^i - \sum_{i=0}^{l}\binom{l}{i}4^i\sum_{p=1}^{i}\binom{n+p-1}{2p-1}$$

$$= 5^l F_{2n-1} - \sum_{i=0}^{l}\sum_{p=1}^{i}\binom{l}{i}4^i\binom{n+p-1}{2p-1}.$$

$$\tilde{f}_{2l+1}(n,m,2) = \sum_{m=1}^{n}\sum_{i=0}^{l}\binom{l}{i}\frac{4^i n}{m+i}\binom{n+m+i-1}{2m+2i-1}$$

$$= \sum_{i=0}^{l}\binom{l}{i}4^i\sum_{m=1}^{n}\frac{n}{m+i}\binom{n+m+i-1}{2m+2i-1}$$

$$= \sum_{i=0}^{l}\binom{l}{i}4^i\{L_{2n} - 2 - \sum_{p=1}^{i}\frac{n}{p}\binom{n+p-1}{2p-1}\}$$

$$= 5^l(L_{2n-1} - 2) - \sum_{i=0}^{l}\sum_{p=1}^{i}\binom{l}{i}\frac{4^i n}{p}\binom{n+p-1}{2p-1}.$$

∎

3.2 The enumeration of $f_j(n,m,3)$

Theorem 3.3 *Let n be a nonnegative integer and let $m \geq 1$ be an integer. Then*

$$f_j(n,m,3) = \sum_{i=0}^{j}\binom{j}{i}f(n,m+i,1)6^i, \quad \textit{for integers } \ 0 \leq j \leq m.$$

The proof of this theorem is similar to that of Theorem 2.9. We won't repeat it here.

Set $\tilde{f}_j(n,m,3) = \sum_{m=1}^{n} f_j(n,m,3)$, then we have the following theorem.

Theorem 3.4

$$\tilde{f}_j(n,m,3) = \begin{cases} F_{2n-1}, & j = 0; \\ 7^l F_{2n-1} - \sum_{i=1}^{j}\sum_{p=1}^{i}\binom{j}{i}6^i\binom{n+p-1}{2p-1}, & \textit{for integers } \ 1 \leq j < m. \end{cases}$$

Proof. By Theorem 3.3 and Lemma 2.10, $\tilde{f}_0(n,m,3) = F_{2n-1}$. For integers $j \geq 1$, we have

$$\tilde{f}_j(n,m,3) = \sum_{m=1}^{n}\sum_{i=0}^{j}\binom{j}{i}6^i\binom{n+m+i-1}{2m+2i-1}$$
$$= \sum_{i=0}^{j}\binom{j}{i}6^i\sum_{m=1}^{n}\binom{n+m+i-1}{2m+2i-1}$$
$$= \sum_{i=0}^{j}\binom{j}{i}6^i\{F_{2n-1} - \sum_{p=1}^{i}\binom{n+p-1}{2p-1}\}$$
$$= 7^l F_{2n-1} - \sum_{i=1}^{j}\sum_{p=1}^{i}\binom{j}{i}6^i\binom{n+p-1}{2p-1}.$$

∎

4 Some further research

For further research, the following questions may be considered:

(1) The explicit formulas for $f(n,m,r)$ for $r \geq 5$;

(2) The explicit formulas for $f_j(n,m,r)$ for $1 \leq j < m$ and $r \geq 5$;

(3) The relation to $f(n,m,r)$ and $f_j(n,m,r)$.

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ON THE POSSIBLE AUTOMORPHISMS OF A 3-(16,7,5) DESIGN

ZIBA ESLAMI

ABSTRACT. The existence question for a 3-(16,7,5) design is open. In this paper, we examine possible automorphisms of this design. We consider a minimum subset of basic permutations consisting of cycles of prime length p and prove that if a 3-(16,7,5) design exists, then it is either rigid or admits basic automorphisms with cycles of length 2 or 3.

