AVOIDABLE PARTIAL LATIN SQUARES OF ORDER $4m+1$

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Abstract: A *partial latin square* P of order n is an $n \times n$ array with entries from the set $\{1, 2, \dots, n\}$ such that each symbol is used at most once in each row and at most once in each column. If every cell of the array is filled we call P a *latin square*. A partial latin square P of order n is said to be *avoidable* if there exists a latin square L of order n such that P and L are disjoint. That is, corresponding cells of P and L contain different entries. In this note we show that, with the trivial exception of the latin square of order 1, every partial latin square of order congruent to 1 modulo 4 is avoidable.

1. Introduction

In what follows, $N(n) = \{1, 2, \dots, n\}$. A *partial latin square* P of order n a set of ordered triples of the form (i, j, k), where $i, j, k \in N(n)$ with the following properties:

- if $(i, j, k) \in P$ and $(i, j, k') \in P$ then $k = k'$,
- if $(i, j, k) \in P$ and $(i, j', k) \in P$ then $j = j'$ and
- if $(i, j, k) \in P$ and $(i', j, k) \in P$ then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set $N(n)$ such that if $(i, j, k) \in P$, the *entry* k occurs in row i and column j.

In this paper we wish to keep an informal tone to the proofs. Nevertheless we make the following points clear to avoid any ambiguity.

Firstly, the term *entry* never refers to a row or a column. Specifically, an *element* of P is a triple $(i,j,k) \in P$, whereas an *entry* in P is a value k, $k \in N(n)$ such that $(i,j,k) \in P$, for some $i,j \in N(n)$. Also, when we say that a row i occurs *in* a latin square P, we mean there is some column j and entry k such that $(i,j,k) \in P$. With similar meaning, we may also say that a column j or entry k occurs *in* a partial latin square P.

Note that a partial latin square has the property that each entry occurs at most once in each row and at most once in each column. If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $N(n)$ in such a way that each entry occurs precisely once in each row and precisely once in each column of the array.

A partial latin square of order n is said to be *completable* if there exists some latin square L of order n such that $P \subseteq L$. Not every partial latin square is completable; the partial latin squares $P2$ and $P3$ below are clearly *incompletable.*

1	
	2

$P2$

1	2	
2	1	3
	3	1

$P3$

For a given partial latin square P the set of cells

$$\mathcal{S}_P = \{(i,j) \mid (i,j,k) \in P, \text{ for some } k \in N(n)\}$$

is said to determine the *shape* of P and $|\mathcal{S}_P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells in the array. A partial latin square P which is not a latin square is said to be *maximal* if for any partial latin square Q, $P \subseteq Q$ implies that $P = Q$. (Thus a maximal partial latin square is incompletable.) The examples $P2$ and $P3$ above are both maximal partial latin squares. A partial latin square P of order n is said to be *avoidable* if there exists a latin square L of order n such that $L \cap P = \emptyset$.

Two partial latin squares P and Q are said to be *isotopic* if there exist three permutations α, β, γ on the set $N(n)$ such that $(i,j,k) \in P$ if and only if $(\alpha(i), \beta(j), \gamma(k)) \in Q$. Since an isotopism just involves a relabelling of entries and a reshuffling of rows and columns, most structural properties of a partial latin square are preserved under isotopism. In particular, if a partial latin square P is avoidable, then any isotopism of P is also avoidable.

There is another kind of equivalence if we transpose columns with rows, or permute row, column and entry labels with each other. For a given partial latin square P, the *parastrophies* (or *conjugates*) of P are the partial latin squares given by:

$$\begin{array}{l}\{(i,k,j) \mid (i,j,k) \in P\}, \{(k,j,i) \mid (i,j,k) \in P\} \\ P^T = \{(j,i,k) \mid (i,j,k) \in P\}, \{(k,i,j) \mid (i,j,k) \in P\}, \\ \{(j,k,i) \mid (i,j,k) \in P\}.\end{array}$$

The parastrophy P^T (labelled above) is also called the *transpose* of P. Again, if a partial latin square P is avoidable, then any parastrophy of P is also avoidable. The proofs in this paper use isotopies and conjugacies freely and frequently.

To date, up to isotopism, there are only three partial latin squares which are known to be unavoidable. One is trivially the latin square of order 1. The remaining two are $P2$ and $P3$, given above. In fact, the following is conjectured:

Conjecture 1. (Chatwynd and Rhodes [1]) If $n > 3$, every partial latin square of order n is avoidable.

Although it was Chatwynd and Rhodes who first formally made this conjecture, the problem originates in work by Häggkvist, who showed the conjecture to be true in the following case:

Theorem 2. (Häggkvist [4]) Let $n = 2^k$ and let P be a partial latin square with an empty last column. Then P is avoidable.

This result was broadened to the following:

Theorem 3. (Chatwynd and Rhodes [1]) Any partial latin square of order n is avoidable, where $n > 3$ and either n is even or n is a multiple of three.

The problem of avoiding partial latin squares is related to the intricacy of avoiding arrays, discussed in [6]. The next lemma has a clear proof.

