# Caterpillar Factorizations of Crowns

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## Abstract

*For positive integers $k \leq n$, the crown $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \cdots, a_{n-1}, b_0, b_1, \cdots, b_{n-1}\}$ and edge set $\{a_ib_j : 0 \leq i \leq n-1, j = i+1, i+2, \cdots, i+k \ (\bmod n)\}$. A caterpillar is a tree of order at least three which contains a path such that each vertex not on the path is adjacent to a vertex on the path. Being a connected bipartite graph, a caterpillar is balanced if the two parts of the bipartition of its vertices have equal size; otherwise, it is unbalanced. In this paper we obtain the necessary and sufficient condition for balanced-caterpillar factorization of crowns. The Criterion for unbalanced-caterpillar factorization of crowns is open. We also obtain the necessary and sufficient condition for directed caterpillar factorization of symmetric crowns.*

# 1 Introduction and preliminaries

Suppose that $G$ and $H$ are multigraphs. A *G-factor* of $H$ is a spanning subgraph of $H$ which is the union of vertex disjoint subgraphs each isomorphic

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to $G$. A *G-factorization* of $H$ is a set of $G$-factors of $H$ which partition the edges of $H$. For multidigraphs $G$ and $H$, a *G-factor* and a *G-factorization* of $H$ are similarly defined.

Let $H$ be a graph. The *symmetric digraph* $H^*$ is the digraph obtained from $H$ by replacing each edge of $H$ by two arcs with opposite directions. For a positive integer $\lambda$, $\lambda H$ is the multigraph obtained from $H$ by replacing each edge of $H$ by $\lambda$ edges.

A *star factorization* of a graph $H$ is a $G$-factorization of $H$ where $G$ is a star. Path factorization, tree factorization, directed star factorization and directed path factorization are similarly defined. Star factorizations were investigated for $K_n$ [24], $K_{m,n}$ [4, 6, 7, 11, 12, 14, 21] and graph products [2]. Directed star factorizations were investigated for $K^*_{m,n}$ [15], $\lambda K^*_{m,n}$ [19], $K^*_{l,m,n}$ [16, 17, 18], and $\lambda K^*_{l,m,n}$ [20]. Path factorizations were investigated for $\lambda K_n$ [1], $K_{m,n}$ [8], $\lambda K_{m,n}$ [5], $\lambda K(n,r)$ [23] ($K(n,r)$ is the complete $n$-partite graph with each part of cardinality $r$), $K_{l,m,n}$ [10], cubic graphs [3] and product graphs [9]. Directed path factorizations were investigated for $\lambda K^*_n$ [22]. Tree factorizations were investigated for $K_n$ [22, 24].

For positive integers $k \leq n$, the *crown* $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \cdots, a_{n-1}, b_0, b_1, \cdots, b_{n-1}\}$ and edge set $\{a_i b_j : 0 \leq i \leq n-1, j = i+1, i+2, \cdots, i+k \pmod{n}\}$.

A *caterpillar* is a tree of order at least three which contains a path such that each vertex not on the path is adjacent to a vertex on the path. In this paper, we investigate the caterpillar factorization of $C_{n,k}$ and the directed caterpillar factorization of $C^*_{n,k}$. For the convenience of discussions, a caterpillar can be defined alternatively as follows. For a positive integer $d$, let $S_d$ denote the star with $d$ edges. For positive integers $d_1, d_2, \cdots, d_v$ ($v \geq 1$), $S(d_1, d_2, \cdots, d_v)$ denotes the graph obtained from the stars $S_{d_1}, S_{d_2}, \cdots, S_{d_v}$ by identifying an endvertex of $S_{d_i}$ with the center of $S_{d_{i+1}}$ for $i = 1, 2, \cdots, v-1$. As an illustration, $S(5,2,1,3)$ is exhibited in Fig. 1. It is easy to see that the graph $S(d_1, d_2, \cdots, d_v)$ has $1 + d_1 + d_2 + \cdots + d_v$ vertices. A caterpillar can be defined as the graph $S(d_1, d_2, \cdots, d_v)$ with order $\geq 3$.

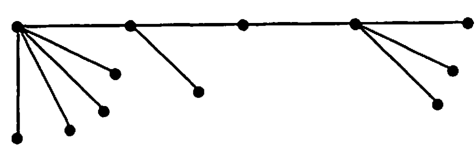

Fig. 1. $S(5,2,1,3)$

Let us begin with some lemmas.