1. INTRODUCTION

For positive integers $1 \le t < k < v$, let $X = \{1,2,\ldots,v\}$, S_X denote the symmetric group on the elements of X, $\binom{X}{k}$ the set of all k-subsets of X and 2^X the power set of X. The elements of X and $\binom{X}{k}$ are called *points* and *blocks* , respectively. If $\sigma \in S_X$, $x \in X$, $B \in \binom{X}{k}$, and $\mathscr{B} \subseteq 2^X$, we denote by $\sigma(x)$, $\sigma(B)$, and $\sigma(\mathscr{B})$ the images under σ of $x, B, \mathscr{B}$, respectively.

A t-(v,k,λ) *design*, or briefly a *t-design*, is a pair $(X,\mathscr{B})$ where $\mathscr{B} \subseteq \binom{X}{k}$, so that for every $T \in \binom{X}{t}$, $|\{B \in \mathscr{B} | T \subseteq B\}| = \lambda$. A t-design is *simple* if no two blocks are identical. In this paper, we consider only simple designs. $(X, \binom{X}{k})$ is called the *complete* design.

For a set of blocks $\mathscr{B} \subseteq \binom{X}{k}$ and $\sigma \in S_X$, let σ act on $\mathscr{B}$. Then $\sigma(\mathscr{B})$ is an *isomorphic* copy of $\mathscr{B}$. If further, $\sigma(\mathscr{B}) = \mathscr{B}$, then σ is called an *automorphism* of $\mathscr{B}$. If G is a subgroup of S_X such that $\sigma(\mathscr{B}) = \mathscr{B}$ for every $\sigma \in G$, we say that $\mathscr{B}$ is *G-invariant*. The set of all automorphisms of $\mathscr{B}$ forms a group, denoted by *Aut*$\mathscr{B}$ and called the *full Automorphism group* of $(X,\mathscr{B})$. $\mathscr{B}$ is called *rigid* if its automorphism group is trivial. We also recall the notion of the *normalizer* of a group G in a bigger group H as the subgroup of H consisting of all elements $g \in H$ such that $g^{-1}Gg = G$.

Let $(X,\mathscr{B})$ be a t-(v,k,λ) design and consider the set $W \subset X$ with $|W| = w < t$. Let $X' = X \setminus W$ and $\mathscr{B}' = \{B \setminus W : B \in \mathscr{B}, W \subseteq B\}$. Then $(X',\mathscr{B}')$ is a $(t-w)$-$(v-w, k-w, \lambda)$ design called the *derived* design with respect to W.

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In order to tackle existence/classification problems for designs, it is often possible to devise techniques for extending smaller designs into larger designs. These approaches may lead to the construction of previously unknown design, or in some cases even to a complete classification of larger designs. A good example of this is [13], where to prove the non-existence of 4-$(12,6,6)$ designs, the authors made use of the classification of the smaller 3-$(10,4,3)$ and 4-$(11,5,3)$ simple designs. For examples of classification achieved through extension, the reader is referred to [4, 5, 8, 7]. In this paper, we pursue the same approach.

The family of 3-$(16,7,5)$ designs is of interest mainly because the existence question for this class is still open. Even for the derived 2-$(15,6,5)$ designs, only a lower bound of 117 is given in [11]. In this paper, we first consider the derived family of 2-$(15,6,5)$ designs and improve the existing bound to 1454 which contains the complete catalogue of 2-$(15,6,5)$ designs admitting automorphisms of order at least 5. Then we consider possible extension of these designs. For a permutation π of prime order p which consists of m disjoint cycles of length p, we say that π is of basic type p^m. Therefore, this approach excludes the existence of a 3-$(16,7,5)$ design with automorphisms of basic types p^m with $p \in \{5,7,11,13\}$ and m a positive integer such that $pm \leq 16$. In other words, we prove that if a 3-$(16,7,5)$ design exists, then it is either rigid or admits automorphisms of basic type 2^m and 3^m. To accomplish this, we employ, with slight modifications, the algorithm presented in [3]. For the sake of completeness, in section 2, we cite from [3] the algorithm , which is an exhaustive technique based on backtracking on the solutions of a matrix-system which generates G-invariant designs with efficient rejection of isomorphic sub-configurations. Using this algorithm, we consider in section 3, the family of 2-$(15,6,5)$ designs with a nontrivial automorphism group and possible extensions of the results.