Lemma 4. Let P and Q be partial latin squares such that Q is avoidable and $P \subseteq Q$. Then Q is also avoidable.

Any latin square of order at least 2 is avoidable (simply relabel the entries). The following theorem is then immediate.

Theorem 5. (Theorem 2.1, Chatwynd and Rhodes [1]) Any partial latin square of order $n > 1$ with unique completion to a latin square is avoidable.

From the above observations, it follows that if we can avoid any maximal partial latin square of order n, we can avoid any partial latin square of order n. With this in mind, the next result will come in handy:

Theorem 6. (Theorem 7, Horak and Rosa [5]) Let P be a maximal partial latin square of order n. Then $0.5n^2 \leq |P| \leq n^2 - 2$.

The following lemma is also useful.

Lemma 7. Let m be some integer such that every partial latin square of order m is avoidable. Then for each integer $k \geq 1$, every partial latin square of order mk is avoidable.

Proof: Let P be a partial latin square of order km and let L be a latin square of order km which partitions into k^2 disjoint latin subsquares, each of order m. Let M be one of these subsquares and suppose that M contains the entries from the set $E \subset N(n)$, where $|E| = m$. Next construct a partial latin square P', by deleting any occurrences of entries $N(n) \setminus E$ from the corresponding cells in P. Rearrange M so that it avoids P'. (This is possible as we are assuming that every partial latin square of order m is avoidable.) Repeat for each $m \times m$ subsquare. The resultant latin square avoids P. □

Before commencing our proofs, we introduce some further notation. A *transversal* is a partial latin square of size n and order n that conatins each row, column and entry exactly once. A *partial transversal* is any subset of a transversal. In this paper a *quasi-transversal* is a partial latin square in which each row and column occurs *at most* once. So in a quasi-transversal, entries may be repeated.

A $k \times l$ partial latin square is one in which entries may occur only within a set of k rows and l columns. Let P be a partial latin square of order n and let R, C be subsets of $N(n)$. Then the partial latin square $P(R \times C)$ is defined as:

$$P(R \times C) = \{(i,j,k) \mid i \in R, j \in C, (i,j,k) \in P\}.$$

Clearly $P(R \times C)$ is a $|R| \times |C|$ partial latin square.

2. Avoiding partial latin squares of order 5

There are, up to isotopism and under parastrophy, exactly two latin squares of order 5 (see, for example, [2]). We label these squares $L5.1$ and $L5.2$:

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L5.1$

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

$L5.2$

We will mainly use isotopisms of $L5.2$ to avoid partial latin squares of order 5.

Lemma 8. Let P be a maximal partial latin square of order 5. Then there exists a row, a column, and an entry which occurs more than once in P.

Proof: From Theorem 6, $|P| \geq 13$. The result then follows. □

Definition 9. We call a 2×3 partial latin square *appropriate* if it contains at least two entries in one of its rows *but* at most three distinct entries altogether.

Lemma 10. Let P be a maximal partial latin square P of order 5. Then either P or the transpose of P contains an appropriate partial latin square.

Proof: Note that an appropriate partial latin square has two opposing (but not contradictory) properties: on the one hand we need to minimize the number of distinct entries to *at most* three; on the other hand we need a row that contains *at least* two entries. Our proof is a careful balancing of these properties.

Now, from Theorem 6, since P is maximal, P has at least two empty cells. We split up our proof into cases depending on the maximum number of empty cells in a row or column.

Case A1: P has a row with exactly one empty cell. Let this row be r and let r' be another row with at least one empty cell. Suppose cells (r, c) and (r', c') are the empty cells. First consider the case $c = c'$. Then there must exist a cell (r', c'') such that $c'' \neq c'$ and either (r', c'') is empty or the entry in cell (r', c'') also occurs in some cell (r, c'''). In either case $P(\{r, r'\} \times \{c, c'', c'''\})$ is appropriate. Otherwise we may assume that $c \neq c'$.

If the entry in (r', c) equals the entry in (r, c'), then for any column $c'' \notin \{c, c'\}$, $P(\{r, r'\} \times \{c, c', c''\})$ is appropriate. Otherwise suppose $1 \in (r, c')$ and $2 \in (r', c)$. If either 1 or 2 (or an empty cell) occurs again in rows $r \cup r'$, we can append an extra column as before. Otherwise, exactly three distinct entries 3, 4 and 5 occur in the columns excluding c and c', and each of them occurs in row r, so $P(\{r, r'\} \times (N(5) \setminus \{c, c'\}))$ is appropriate.

Case A2: P has a column with exactly one empty cell. Here we apply the proof for Case A1 to the transpose of P.

Case B: P has a row with exactly two empty cells. Let this row be r and suppose $1 \in (r, 1)$, $2 \in (r, 2)$, $3 \in (r, 3)$ and $(r, 4)$ and $(r, 5)$ are both empty. Take any other row r'. Firstly, suppose that at least one of the entries 1, 2 or 3 occurs in cell $(r', 4)$ or cell $(r', 5)$. Without loss of generality assume that $(r', 4)$ contains entry 1. Then if $(r', 1)$ is non-empty, $P(\{r, r'\} \times \{1, 4, 5\})$ is appropriate. On the other hand if $(r', 1)$ is empty, $P(\{r, r'\} \times \{1, 2, 4\})$ is appropriate.