**Lemma 1.1** *Suppose that a multigraph $G$ has $n_1$ vertices, $e_1$ edges, and a multigraph $H$ has $n_2$ vertices, $e_2$ edges. We have the following.*

(1) *If $H$ has a $G$-factor, then $n_2 \equiv 0 \pmod{n_1}$.*
(2) *If $H$ has a $G$-factorization, then $n_1 e_2 \equiv 0 \pmod{n_2 e_1}$.*

**Proof.** Let $t$ be the number of copies of $G$ in a $G$-factor of $H$. Then $n_2 = tn_1$. Thus
(1) $n_2 \equiv 0 \pmod{n_1}$.
(2) A $G$-factor of $H$ has $te_1 = \frac{n_2}{n_1} e_1$ edges. Let $r$ be the number of $G$-factors in the $G$-factorization of $H$. Then $e_2 = r(\frac{n_2}{n_1} e_1)$, which implies $r = \frac{n_1 e_2}{n_2 e_1}$. Thus $n_1 e_2 \equiv 0 \pmod{n_2 e_1}$. □

**Lemma 1.2** *Let $T$ be a tree of order $t$. If $\lambda C_{n,k}$ has a $T$-factorization where $\lambda = 1$ or $2$, then $k \equiv 0 \pmod{t-1}$.*

**Proof.** By Lemma 1.1(2), $t \cdot \lambda nk \equiv 0 \pmod{2n(t-1)}$. Thus $t\lambda k \equiv 0 \pmod{2(t-1)}$.
Case 1. $\lambda = 1$.
Then $tk \equiv 0 \pmod{2(t-1)}$. Hence $tk \equiv 0 \pmod{t-1}$, which implies $k \equiv 0 \pmod{t-1}$.
Case 2. $\lambda = 2$.
Then $2tk \equiv 0 \pmod{2(t-1)}$. Hence $tk \equiv 0 \pmod{t-1}$, which implies $k \equiv 0 \pmod{t-1}$. □

A connected bipartite multigraph is *balanced* if it has a bipartition $(C, D)$ with $|C| = |D|$. It is easy to see that $S(d_1, d_2, \cdots, d_v)$ is balanced if and only if $d_1 + d_3 + d_5 + \cdots + d_{2\lfloor \frac{v-1}{2} \rfloor + 1} = 1 + d_2 + d_4 + \cdots + d_{2\lfloor \frac{v}{2} \rfloor}$. A connected bipartite multigraph is *unbalanced* if it is not balanced.

**Lemma 1.3** *Suppose that $G$ is an unbalanced connected bipartite multigraph of order $t$ and $H$ is a balanced connected bipartite multigraph of order $2n$. If $H$ has a $G$-factor, then $n \equiv 0 \pmod{t}$.*

**Proof.** Let $(A, B)$ be a bipartition of $G$, and let $a = |A|$, $b = |B|$. Then $a + b = t$, $a \neq b$. Let $(C, D)$ be a bipartition of $H$ with $|C| = |D| = n$. Suppose that in a $G$-factor of $H$, there are $l$ copies of $G$ with $A$-part in $C$, and $m$ copies of $G$ with $B$-part in $C$. Then $la + mb = |C| = n$ and $lb + ma = |D| = n$. Thus $bn - an = m(b^2 - a^2)$. Since $a \neq b$, we have $n = m(b + a)$. Thus $n \equiv 0 \pmod{t}$. □

**Lemma 1.4** *Let $G$ be a graph. Suppose that the crown $C_{n,p}$ has a $G$-factorization. If $k$ is a positive integer with $k \leq n$ and $k \equiv 0 \pmod{p}$, then $C_{n,k}$ has a $G$-factorization.*

**Proof.** Let $k = pq$ for some positive integer $q$. It is easy to see that $C_{n,pq}$ can be decomposed into $q$ spanning subgraphs of which each is isomorphic to $C_{n,p}$. Since, by assumption, $C_{n,p}$ has a $G$-factorization, so does $C_{n,pq}$. □

The following is the directed version of Lemma 1.4. The proof is similar and hence omitted.