2. The Algorithm

For a subgroup G of S_X, let $(X,\mathscr{B})$ be a G-invariant t-(v,k,λ) design. Let $\tau_1,\tau_2,\ldots,\tau_m$ and $\kappa_1,\kappa_2,\ldots,\kappa_n$ be the orbits of $\binom{X}{t}$ and $\binom{X}{k}$ under the action of G, respectively. Define the $m\times n$ matrix $A(G|X)$ whose (i,j)th entry is $|\{K \in \kappa_j | T \subseteq K\}|$, where T is any representative in τ_i. It is known [9] that there exists a G-invariant t-(v,k,λ) design $(X,\mathscr{B})$ if and only if there exists a vector $\mathbf{u} \in \{0,1\}^n$ satisfying the equation

$$A(G|X)\mathbf{u} = \lambda j_m \qquad (*)$$

where j_m is the m-dimensional all-one vector. Note that $\mathbf{u}$ is indeed the vector representation of $(X,\mathscr{B})$, i.e., $\mathbf{u}$ is a column vector whose rows are indexed by the elements of the orbits of $\binom{X}{k}$ such that $\mathbf{u}_i = 1$ if and only if $\mathscr{B}$ contains the i-th orbit of $\binom{X}{k}$.

Now, since finding a G-invariant t-design can be reduced to solving a matrix problem, it is important to devise an efficient procedure for solving this system.

In this respect, the role of backtracking is widely recognized [1, 2, 6, 10, 14]. In backtracking, series of partial feasible solutions are constructed one step at a time in an orderly fashion. At each step, we test to see if a partially constructed solution has any chance to be extended. If not, we immediately reject the partial solution and go to the next one, thereby saving the effort of constructing the descendants of a clearly unsuitable partial vector. The key to the success of backtracking lies in how we restrict the number of candidates for extension, especially at the earlier steps of the process.

We define the following ordering on the solutions of the system $A(G|X)\mathbf{u} = \lambda j_m$, i.e., G-invariant t-(v,k,λ) designs. Let $\mathbf{u} = \{\mathbf{u}_i\}_{i=1}^n$ and $\mathbf{u}' = \{\mathbf{u}'_i\}_{i=1}^n$ be two such solutions, and let j be the smallest value for which $\mathbf{u}_j \neq \mathbf{u}'_j$. Whichever of the solutions contain a 1 in this position is defined to be the *smaller* of the two. The algorithm initially constructs the smallest possible solution and then proceeds to exhaustively generate a sequence of solutions in strictly increasing order. This ordering can be used to cut down on the number of solutions the algorithm has to deal with, as follows. Let $\mathbf{u}_i$ represent a partial design validly constructed on the first i orbits, $i \leq n$. If there exists another partial solution which is both isomorphic to and less than $\mathbf{u}_i$, it would have already been considered at some point earlier in the enumeration and there is no point in examining $\mathbf{u}_i$ or its possible extensions any further. However, if no such configuration exists, the partial solution has to be extended to the next level, $i+1$. We are therefore interested in partial solutions for which no smaller isomorphic configuration exists.

For a given isomorphism class, $\mathscr{C}$, the smallest t-design belonging to $\mathscr{C}$ is said to be the *canonical* representative of its class, and the design itself is said to be in canonical form. The aim of the isomorph rejection technique is to reduce the number of partial solutions through collapsing the ones which are not in canonical form. Obviously, if a partial solution $\mathbf{u}_i$ is in canonical form and $j \leq i$, then $\mathbf{u}_j$ is also canonical. However, if $\mathbf{u}_i$ is not in canonical form, then any solution extended from $\mathbf{u}_i$ is not canonical either. The following lemma [3] shows how we can restrict the elements of S_X that are to be considered to detect if a smaller isomorphic copy of a partial solution exists.