Secondly, suppose that at least one of cells $(r', 4)$ and $(r', 5)$ is empty. Without loss of generality let $(r', 5)$ be empty. Consider the cells $(r', 1)$, $(r', 2)$ and $(r', 3)$. Since they lie in the same row, at least one contains neither the entry 4 nor the entry 5. Suppose without loss of generality that $(r', 1)$ is one such cell and that $(r', 1)$ contains the entry 2. Then $P(\{r, r'\} \times \{1, 2, 5\})$ is appropriate.

Thirdly and finally, the cells $(r', 4)$ and $(r', 5)$ contain the entries 4 and 5, in some order. In this case $P(\{r, r'\} \times \{1, 2, 3\})$ is appropriate.

Case B2: P has a column with exactly two empty cells. Here we may use the transpose of P.

Case C: Every row or column has at most two filled cells. Under the appropriate conjugacy/parastrophy, we can assume that $1 \in (r, 1)$, $2 \in (r, 2)$ and each other cell in row r is empty. But each column has at most two filled cells so there exists a row $r' \neq r$ such that $(r', 1)$ and $(r', 2)$ are empty. Then $P(\{r, r'\} \times \{1, 2, 3\})$ must be appropriate. □

Theorem 11. Every maximal partial latin square of order 5 can be avoided.

Proof: Let P be a maximal partial latin square of order 5. From the previous lemma, we may assume P contains an appropriate 2×3 partial latin square. So, by transpose and isotopisms, we may assume that $(1,3,1),(1,4,2) \in P$ and that $P_2 = P(\{1,2\} \times \{3,4,5\})$ contains no occurrences of the entries 4 and 5. We then define $P_1 = P(\{1,2\} \times \{1,2\})$, $P_3 = P(\{3,4,5\} \times \{1,2\})$ and $P_4 = P(\{3,4,5\} \times \{3,4,5\})$.

We will construct a latin square Q of order 5 such that $Q \cap P = \emptyset$. Incidentally to the lemma, Q will be isotopic to the latin square $L5.2$.

We first define Q_1, Q_2, Q_3 and Q_4 on the same sets of cells as P_1, P_2, P_3 and P_4, respectively, except as subsets of Q rather than P. First consider Q_1. In this subarray we will place a latin subsquare of order 2 based on the entries 1 and 2. Since the entries 1 and 2 occur neither in cell $(1,1)$ nor in cell $(1,2)$ of P, this subsquare can be made to avoid the corresponding subarray P_1 of P.

We next fill rows 3 to 5 of Q. Consider the subarray P_4. Let $P_4' \subset P_4$ be the set of cells in P_4 containing either entry 1 or entry 2. Since $(1,3,1),(1,4,2) \in P$, $|P_4'| \leq 4$.

Claim: We can locate a quasi-transversal R in P_4 of size 3 such that $(P_4 \setminus R) \cap S = \emptyset$, where S is a partial latin square with the same shape as $P_4 \setminus R$ and S contains exactly three occurrences of 1 and three occurrences of 2. Moreover, either:

(A) at most one cell of R contains an entry from $\{3,4,5\}$; or

(B) at least two rows of P_4 each contains each of the entries 3, 4 and 5; or

(C) at least two columns of P_4 each contains each of the entries 3, 4 and 5.

Suppose, in the first instance, that $|P_4'| \geq 2$ and there exist two cells in P_4' that share neither a common row nor a common column. Then the following diagrams show each possible configuration of P_4', up to any row-relabelling.

	1	2
2		
		1

	1	
2		1
		2

2	1	
		1
		2

	1	2
2		1

2	1	
		1

2	1	
		2

2		
	1	
		1

2		
	1	
		2

2		
		1
		2

		2
	1	
		1

2		
	1	

2		
		1

	1	
		2

Let

$$S_1 = \begin{array}{|c|c|c|}\hline 2 & * & 1 \\ \hline * & 1 & 2 \\ \hline 1 & 2 & * \\ \hline \end{array} \quad \text{and} \quad S_2 = \begin{array}{|c|c|c|}\hline * & 2 & 1 \\ \hline 1 & * & 2 \\ \hline 2 & 1 & * \\ \hline \end{array}.$$

Then letting $S = S_1$ for the first two examples and $S = S_2$ for the remainder, S always avoids $P_4 \setminus R$, where the cells of R are marked with $*$. Moreover in all of these cases at least two cells from R contain either 1 or 2. Thus at most one cell from R contains an entry from $\{3,4,5\}$. In other words all of these cases satisfy Case A.

Otherwise, $|P_4'| \leq 2$ and any two cells of P_4' lie in either a common row or a common column. In this case no matter which quasi-transversal R we pick, an appropriate S can be constructed to avoid $P_4 \setminus R$. If we can construct an R that intersects at most once with entries 3, 4 and 5, we have Case A. Otherwise, all the cells that are either empty or contain 1 or 2 must lie in a single row or column of P_4, implying Cases B or C. Thus, our claim is proven!