**Lemma 1.5** *Let $G$ be a digraph. Suppose that the symmetric crown $C^*_{n,p}$ has a $G$-factorization. If $k$ is a positive integer with $k \leq n$ and $k \equiv 0 \pmod{p}$, then $C^*_{n,k}$ has a $G$-factorization.*

## 2 Caterpillar factorization of crowns

In this section we investigate the caterpillar factorization of crowns. In the sequel of the paper, $a_0, a_1, \cdots, a_{n-1}$ and $b_0, b_1, \cdots, b_{n-1}$ are the vertices of $C_{n,k}$ as given in the definition, and the subscripts of $a_i$ and $b_j$ are always taken modulo $n$. Each edge $a_i b_{i+p}$ $(1 \leq p \leq k)$ in $C_{n,k}$ is assigned the *label* $p$. For example, in $C_{5,5}$ the edges $a_2b_3, a_2b_4, a_2b_0, a_2b_1, a_2b_2$ have labels $1, 2, 3, 4, 5$ respectively.

**Lemma 2.1** *Let $G = S(d_1, d_2, \cdots, d_v)$ be a caterpillar of order $t$. Let $n$ be a positive integer with $n \geq t-1$. Then there exists a subgraph $G_1$ of the crown $C_{n,t-1}$ such that*

(i) $G_1 \cong G$,

(ii) *the vertex set of* $G_1 = \{a_i : i = 0, -1, -2, \cdots, -d_2 - d_4 - d_6 - \cdots - d_{2\lfloor \frac{v}{2} \rfloor}\} \cup \{b_j : j = 1, 2, \cdots, d_1 + d_3 + d_5 + \cdots + d_{2\lfloor \frac{v-1}{2} \rfloor + 1}\}$,

(iii) *the edges of* $G_1$ *have labels* $1, 2, 3, \cdots, t-1$.

**Proof.** Let $G = S(d_1, d_2, \cdots, d_v)$ be a caterpillar of order $t = 1 + d_1 + d_2 + \cdots + d_v$. Let $G_1$ be the subgraph of $C_{n,t-1}$ induced by the following edge set

$$\begin{array}{ll} & \{a_0 b_i : i = 1, 2, \cdots, d_1\} \\ \cup & \{a_i b_{d_1} : i = -1, -2, \cdots, -d_2\} \\ \cup & \{a_{-d_2} b_i : i = d_1 + 1, d_1 + 2, \cdots, d_1 + d_3\} \\ \cup & \{a_i b_{d_1+d_3} : i = -d_2 - 1, -d_2 - 2, \cdots, -d_2 - d_4\} \\ \cup & \{a_{-d_2-d_4} b_i : i = d_1 + d_3 + 1, d_1 + d_3 + 2, \cdots, d_1 + d_3 + d_5\} \\ \cup & \cdots. \end{array}$$

We see that $G_1$ has the required properties. □

For a subgraph $S$ of $C_{n,k}$ and an integer $v$, let $S+v$ denote the subgraph of $C_{n,k}$ which is induced by $\{a_{i+v} b_{j+v} : a_i b_j \in E(S)\}$.

**Lemma 2.2** *Let $G$ be a balanced-caterpillar of order $t$. Suppose that $n$ is a positive integer $\geq t-1$ such that $n \equiv 0 \pmod{t/2}$. Then $C_{n,t-1}$ has a $G$-factorization.*

**Proof.** Let $G = S(d_1, d_2, \cdots, d_v)$. Since $G$ is balanced and of order $t$, $d_1 + d_3 + d_5 + \cdots + d_{2\lfloor \frac{v-1}{2} \rfloor + 1} = 1 + d_2 + d_4 + d_6 + \cdots + d_{2\lfloor \frac{v}{2} \rfloor} = t/2$. Let $n' = \frac{n}{t/2}$. Let $G_1$ be the subgraph of $C_{n,t-1}$ described in Lemma 2.1.

Note that $G_1$ has $t/2$ vertices in $A = \{a_0, a_1, \cdots, a_{n-1}\}$ and $t/2$ vertices in $B = \{b_0, b_1, \cdots, b_{n-1}\}$. Let $F$ be the subgraph of $C_{n,t-1}$ such that

$$F = G_1 \cup (G_1 + \frac{t}{2}) \cup (G_1 + 2 \cdot \frac{t}{2}) \cup \cdots \cup (G_1 + (n' - 1)\frac{t}{2}).$$

From Lemma 2.1(i) and (ii), we see that $F$ is a $G$-factor of $C_{n,t-1}$. From Lemma 2.1 (iii), we see that $C_{n,t-1}$ can be decomposed into $F, F+1, F+2, \cdots, F+(\frac{t}{2}-1)$. Hence $C_{n,t-1}$ has a $G$-factorization. □

Now we have the necessary and sufficient condition for the balanced-caterpillar factorization of crowns.