Lemma 2.1. *Let $(X,\mathscr{B})$ be a t-(v,k,λ) design with a nontrivial automorphism π, i.e., $\pi(\mathscr{B}) = \mathscr{B}$. If $\sigma \in N(\langle\pi\rangle)$, then $\sigma(\mathscr{B})$ admits π as an automorphism. Further, if $(\langle\pi\rangle)$ is the full automorphism group of $(X,\mathscr{B})$, then the converse is also true.*

Now, Suppose that we want to find t-(v,k,λ) designs with a given automorphism π of prime order p. We backtrack on the solutions of the system (*) using the approach presented in [5], and produce all partial solutions, i.e. partial designs, up to some specified level (orbit) l. We then use isomorphism rejection based on the normalizers according to Lemma 2.1 to detect solutions which can not extend to canonical form. To do so, we must apply elements of the normalizers of π in S_X to the solutions and if a smaller copy is produced, we reject the solution. We then extend the remaining partial solutions and if necessary (i.e., if there are many extensions) we also repeat this procedure for some other levels. Clearly, there might still be isomorphic copies among complete solutions which have to be discarded. Hence, we first reject isomorphisms under normalizers. Second we determine the order of the full automorphism group of each of the remaining designs. According to Lemma 2.1, at this step any two designs with exactly p automorphisms are non-isomorphic. Thus in the last step we concentrate on designs with more than p automorphisms and extract non-isomorphic designs among them.

3. THE DERIVED 2-(15,6,5) DESIGNS

In [11] a lower bound of 117 is given for the number of 2-(15,6,5) designs. In this section we improve this bound and consider possible automorphisms of this class of designs.

Let π be an automorphism of a t-(v,k,λ). Clearly, we can take a suitable power of π of prime order p. If π consists of m disjoint cycles of length p, we say that π is of basic type p^m. We first prove that for a 2-(15,6,5) design, the types 13^1, 11^1 and 7^1 are infeasible.

Theorem 3.1. *For a* 2-(15,6,5) *design, automorphisms of basic type* 7^1, 11^1 *and* 13^1 *are not possible.*

Proof. Consider a 2-(15,6,5) design $D = (X, \mathscr{B})$ with a nontrivial automorphism π of type 7^1, 11^1 or 13^1. Without loss of generality, we can take the points 1 and 2 as the fixed points of π. Then blocks of D containing the pair $\{1,2\}$ are fixed by π. Further, since $\lambda = 5$, then every such block must also be fixed by π. This forces π to have at least 8 more fixed points, which is a contradiction in all three cases. □

For a 2-(15,6,5) design we have $\binom{15}{6} = 5005$ and $\binom{15}{2}) = 105$. If we assume an automorphism $\pi = (1)\ (2\ldots8)(9\ldots15)$ of type 7^2, the action of π on 2-subsets of $\{1,2,\ldots,15\}$ produces 715 orbits. Here, using group actions alone does not suffice to reduce the size of the problem. Hence, it is a challenging task to find solutions of the system (*) of previous section for this class and we definitely need an algorithm to do this. One approach to get round this problem is using the derived designs to prune the search space, and then proceed according to the algorithm presented in Section 2 to identify and reject derived solutions which can not extend to canonical form.

Let D denote a 2-(15,6,5) design with automorphism π and let D_1 be its derived design with respect to the point 1. Therefore, π is an automorphism of D_1 as well. According to Lemma 2.1, for each design D_1 and each $\sigma \in N(\langle\pi\rangle)$, there exists also $\sigma(D_1)$ among solutions of (*) (i.e. designs admitting π as automorphism) which extend to $\sigma(D)$ and can therefore be discarded. Hence, we first employ the method of [5] to solve the equation (*) to produce all candidates for D_1. The action of π on the points of $\{1,2,\ldots,14\}$ has 286 orbits and there are 12,259 solutions for this step. In the next step, we use normalizers as in Lemma 2.1 to delete additional copies of solutions and this results in 170 derived designs which are canonical and are to be extended to a 2-$(15,6,5)$. For the extension, we note that already 286 orbits are dealt with and to determine the remaining orbits, we proceed as before and obtain extensions of the derived designs which admit π as automorphism. Using *nauty* [12] to reject isomorphic copies we get:

Theorem 3.2. *Up to isomorphism, the number of* 2-$(15,6,5)$ *designs admitting an automorphism of order* 7 *is as follows:*

|Aut|	#Designs
7	66
14	4
21	8
42	2
336	1
	total:81

Examples of designs with 336 and 42 automorphisms are given in the Appendix.

We now consider the type 5^m for $m = 1,2,3$. We prove first that $m = 1$ and $m = 2$ are impossible.

Theorem 3.3. *For a* 2-$(15,6,5)$ *design, automorphisms of basic type* 5^1 *and* 5^2 *are not possible.*

Proof. Consider a 2-$(15,6,5)$ design $D = (X,\mathscr{B})$ with nontrivial automorphism

$$\pi = (f_1)\ldots(f_5)\ (\alpha_1\ldots\alpha_5)(\alpha_6\ldots\alpha_{10})$$

Let S consist of all 14 blocks of $\mathscr{B}$ containing f_1. Since $5 \nmid 14$, at least 4 elements of S must be fixed by π. However, at most 2 blocks of S can be of form $f_1\alpha_{i1}\alpha_{i2}\alpha_{i3}\alpha_{i4}\alpha_{i5}$ and therefore the other two blocks must consist only of fixed points f_i which is not possible. For automorphism $\sigma = f_1\ldots f_{10}\ (\gamma_1\cdots\gamma_5)$, we have one block of form $f_1\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$ and need three blocks of fixed points which is again impossible. □

We now consider 2-$(15,6,5)$ design $D = (X,\mathscr{B})$ admitting an automorphism $\pi = \sigma_1\sigma_2\sigma_3$ of type 5^3, where $\sigma_i = (1+5i\ \cdots\ 5+5i)$, $0 \le i \le 2$. In this case, the number of orbits for the action of π on 2-subsets of $\{1,2,\ldots,15\}$ equals 1001 and we have to consider some modifications to the algorithm of Section 2 to categorize this class. Hence we prefer to employ the properties of the normalizers of $<$

$\pi >$ such that isomorphism test using normalizers can reject partially-completed isomorphic solutions. The details of this approach is as follows.

Let $\Gamma = \{\Gamma_i\}_1^{1001}$ be the set of orbits of the blocks of D under the action of $< \pi >$ and let $\Gamma^j = \{\Gamma_i :$ there exists a block $B \in \Gamma_i$ such that $|B \cap point(\sigma_l)| = j, 1 \leq l \leq 3\}$, $1 \leq j \leq 3$. Clearly Γ^j is a well defined subset of Γ and we have $|\Gamma^1| = 6$, $|\Gamma^2| = |\Gamma^3| = 60$. Furthermore, for any $\sigma \in N(< \pi >)$, it holds that $\sigma(\Gamma^j) = \Gamma^j$. Let **u** be the vector representation of D and permute the rows of **u** so that the orbits of Γ^1, Γ^2, and Γ^3 appear one group after another. We again consider (*) where the columns of matrices are permuted accordingly and obtain all partial solutions of length $|\Gamma^1|$, $|\Gamma^1|+|\Gamma^2|$, $|\Gamma^1|+|\Gamma^2|+|\Gamma^3|$, and $|\Gamma|$, respectively. As before, we can employ normalizers to reduce the number of solutions at each step. Note that considering a different ordering of orbits for (*) is such that application of Lemma 2.1 is more efficient. We can now proceed as before and the final results are as follows.