Next, since we can still reorder rows 3, 4 and 5 without any loss of generality, assume that $R = \{(3,3),(4,4),(5,5)\}$. We construct a 3×3 partial latin square M as follows. For convenience, we will break convention and label the rows of M with the set $\{3,4,5\}$ and the columns with the set $\{1,2,3\}$. In columns 1 and 2, we place the corresponding entries from P. (So the first two columns are equal to P_3.) In column 3 of M, we place the entries from cells $(3,3)$, $(4,4)$ and $(5,5)$ from P into the cells $(3,3)$, $(4,3)$ and $(5,3)$ of M, respectively. Finally, delete from M any occurrences of the entries 1 and 2.

First, note that M is a partial latin square. Secondly, note that for Case A, M has two empty cells within column 3. For Case B, up to isotopism, P_4 must be the following:

3	4	5
4	5	3

.

Thus M will have two empty cells within a row. Similarly, for Case C, M will also have two empty cells within a row. It follows that for any case, M is avoidable by a 3×3 latin square M' based on the entries 3, 4 and 5. (Recall from the Introduction that the only unavoidable partial latin square of order 3 has at most one empty cell in each row and column.)

We let the first two columns of M' become Q_3. The entries in cells $(3,3)$, $(4,3)$ and $(5,3)$ of M' are next placed in cells $(3,3)$, $(4,4)$ and $(5,5)$ of Q. By construction these will avoid the corresponding entries of P. Finally we fill in the remaining cells of Q_4 with the isotopism of S that avoids $P_4 \setminus R$.

Thus we have determine subarrays Q_1, Q_3 and Q_4 which each avoid P_1, P_3 and P_4, respectively. So, up to isotopism (and possible rearrangements of entries 1 and 2) $Q_1 \cup Q_3 \cup Q_4$ looks like the following:

1	2			
2	1			
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

So in order to complete Q to a latin square, Q_2 is equal to either

3	4	5
4	5	3

or

4	5	3
3	4	5

But as P_2 contains no occurrences of 4 or 5, at least one of these will avoid P_2. □

Applying Lemma 7 to Theorem 11, we obtain the following corollary.

Corollary 12. Any partial latin square of order $5k$ is avoidable, for each $k \geq 1$.

3. The case $n \geq 13$

Here we prove that any partial latin square of order n is avoidable, where $n \geq 13$ and $n \equiv 1 \pmod 4$. Note that the case $n = 9$ is done by Theorem 3 ([1]).

First we observe a few properties of latin squares of order 4. There are, up to isotopism and parastrohpy, two latin squares of order 4 ([2]):

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

$L4.1$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$L4.2$

Given a latin square L of order 4, how can we quickly check whether L is isotopic to $L4.1$ or $L4.2$? The quickest way is to count the number of 2×2 latin subsquares. The latin square $L4.1$ has four of these, while $L4.2$ has twelve.

The subsquares of $L4.1$ are: $L4.1a = L4.1(\{1,2\} \times \{1,2\}), L4.1b = L4.1(\{1,2\} \times \{3,4\}), L4.1c = L4.1(\{3,4\} \times \{1,2\})$ and $L4.1d = L4.1(\{3,4\} \times \{3,4\})$. If we permute the entries within *just one* of these subsquares, we always obtain a latin square isotopic to $L4.2$. To illustrate this, swap the entries 1 and 2 within $L4.1(\{3,4\} \times \{3,4\})$.

Another important property that will exploit is that $L_{4.2}$ partitions into 4 disjoint transversals:

1			
			3
	4		
		2	

+

	2		
		4	
3			
			1

+

		3	
	1		
			2
4			

+

			4
2			
		1	
	3		

.

Finally, observe that there exists a latin square that is isotopic to $L_{4.2}$ and avoids both $L_{4.1}$ and $L_{4.2}$:

3	4	1	2
4	3	2	1
1	2	3	4
2	1	4	3

.

Definition 13. Given a partial latin square P, let $R(i) = \{k \mid (i,j,k) \in P\}$ and $C(j) = \{k \mid (i,j,k) \in P\}$. That is, $R(i)$ is the set of entries that occur in row i in P and $C(j)$ is the set of entries that occur in column j of P.

Lemma 14. Let P be a maximal partial latin square of order 4. Suppose furthermore that for each cell (i, j) that is empty in P, $R(i) \cap C(j) = \emptyset$. Then P, up to isotopism and parastrophy, is equal to either:

1	2		
2	1		
		3	4
		4	3

$P4.1$

or

1	2	3	
2	3	1	
3	1	2	
			4

$P4.2$

.

Proof: If P is maximal and of order 4, then for each empty cell (i, j) of P, $R(i) \cup C(j) = N(4)$. If P satisfies the conditions of this lemma then we also have $R(i) \cap N(j) = \emptyset$. Thus, without loss of generality, let the cell $(1, 4)$ be empty and either:

(1) $R(i) = \{1, 2, 3\}$ and $C(j) = \{4\}$; or

(2) $R(i) = \{1, 2\}$ and $C(j) = \{3, 4\}$.