**Theorem 2.3** *Let $G$ be a balanced-caterpillar of order $t$. Then $C_{n,k}$ has a $G$-factorization if and only if $n \geq k$ are positive integers and $n \equiv 0 \ (mod\ t/2)$, $k \equiv 0 \ (mod\ t-1)$.*

**Proof.** (*Necessity*) Since $G$ is balanced, $t$ is an even integer. By Lemma 1.1 (1), we obtain $2n \equiv 0 \pmod t$, which implies $n \equiv 0 \pmod{t/2}$. By Lemma 1.2 with $\lambda = 1$, we obtain $k \equiv 0 \pmod{t-1}$.

(*Sufficiency*) Since $n \geq k \geq t-1$, it follows from Lemma 2.2 that $C_{n,t-1}$ has a $G$-factorization. Since $n \geq k$ and $k \equiv 0 \pmod{t-1}$, it follows from Lemma 1.4 that $C_{n,k}$ has a $G$-factorization. □

The necessary and sufficient condition for unbalanced-caterpillar factorization of crowns is open.

# 3 Directed caterpillar factorization of symmetric crowns

A *directed caterpillar* is the digraph obtained from a caterpillar by orienting each edge arbitrarily. In this section the directed caterpillar factorization of symmetric crowns is investigated. We assign the arc $\overrightarrow{a_ib_j}$ and the arc $\overrightarrow{b_ja_i}$ in $C^*_{n,k}$ with the same label of $a_ib_j$ in $C_{n,k}$, i.e., for $1 \leq t \leq k$, $\overrightarrow{a_ib_{i+t}}$ has label $t$, and $\overrightarrow{b_{i+t}a_i}$ also has label $t$.

Suppose that $G$ is a subdigraph of $C^*_{n,k}$ and $G$ has no isolated vertex. The *dual* of $G$, denoted by $Dual(G)$, is the subdigraph of $C^*_{n,k}$ induced by the arc set $\{ \overrightarrow{b_{-i}a_{-j}}\colon \overrightarrow{a_ib_j} \in E(G)\} \cup \{\overrightarrow{a_{-j}b_{-i}} : \overrightarrow{b_ja_i} \in E(G)\}$. Note that $Dual(G) \cong G$. Note also that the arc $\overrightarrow{a_ib_j}$ in $G$, which is from $A$ to $B$, and the corresponding arc $\overrightarrow{b_{-i}a_{-j}}$ in $Dual(G)$, which is from $B$ to $A$, have the same label. Similarly, the arc $\overrightarrow{b_ja_i}$ in $G$, which is from $B$ to $A$, and the corresponding arc $\overrightarrow{a_{-j}b_{-i}}$ in $Dual(G)$, which is from $A$ to $B$, have the same label.

For a subdigraph $S$ of $C^*_{n,k}$ and an integer $v$, let $S+v$ denote the subdigraph of $C^*_{n,k}$ which is induced by $\{\overrightarrow{a_{i+v}b_{j+v}} : \overrightarrow{a_ib_j} \in E(S)\} \cup \{\overrightarrow{b_{j+v}a_{i+v}} : \overrightarrow{b_ja_i} \in E(S)\}$.

**Lemma 3.1** *Let $G$ be a caterpillar of order $t$, and the directed caterpillar $\hat{G}$ be a digraph obtained from $G$ by orienting each edge of $G$ arbitrarily. We have the following.*
*(1) If $G$ is balanced, and $n$ is an integer $\geq t-1$ such that $n \equiv 0 \pmod{t/2}$, then $C^*_{n,t-1}$ has a $\hat{G}$-factorization.*
*(2) If $G$ is unbalanced, and $n$ is a positive integer such that $n \equiv 0 \pmod{t}$, then $C^*_{n,t-1}$ has a $\hat{G}$-factorization.*

**Proof.** Let $G = S(d_1, d_2, \cdots, d_v)$ be a caterpillar of order $t$. By Lemma 2.1, there exists a subgraph $G_1$ of the crown $C_{n,t-1}$ such that

(i) $G_1 \cong G$,

(ii) the vertex set of $G_1 = \{a_i : i = 0, -1, -2, \cdots, -d_2 - d_4 - d_6 - \cdots - d_{2\lfloor \frac{v}{2} \rfloor}\} \cup \{b_j : j = 1, 2, \cdots, d_1 + d_3 + d_5 + \cdots + d_{2\lfloor \frac{v-1}{2} \rfloor + 1}\}$,

(iii) the edges of $G_1$ have labels $1, 2, 3, \cdots, t-1$.