Theorem 3.4. *Up to isomorphism, the number of* 2-(15,6,5) *designs invariant under an automorphism of order* 5 *is as follows:*

|Aut|	*#Designs*
5	*1317*
10	*43*
20	*9*
30	*1*
40	*1*
60	*1*
120	*1*
	total:1373

Examples of designs with more than 40 automorphisms are given in the Appendix.

The remaining types, namely 3^m and 2^m, can not be classified in reasonable time with our algorithm. Therefore, we consider possible extension of the obtained designs to 3-(16,7,5) designs. Let D denote a 3-(16,7,5) design which is the extension of one of the obtained 2-(15,6,5) designs, say D_1. Hence, , we can employ the method of [5] to solve the equation (*) to produce all candidates for D where some of the elements of **u** are already determined by D_1. The computational results show (in a few seconds, on a 3.4GHz Pentium 4 running a C program) that none of the obtained designs extend to a 3-(16,7,5) design. This means that a 3-(16,7,5) design with automorphisms of basic types p^m with $p \in \{5,7,11,13\}$ does not exist. Therefore, we have proved the following theorem.

Theorem 3.5. *if a* 3-(16,7,5) *design exists, then it is either rigid or admits automorphisms of basic types* 2^m *or* 3^m.

Appendix

Examples of 2-$(15,6,5)$ designs are given below.

The point set is $V = \{1,\ldots,9,A,\ldots,F\}$.

|Aut|=40

12346B	1235AF	12459E	1267BD	1289CE	13458D	1369BC
137AEF	1479EF	148ACD	156ACF	1578BD	1689AC	17BDEF
23457C	2378CE	239ADF	2468BF	247ACD	2569DE	258ABF
2679AD	28BCEF	346ABE	3489DF	3568DE	3579BC	3678AE
39BCDF	4567CF	459ABE	46789F	4ABCDE	56CDEF	5789AB

|Aut|=42

123469	12358F	12478E	12567C	129ABD	13457A	13678D
13ABCE	14568B	14BCDF	159CDE	16ADEF	179BEF	189ACF
235BDE	2379AE	237CDF	2459BC	245AEF	248ACD	267ABF
2689DF	268BCE	346CEF	3489DE	348ABF	3569BF	356ACD
3789BC	4579DF	4679AC	467BDE	5689AE	578ABD	578CEF

|Aut|=42

123469	12358F	12478E	12567C	129ABD	13457A	13678D
13ABCE	14568B	14BCDF	159CDE	16ADEF	179BEF	189ACF
2359AC	235BDE	237CDF	245AEF	2489BF	248ACD	2679DE
267ABF	268BCE	346ABD	346CEF	3489DE	3569BF	3789BC
378AEF	4579DF	457BCE	4679AC	5689AE	568CDF	578ABD

|Aut|=60

12346B	1235AF	12459E	1269BE	1278CD	13458D	136ACE
1379BF	147ADE	1489CF	1567BC	158ADF	1689AC	17BDEF
23457C	237ACF	2389DE	2467DF	248ABC	2568EF	259ABD
2679AD	28BCEF	3468BD	349AEF	3569CD	3578BE	3678AE
39BCDF	456ABF	4579CE	46789F	4ABCDE	56CDEF	5789AB

|Aut|=120

123679	1248BD	1249CF	12568A	126DEF	1346CE	134ADF
1357CF	1358BE	14579A	15ACDE	1678BC	179BDE	189ABF
23478A	2359CE	235ABD	237BEF	2456BE	2457DF	269ABC
2789CD	28ACEF	345689	348BCF	367ACD	369BDF	389ADE
459BCD	4678DE	469AEF	47ABCE	567ABF	568CDF	5789EF

|Aut|=336

123469	12358F	12478E	12567C	129CEF	13457A	13678D
139ADF	14568B	149ABE	15ABCF	169BCD	17ACDE	18BDEF
2379BF	237ABD	237BCE	2459DE	245ABD	245CDF	2689AC
268ABD	268AEF	3489AC	348BCE	348CDF	3569DE	356AEF
356BCE	4679BF	467AEF	467CDF	5789AC	5789BF	5789DE

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