Assume the first case. Without loss of generality, $(1, 1, 1)$, $(1, 2, 2)$, $(1, 3, 3)$, $(4, 4, 4) \in P$ and cells $(1, 4)$, $(2, 4)$ and $(3, 4)$ are empty. Since $(2, 4)$ and $(3, 4)$ are empty, we must have $R(2) = R(3) = \{1, 2, 3\}$. It follows that $P(\{1, 2, 3\} \times \{1, 2, 3\})$ is a latin subsquare of order 3. Thus P is isotopic to $P4.2$ above.

Otherwise, without loss of generality, $(1, 1, 1)$, $(1, 2, 2)$, $(3, 4, 4)$, $(4, 4, 3) \in P$ and cells $(1, 3)$, $(1, 4)$ and $(2, 4)$ are empty. Since $(2, 4)$ and $(1, 3)$ are empty, $R(2) = \{1, 2\}$ and $C(3) = \{3, 4\}$, respectively. Since $R(2) \cap C(3) = \emptyset$, the cell $(2, 3)$ must be empty. It follows that $(2, 1, 2), (2, 2, 1), (3, 3, 3), (4, 3, 4) \in P$, and P is isotopic to $P4.1$ above. □

Lemma 15. Let P be a partial latin square of order n such that every row (column) of P has either 0 or n empty cells. Then P is completable to a latin square of order n.

Proof: This is a simple application of Hall's Condition. (See [2], for instance, for a proof.) □

Lemma 16. Let P be a partial latin square of order 4. Then some isotope or parastrophy of P is either equal to $P4.2$ or avoids $L4.2$.

Proof: Suppose first that P is completable to a latin square. Then P is completable to an isotope of either $L4.1$ or $L4.2$. However, an isotope of $L4.2$ may avoid either $L4.1$ or itself (from the preamble in this section), so we are done.

Henceforth P is a maximal partial latin square. Suppose that for each empty cell (i, j) in P, $R(i) \cap C(j) = \emptyset$. Then, from Lemma 14, P is isotopic to either $P4.1$ or $P4.2$. If P is isotopic to $P4.2$, this satisfies the conditions of the lemma. Otherwise observe that $P4.1$ is avoided by:

2	1	3	4
1	2	4	3
3	4	2	1
4	3	1	2

,

which is isotopic to $L4.2$.

We may thus assume from now on that there is some empty cell (i, j) with $R(i) \cap C(j) \neq \emptyset$. So, without loss of generality, let $(1, 2, 1), (2, 1, 1) \in P$ and let the cell $(1, 1)$ be empty.

Suppose in the first instance that cell $(2, 2)$ is empty in P. We construct a latin square Q that avoids P as follows. Observe that there are no occurrences of the entry 1 within $P(\{1, 2\} \times \{3, 4\})$ or $P(\{3, 4\} \times \{1, 2\})$. It follows that we can fill the cells of $Q(\{1, 2\} \times \{3, 4\}) \cup Q(\{3, 4\} \times \{1, 2\})$ with entries 1 and z, for any $z \in \{2, 3, 4\}$, in such a way that there is no intersection so far between P and Q. Now consider $P(\{3, 4\} \times \{3, 4\})$.

We wish $Q(\{3, 4\} \times \{3, 4\})$ to be a 2×2 latin subsquare based on two entries a and b. However, if there is a partial transversal of size 2 containing entries a and b within $P(\{3, 4\} \times \{3, 4\})$, then $Q(\{3, 4\} \times \{3, 4\})$ cannot avoid $P(\{3, 4\} \times \{3, 4\})$.

There are at most two partial transversals of size two within $P(\{3, 4\} \times \{3, 4\})$. If there are exactly two, then by the definition of a partial transversal, $P(\{3, 4\} \times \{3, 4\})$ must contain each of the entries 1, 2, 3 and 4 exactly once. It follows that there is at most one partial transversal in $P(\{3, 4\} \times \{3, 4\})$ that *does not* contain the entry 1. Thus there are at least two possible pairs $\{x, y\}$ from the set $\{\{2, 3\}, \{2, 4\}, \{3, 4\}\}$ such that we can fill $Q(\{3, 4\} \times \{3, 4\})$ with entries x and y, whilst still avoiding $P(\{3, 4\} \times \{3, 4\})$. For any such pair $\{x, y\}$, we can also fill $Q(\{1, 2\} \times \{1, 2\})$ with entries x and y, avoiding $P(\{1, 2\} \times \{1, 2\})$ in the process.

So, letting $z = N(4)\setminus\{1, x, y\}$, we have a latin square Q that avoids P. If Q is isotopic to $L4.2$ we are done. Otherwise, since cell $(2,2)$ is empty in P, we can swap the entries x and y within $Q(\{1,2\}\times\{1,2\})$, creating a Q that both avoids P and is isotopic to $L4.2$.