Since $G_1 \cong G$, we can orient the edges of $G_1$ so that the resulting digraph, say $G_2$, is isomorphic to $\hat{G}$. Hence $G_2$ is a subdigraph of $C^*_{n,t-1}$ such that

(i) $G_2 \cong \hat{G}$,

(ii) the vertex set of $G_2 = \{a_i : i = 0, -1, -2, \cdots, -d_2 - d_4 - d_6 - \cdots - d_{2\lfloor \frac{v}{2} \rfloor}\} \cup \{b_j : j = 1, 2, \cdots, d_1 + d_3 + d_5 + \cdots + d_{2\lfloor \frac{v-1}{2} \rfloor + 1}\}$,

(iii) the arcs of $G_2$ have labels $1, 2, 3, \cdots, t-1$.

Let $G_3 = Dual(G_2)$. Then

(i) $G_3 \cong \hat{G}$,

(ii) the vertex set of $G_3$ is $\{a_i : i = -1, -2, \cdots, -d_1 - d_3 - d_5 - \cdots - d_{2\lfloor \frac{v-1}{2} \rfloor + 1}\} \cup \{b_j : j = 0, 1, 2, \cdots, d_2 + d_4 + d_6 + \cdots + d_{2\lfloor \frac{v}{2} \rfloor}\}$,

(iii) the arcs of $G_3$ have labels $1, 2, 3, \cdots, t-1$.

Now we prove (1). Since $G$ is balanced, we have $d_1 + d_3 + d_5 + \cdots + d_{2\lfloor \frac{v-1}{2} \rfloor + 1} = 1 + d_2 + d_4 + \cdots + d_{2\lfloor \frac{v}{2} \rfloor} = t/2$. Let $n' = \frac{n}{t/2}$ and let $F_1$ and $F_2$ be subdigraphs of $C^*_{n,t-1}$ such that $F_1 = G_2 \cup (G_2 + \frac{t}{2}) \cup (G_2 + 2 \cdot \frac{t}{2}) \cup (G_2 + 3 \cdot \frac{t}{2}) \cup \cdots \cup (G_2 + (n'-1)\frac{t}{2})$ and $F_2 = G_3 \cup (G_3 + \frac{t}{2}) \cup (G_3 + 2 \cdot \frac{t}{2}) \cup (G_3 +$

$3\cdot\frac{t}{2})\cup\cdots\cup(G_3+(n'-1)\frac{t}{2})$. Then both $F_1$ and $F_2$ are $\hat{G}$-factors of $C^*_{n,t-1}$. We can see that $C^*_{n,t-1}$ can be decomposed into the following $\hat{G}$-factors: $F_1, F_1+1, F_1+2, \cdots, F_1+(\frac{t}{2}-1)$ and $F_2, F_2+1, F_2+2, \cdots, F_2+(\frac{t}{2}-1)$. Thus $C^*_{n,t-1}$ has a $\hat{G}$-factorization.

Next we prove (2). Let $G_4 = G_3 + (1 + d_1 + d_3 + d_5 + \cdots + d_{2\lfloor\frac{v-1}{2}\rfloor+1})$. Then

(i) $G_4 \cong \hat{G}$,

(ii) the vertex set of $G_4$ is $\{a_i : i = 1, 2, \cdots, -1 + d_1 + d_3 + d_5 + \cdots + d_{2\lfloor\frac{v-1}{2}\rfloor+1}, d_1 + d_3 + d_5 + \cdots + d_{2\lfloor\frac{v-1}{2}\rfloor+1}\} \cup \{b_j : j = 1 + d_1 + d_3 + d_5 + \cdots + d_{2\lfloor\frac{v-1}{2}\rfloor+1}, 2 + d_1 + d_3 + d_5 + \cdots + d_{2\lfloor\frac{v-1}{2}\rfloor+1}, \cdots, 1 + d_1 + d_2 + d_3 + d_4 + \cdots + d_v = t\}$,

(iii) the arcs of $G_4$ have labels $1, 2, 3, \cdots, t-1$.