Otherwise cell $(2,2)$ is non-empty in Q. Without loss of generality let $(2,2,2) \in P$. Suppose firstly that there is no partial transversal in $P(\{3,4\}\times\{3,4\})$ with entries 3 and 4. Then we can avoid P with a latin square Q formed by putting entries 1 and 2 in the cells of $Q(\{1,2\}\times\{3,4\}) \cup Q(\{3,4\}\times\{1,2\})$ and entries 3 and 4 in the cells of $Q(\{1,2\}\times\{1,2\}) \cup Q(\{3,4\}\times\{3,4\})$. Moreover we can make Q' isotopic to $L4.2$ by swapping entries 3 and 4 within $Q(\{1,2\}\times\{1,2\})$, if necessary.

Otherwise, without loss of generality, $(3,3,3), (4,4,4) \in P$. Since cell $(1,1)$ is empty and P is maximal, the entry 2 must occur within $R(1) \cup C(1)$. So, without loss of generality, (since 3 and 4 are so far interchangeable and P is so far symmetric) let $(1,3,2) \in P$. Similarly, the entry 4 must occur within $R(1) \cup C(1)$, so $(3,1,4) \in P$. Next, if cell $(2,3)$ is empty, entry 4 must occur within $R(2) \cup C(3)$, which is impossible. Thus we may infer that $(2,3,4) \in P$. By similar reasoning $(4,3,1) \in P$.

If entry 1 does not occur in cell $(3,4)$ of P, the latin square

2	3	1	4
4	1	3	2
3	2	4	1
1	4	2	3

avoids P. Note that this square is isotopic to $L4.2$. Otherwise $(3,4,1) \in P$. If cell $(1,4)$ is empty, the subarray $P(\{1,3\}\times\{2,4\})$ contains only one entry. So we have an isotopism of the case when $(2,2)$ is empty. Otherwise $(1,4,3) \in P$. Considering all possible attempted completions of P, we have that P is either:

	1	2	3
1	2	4	
4		3	1
3		1	4

or

	1	2	3
1	2	4	
4		3	1
2	3	1	4

.

These partial latin squares are avoided by

1	2	3	4
4	3	2	1
3	4	1	2
2	1	4	3

and

4	3	1	2
2	1	3	4
1	2	4	3
3	4	2	1

,

respectively, each of which are isotopic to $L4.2$, as required. □

So, from the previous lemma, $P4.2$ is the unique partial latin square (up to isotopism and parastrophy) that cannot be avoided by $L4.2$.

Lemma 17. Consider the partial latin square $P4.2$. Let $P4.2'$ be the partial latin square created by deleting the entry k from any two cells of $P4.2$; for some fixed $k \in \{1, 2, 3\}$. Then $P4.2'$ may be avoided by an isotope of $L4.2$. Moreover there is a transversal in this isotope of $L4.2$ that includes the two cells from which entries have been deleted.

Proof: First note that there are isotopisms of $P4.2$ which rearrange the entries $1, 2, 3$ in any order, while leaving 4 fixed. Thus, without loss of generality, we may assume that $P4.2'$ is the partial latin square below on the left, which is avoided by $L4.2'$ (which is isotopic to $L_{4.2}$).

1		3	
	3	1	
3	1	2	
			4

$P4.2'$

3	*1*	2	4
2	4	3	1
4	2	1	*3*
1	3	*4*	2

$L4.2'$

The required transversal is shown in italics. □

Lemma 18. For every integer $m \geq 3$, there exists a latin square M of order m, such that M contains a transversal on the set of cells $\{(i, i) \mid 1 \leq i \leq m\}$.

Proof: For $m = 3$, a latin square M with the required properties is the following:

2	1	3
1	3	2
3	2	1

.

For $m > 3$, we can use the fact that there exists a *diagonal* latin square M of order m ([3]). (A diagonal latin square is one which contains a transversal on both the main diagonal $\{(i,i) \mid 1 \le i \le n\}$ and the back diagonal $\{(i, m-i+1) \mid 1 \le \imath \le m\}$.) □

Given M as in the previous lemma, a latin square L of order $4m+1$ may be constructed as follows. Let $M_k = \{4k-2, 4k-1, 4k, 4k+1\}$ for each k between 1 and m. Let $L(\{1,2,3,4,5\} \times \{1,2,3,4,5\})$ be a latin square of order 5 based on the set of entries $\{1\} \cup M_k$, where $(1,1,k) \in M$. For each i and j such that $i \ne j$, let $L(M_i \times M_j)$ be a latin square of order 4 based on the set of entries M_k, where $(i,j,k) \in M$. For each i between 2 and m, let $L(M_i \times M_i)$ be a latin square isotopic to $L4.2$ based on the set of entries M_k, where $(i,i,k) \in M$. Next, replace a transversal from each $L(M_i \times M_i)$ with four occurrences of the entry 1. Finally, there is a unique way to complete row 1 and column 1 of L.

It is this form of construction that is used in the following theorem, to generate a latin square L which avoids a given partial latin square of order $4m+1$. The theorem is followed by an example in order to clarify the process to the reader.

Theorem 19. Let P be a partial latin square of order $n = 4m+1$, where n and m are integers and $m > 0$. Then P is avoidable by some latin square L of order n.

Proof: The cases $n = 5$ and $n = 9$ are done by Theorem 11 and Theorem 3, respectively, so we may assume that $m \ge 3$. From the commentary in the introduction, we may assume that P is maximal. It follows from Theorem 6 that $|P| \ge n^2/2$. Thus there is some entry (say, 1) that occurs at least $n/2 > m-1$ times in P.

So choose a set $Q = \{(r_i, c_i) \mid 1 \le i \le m-1\}$ of cells which each contain the entry 1. Next select any row $r \notin \{r_i \mid 1 \le i \le m-1\}$. Let $E = \{e \mid (r, c_i, e) \in P, 1 \le i \le m-1\}$. Since some cells of P may be empty, the size of E is at most $m-1$. Next, locate a column c such that:

(1) $c \notin \{c_i \mid 1 \le i \le m-1\}$; and
(2) the entry in (r_i, c) is not equal to the entry in (r, c_i), for each i, $1 \le i \le m-1$; and
(3) there exists at least two distinct r_i such that either (r_i, c) is empty or (r_i, c) contains some entry $e \notin E$.

Why does such a column exist? There are $4m+1$ columns altogether. Conditions 1 and 2 each rule out at most $m-1$ choices for c. Since $|E| \le m-1$, the entries from E occur at most $(m-1)^2$ times within the set of rows $\{r_i \mid 1 \leq i \leq m-1\}$. From Condition 3 we wish to rule out columns that contain at least $m-2$ entries from E within this set of rows, so this excludes at most a further

$$\frac{(m-1)^2}{(m-2)} = m + \frac{1}{(m-2)} < m+1$$

columns. But $4m+1-2(m-1)-(m+1) = m+3 \geq 1$ so there certainly exists such a column.

Next we set up a one-to-one matching between the set of cells $\{(r, c_i) \mid 1 \leq i \leq m-1\}$ and $\{(r_i, c) \mid 1 \leq i \leq m-1\}$ as follows. Firstly, if (r_i, c) and (r, c_j) contain the same entry, we match these. From Condition 3 above, there are at least two remaining unmatched pairs of cells. It is thus possible to match these so that for each i, (r_i, c) is not matched to (r, c_i).

Let e_0 be the entry in cell (r, c) of P. Partition $N(n) \setminus \{e_0\}$ into m disjoint subsets $E_1, E_2, \ldots, E_m$, each of size 4 such that:

1. for each matching $\{(r_i, c), (r, c_j)\}$, there exists a unique subset E_k, where
2. if (r_i, c) contains an entry e_1, $e_1 \in E_k$, and
3. if (r, c_j) contains an entry e_2, $e_2 \in E_k$.

Note there exists just $m-1$ pairs in the matching, so there is one subset in the partition (say E_1) which does not correspond to a matching.

Next, partition the columns of $N(n) \setminus \{c\}$ into m disjoint subsets $C_1, C_2, \ldots, C_m$, each of size 4, such that whenever $(r, c_i, e_j) \in P$, if $e_j \in E_k$ then $c_i \in C_k$. (If row r is full in P there is a unique such partition of the columns; otherwise the columns which are empty in row r may be placed arbitrarily.) Similarly, partition the rows of $N(n) \setminus \{r\}$ into m disjoint subsets $R_1, R_2, \ldots, R_m$, each of size 4, such that whenever $(r_i, c, e_j) \in P$, if $e_j \in E_k$ then $r_i \in R_k$. (If column C is full in P there is a unique such partition of the rows; otherwise the rows which are empty in column c may be placed arbitrarily.)

Without loss of generality, we may assume that $r = c = 1$ and $R_k = C_k = \{4k-2, 4k-1, 4k, 4k+1\}$ for each k between 1 and m. Observe that, by construction, for each k, $1 \leq k \leq m$, there exists an occurrence of the entry 1 within cells (r, c') and (r', c), where $r \in R_k$, $r' \notin R_k$, $c \in C_k$ and $c \notin C_k$. This implies, in turn, that the entry 1 appears at most three times within the partial latin square

$P(R_k \times C_k)$, for each k, $2 \le k \le m$. Suppose that $e_0 \ne 1$ and $1 \in E_k$ for some k. We then replace E_k with $(E_k \setminus \{1\}) \cup \{e_0\}$.

This finishes our "fiddling" with P. We are now ready to construct a latin square L which avoids P. Let M be a latin square of order m containing a transversal. (Such a latin square exists from the previous lemma as $m \ge 3$.) Relabel the entries of M so that cell (i, i) never contains the entry i.

Let $L(\{1,2,3,4,5\} \times \{1,2,3,4,5\})$ be a latin square of order 5 based on the entries $\{1\} \cup E_k$, where k is the entry in cell $(1,1)$ of M. From Theorem 11 in the previous section, this can be made to avoid the corresponding cells of P. For each i and j such that $i \ne j$, place a latin square of order 4 based on the entries E_k on $L(R_i \times C_j)$, where k is the entry in cell (i, j) of M. Since any partial latin square of order 4 is avoidable, each of these latin squares may be placed in such a way to avoid the corresponding cells of P.