Let $n' = \frac{n}{t}$ and let $G_5$ be a subdigraph of $C^*_{n,k}$ such that $G_5 = G_2 \cup G_4$,. Note the vertex set of $G_5$ is $\{a_i : -d_2 - d_4 - d_6 - \cdots - d_{2\lfloor\frac{v}{2}\rfloor} \le i \le d_1+d_3+d_5+\cdots+d_{2\lfloor\frac{v-1}{2}\rfloor+1}\} \cup \{b_j : 1 \le j \le 1+d_1+d_2+d_3+d_4+\cdots+d_v = t\}$. Let $F$ be a subdigraph of $C^*_{n,k}$ such that $F = G_5 \cup (G_5+t) \cup (G_5+2t) \cup \cdots \cup (G_5 + (n'-1)t)$. Then $F$ is a $\hat{G}$-factor of $C^*_{n,t-1}$, and $C^*_{n,k}$ is decomposed into the following $\hat{G}$-factors: $F, F+1, F+2, \cdots, F+(t-1)$. □

Now we have the necessary and sufficient condition for the directed caterpillar factorization of symmetric crowns.

**Theorem 3.2** *Let $G$ be a caterpillar of order $t$, and the directed caterpillar $\hat{G}$ be a digraph obtained from $G$ by orienting each edge of $G$ arbitrarily. We have the following.*
*(1) Suppose that $G$ is balanced. Then the following conditions are equivalent.*
*A. $C^*_{n,k}$ has a $\hat{G}$-factorization.*
*B. $2C_{n,k}$ has a $G$-factorization.*
*C. $n \ge k$ are positive integers and $n \equiv 0 \pmod{t/2}$, $k \equiv 0 \pmod{t-1}$.*
*(2) Suppose that $G$ is unbalanced. Then the following conditions are equivalent.*
*A. $C^*_{n,k}$ has a $\hat{G}$-factorization.*
*B. $2C_{n,k}$ has a $G$-factorization.*
*C. $n \ge k$ are positive integers and $n \equiv 0 \pmod{t}$, $k \equiv 0 \pmod{t-1}$.*

**Proof.** We first prove (1).
(A ⇒ B) This is trivial.
(B ⇒ C) It is trivial that $n \ge k$. Since $G$ is balanced, $t$ is an even integer. By Lemma 1.1 (1), $2n \equiv 0 \pmod{t}$. Hence $n \equiv 0 \pmod{t/2}$. By Lemma 1.2 with $\lambda = 2$, we have $k \equiv 0 \pmod{t-1}$.

(C $\Rightarrow$ A) Since $n \geq k \geq t-1$, $n \equiv 0 \pmod{t/2}$, it follows from Lemma 3.1(1) that $C^*_{n,t-1}$ has a $\hat{G}$-factorization. Since $n \geq k$ and $k \equiv 0 \pmod{t-1}$, $C^*_{n,k}$ has a $\hat{G}$-factorization by Lemma 1.5.

Next we prove (2).
(A $\Rightarrow$ B) This is trivial.
(B $\Rightarrow$ C) Since $2C_{n,k}$ is a balanced connected bipartite multigraph of order $2n$, and the caterpillar $G$ is an unbalanced connected bipartite graph of order $t$, by Lemma 1.3 the existence of $G$-factor of $2C_{n,k}$ implies $n \equiv 0 \pmod{t}$. By Lemma 1.2 with $\lambda = 2$, we have $k \equiv 0 \pmod{t-1}$.
(C $\Rightarrow$ A) Since $G$ is an unbalanced caterpillar of order $t$ and $n$ is a positive integer with $n \equiv 0 \pmod{t}$, by Lemma 3.1(2) $C^*_{n,t-1}$ has a $\hat{G}$-factorization. Since $k \equiv 0 \pmod{t-1}$ and $k \leq n$, $C^*_{n,k}$ has a $\hat{G}$-factorization by Lemma 1.5. □

# References


[1] J.-C. Bermond, K. Heinrich, and M.-L. Yu, "Existence of Resolvable Path Designs," *European J. Combin.*, Vol. 11, pp. 205-211, 1990.

[2] D.E. Bryant, S.I. El-Zanati, and C.V. Eynden, "Star Factorizations of Graph Products," *J. Graph Theory*, Vol. 36, pp. 59-66, 2001.

[3] C-J. Colbourn, B-D. McKay, "Cubic Neighbourhoods in triple systems," *Combinatorial Design Theory*, North-Holland Math.Stud., 149, North-Holland, Amsterdam, pp. 119-136, 1987.

[4] B. Du, "$K_{1,p^2}$-Factorization of Complete Bipartite Graphs," *Discrete Math.*, Vol. 187, pp. 273-279, 1998.

[5] B. Du, "$P_{2k}$-Factorization of Complete Bipartite Multigraphs," *Austral. J. Combin.*, Vol. 21, pp. 197-199, 2000.

[6] B. Du, "$K_{1,pq}$-Factorization of Complete Bipartite Graphs," *Austral. J. Combin.*, Vol. 26, pp. 85-92, 2002.

[7] B. Du and J. Wang, "$K_{1,k}$-Factorization of Complete Bipartite graphs," *Discrete Math.*, Vol. 259, pp. 301-306, 2002.

[8] B. Du and J. Wang, "The Proof of Ushio's Conjecture Concerning Path Factorization of Complete Bipartite Graphs," *Sci. China Ser.A*, Vol. 49, pp. 289-299, 2006.

[9] A. Muthusamy and P. Paulraja, "Path Factorizations of Product Graphs," *Graph Theory and its Applications*, Tata McGraw-Hill, New Delhi, pp. 77-88, 1997.

[10] A. Muthusamy and P. Paulraja, "Path Factorizations of Complete Multipartite Graphs," *Discrete Math.*, Vol. 195, pp. 181-201, 1999.

[11] N. Martin, "Complete Bipartite Factoriztions by Complete Bipartite graphs," *Discrete Math.*, Vol. 167/168, pp. 461-480, 1997.

[12] N. Martin, "Balanced Bipartite Graphs May Be Completely Star-Factored," *J. Comb. Des.*, Vol. 5, pp. 407-415, 1997.

[13] N. Martin, R. Stong, "Permutation Separations and Complete Bipartite Factoriztions of $K_{n,n}$," *Electron. J. Combin.*, Vol. 10, pp. 26,2003.

[14] N. Martin, "Unbalanced Star-Factorizations of Complete Bipartite Graphs," *Discrete Math.*, Vol. 283, pp. 159-165, 2004.

[15] K. Ushio, "Star-Factorization of Symmetric Complete Bipartite Digraphs," *Discrete Math.*, Vol. 167/168, pp. 593-596, 1997.

[16] K. Ushio, "$\hat{S}_k$-Factorization of Symmetric Complete Tripartite Digraphs," *Discrete Math.*, Vol. 197/198, pp. 791-799, 1999.

[17] K. Ushio, "Semi-Evenly Partite Star-Factorization of Symmetric Complete Tripartite Digraphs," *Australas. J. Combin.*, Vol. 20 , pp. 69-75, 1999.

[18] K. Ushio, "$\bar{S}_k$-Factorization of Symmetric Complete Tripartite Digraphs," *Discrete Math.*, Vol. 211, pp. 281-286, 2000.

[19] K. Ushio, "Star-Factorization of Symmetric Complete Bipartite Multi-Digraphs," *Discrete Math.*, Vol. 215, pp. 293-297, 2000.

[20] K. Ushio, "Evenly Partite Star-Factorization of Symmetric Complete Tripartite Multi-Digraphs," *Electron. Notes Discrete Math.*, Vol. 11, pp. 1-4, 2002.

[21] H. Wang, "On $K_{1,k}$-Factorizations of a Complete Bipartite Graph," *Discrete Math.*, Vol. 126, pp. 359-364, 1994.

[22] M.-L. Yu, "Resolvable Tree Designs," *J. Combin. Theory Ser.A*, Vol. 61, pp. 302-308, 1992.

[23] M.-L. Yu, "On Path Factorizations of Complete Multipartite Graphs," *Discrete Math.*, Vol. 122, pp. 325-333, 1993.

[24] M.-L. Yu, "On Tree Factorizations of $K_n$," *J. Graph Theory*, Vol. 17, pp. 713-725, 1993.