Next consider $P(R_2 \times C_2)$ and suppose that $(2, 2, k) \in M$. (Note that $k \ne 2$ by construction.) Delete any occurrences of entries in $P(R_2 \times C_2)$ which do not occur in R_k. Let P' be the resultant latin square of order 4. If P' is not isotopic to $P4.2$, from Lemma 16 it is avoidable by a latin square S isotopic to $L4.2$. In this case S partitions into 4 transversals. Moreover, since 1 occurs at most three times in $P(R_2 \times C_2)$, at least one of these transversals avoids all occurrences of 1. Remove this transversal from S and fill the missing cells with entry 1. We place this structure on $L(R_2 \times C_2)$.

Otherwise $P' = P_{4.2}$. As before, we will place entry 1 in four cells of $L(R_2 \times C_2)$. First we place entry 1 once in the set of cells $\{(9,6), (9,7), (9,8)\}$ and once in the set of cells $\{(6,9), (7,9), (8,9)\}$ so that any occurrence of 1 in P is avoided. The remaining two entries 1 are then placed in cells that contain the *same* entry in P. (This is possible as any 2×2 subarray in $P4.2(\{1,2,3\} \times \{1,2,3\})$ must contain some entry at least twice.) Thus, from Lemma 17, we can delete a transversal from some isotope of $L4.2$, replace the entries of the transversal with four occurrences of the entry 1, avoiding P.

In either case there are then unique choices for the entries in cells

$$(1,6), (1,7), (1,8), (1,9), (6,1), (7,1), (8,1), (9,1)$$

in L. Each of these cells must contain an entry from E_k. However, in P the entries in these cells are a subset of $E_2 \cup \{1\}$. Since $k \ne 2$, $E_k \cap E_2 = \emptyset$, so P will be avoided here. We repeat for each $P(R_k \times C_k)$, $2 < k \le m - 1$, and our construction is complete. □

Example 20. Consider the following maximal partial latin square P of order 13:

4	5	6	7	8	9	10	11	12	1	2	3	
5	6	4	8	9	7	11	12	10	2	3	1	
6	4	5	9	7	8	12	10	11	3	1	2	
7	8	9	10	11	12	1	2	3	4	5	6	
8	9	7	11	12	10	2	3	1	5	6	4	
9	7	8	12	10	11	3	1	2	6	4	5	
10	11	12	1	2	3	4	5	6	7	8	9	
11	12	10	2	3	1	5	6	4	8	9	7	
12	10	11	3	1	2	6	4	5	9	7	8	
1	2	3	4	5	6	7	8	9	10	11	12	
2	3	1	5	6	4	8	9	7	11	12	10	
3	1	2	6	4	5	9	7	8	12	10	11	
												13

.

Using the previous lemma, we shall construct a latin square L of order 13 which avoids P:

1	12	11	10	9	3	4	13	2	6	7	8	5
12	1	10	9	11	5	6	7	8	4	2	3	13
11	10	9	1	12	6	5	8	7	2	3	13	4
10	9	12	11	1	7	8	5	6	3	13	4	2
9	11	1	12	10	8	7	6	5	13	4	2	3
3	5	6	7	8	1	2	4	13	10	9	11	12
13	6	5	8	7	2	3	1	4	9	10	12	11
2	7	8	5	6	4	13	3	1	12	11	9	10
4	8	7	6	5	13	1	2	3	11	12	10	9
7	4	2	3	13	10	9	11	12	5	1	6	8
6	2	3	13	4	9	10	12	11	1	8	5	7
5	3	13	4	2	12	11	9	10	8	6	7	1
8	13	4	2	3	11	12	10	9	7	5	1	6

.

Indexing the rows and columns with $N(13)$, we choose $r_1 = 3$, $c_1 = 11$, $r_2 = 11$ and $c_2 = 3$ so that $Q = \{(3,11),(11,3)\}$. Then we choose $r = 1$, which forces $E = \{2,6\}$. Next, $c = 1$ is an appropriate choice, as are the matchings $\{(1,11),(11,1)\}$, $\{(3,1),(1,3)\}$ and sets $R_1 = C_1 = \{2,3,4,5\}$, $R_2 = C_2 = \{6,7,8,9\}$, $R_3 = C_3 = \{10,11,12,13\}$, $E_1 = \{5,6,7,8\}$, $E_2 = \{9,10,11,12\}$ and $E_3 = \{2,3,4,13\}$. We choose M to be the latin square of order 3 from the proof of Lemma

18. Finally, we use Theorem 11 and Lemmata 16 and 17 to construct the latin square L above. The doubled lines in each square are a visual aid for checking that L avoids P.

The following problem remains open; from the results in this paper it seems feasible.

Open Problem 21. Prove that any partial latin square of order $n > 3$ is avoidable, where $n \equiv 3 \pmod 4$ and n is not divisible by 5.

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