On the Disarrangements with k Cycles

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Abstract Let $d(n,k)$ denote the number of disarrangements (permutations without fixed points) with k cycles of the set $[n] = \{1, 2, \cdots, n\}$. In this paper, a new explicit expression for $d(n,k)$ is presented by graph theoretic method, and a concise regular binary tree representation for $d(n,k)$ is provided.

Keywords disarrangements; associated Stirling number; regular binary tree; indefinite equation.

1 Introduction

The Stirling number of the first kind plays an important role in combinatorial analysis, probability, permutation group, etc.. The unsigned Stirling number of the first kind is defined as follows.

Definition 1.1([6]). *The unsigned Stirling numbers of the first kind $s(n,k)$ are defined by*

$$\sum_{k=1}^{n} s(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1).$$

There is an equivalent definition.

Definition 1.2 ([2]). *The unsigned Stirling number of the first kind $s(n,k)$ is defined as the number of permutations of the set $[n] = \{1, 2, \cdots, n\}$ with k cycles.*

According to the definition 1.2, the r-associated Stirling number of the first kind is defined naturally as follows.

Definition 1.3([2]). *The $r-$associated Stirling number of the first kind*

$d_r(n,k)$ is defined as the number of permutations of the set $[n]=\{1,2,\cdots,n\}$ with k cycles of length $\geq r$.

It is clear that $d_1(n,k)=s(n,k)$ is the unsigned Stirling number of the first kind. We denote $d_2(n,k)$ by $d(n,k)$ and $d(n,k)$ is the number of disarrangements with k cycles of the set $[n]=\{1,2,\cdots,n\}$([2]).

From the vertical generating function of $d(n,k)$([2])

$$\sum_{n\geq k} d(n,k)\frac{t^n}{n!}=\frac{1}{k!}(\sum_{n\geq 2}\frac{t^n}{n})^k,$$

it follows that

$$d(n,k)=\frac{n!}{k!}\sum_{\substack{i_1+i_2+\cdots+i_k=n\\ i_j\geq 2,\ j=1,2,\cdots,k}}\frac{1}{i_1 i_2\cdots i_k}. \tag{1}$$

In this paper, we find a new explicit expression for $d(n,k)$ by making use of graph theoretic approach (which is proved by induction method), and give a concise regular binary tree representation for $d(n,k)$.

2 A New Explicit Expression for d(n,k)

For $d(n,n)=0$, we may suppose that $n\geq 2$ and $1\leq k<n$ in the following.

Theorem 2.1. *For the numbers of disarrangements with k cycles of the set $[n]=\{1,2,\cdots,n\}$, there is an explicit expression*

$$d(n,k)=\sum_{\substack{1=i_1<i_2<\cdots<i_{n-k}=n-1\\ i_{j+1}-i_j\leq 2,\ j=1,2,\cdots,n-k-1}} i_1 i_2\cdots i_{n-k}. \tag{2}$$

For example,

$$\begin{aligned} d(7,3) &= \sum_{\substack{1=i_1<i_2<i_3<i_4=6\\ i_{j+1}-i_j\leq 2, j=1,2,3}} i_1 i_2 i_3 i_4 \\ &= 1\cdot 2\cdot 4\cdot 6+1\cdot 3\cdot 4\cdot 6+1\cdot 3\cdot 5\cdot 6 \\ &= 210. \end{aligned}$$

The proof of Theorem 2.1 is based on the triangular recurrence relation for $d(n,k)$.

Lemma 2.2([2]). *The numbers of disarrangements with k cycles of the set $[n]=\{1,2,\cdots,n\}$ satisfy the triangular recurrence relation*

$$d(n+1,k)=nd(n,k)+nd(n-1,k-1).$$

The Proof of Theorem 2.1. We use induction on $n-k$.

When $n-k=1$, $d(n,k)=d(n,n-1)$. From the definition of $d(n,k)$ we know that

$$d(n,n-1)=\begin{cases}1, & n=2\\ 0, & n>2.\end{cases}$$

On the other hand,

$$\sum_{1=i_1=n-1} i_1=\begin{cases}1, & n=2\\ 0, & n>2.\end{cases}$$

Thus Theorem 2.1 holds for $n-k=1$.

Suppose that Theorem 2.1 holds for $n-k=r$. We now consider $d(n+1,k)$, where $(n+1)-k=r+1$. From Lemma 2.2 and induction hypothesis, we have

$$\begin{aligned}
d(n+1,k) &= nd(n,k)+nd(n-1,k-1)\\
&= n\cdot(\sum_{\substack{1=i_1<i_2<\cdots<i_r=n-1\\ i_{j+1}-i_j\le 2,\ j=1,2,\cdots,r-1}} i_1i_2\cdots i_r)\\
&\quad + n\cdot(\sum_{\substack{1=i_1<i_2<\cdots<i_r=n-2\\ i_{j+1}-i_j\le 2,\ j=1,2,\cdots,r-1}} i_1i_2\cdots i_r)\\
&= [r+(n-r)]\cdot(\sum_{\substack{1=i_1<i_2<\cdots<i_r=r+(n-r-1)\\ i_{j+1}-i_j\le 2, j=1,2,\cdots,r-1}} i_1i_2\cdots i_r)\\
&\quad + [r+(n-r)]\cdot(\sum_{\substack{1=i_1<i_2<\cdots<i_r=r+(n-r-2)\\ i_{j+1}-i_j\le 2,\ j=1,2,\cdots,r-1}} i_1i_2\cdots i_r)\\
&= \sum_{\substack{1=i_1<i_2<\cdots<i_r<i_{r+1}=r+(n-r)\\ i_{j+1}-i_j\le 2,\ j=1,2,\cdots,r}} i_1i_2\cdots i_ri_{r+1}\\
&= \sum_{\substack{1=i_1<i_2<\cdots<i_{(n+1)-k}=(n+1)-1\\ i_{j+1}-i_j\le 2,\ j=1,2,\cdots,(n+1)-k-1}} i_1i_2\cdots i_{(n+1)-k}.
\end{aligned}$$

Hence Theorem 2.1 holds for $n-k=r+1$.
This completes the proof. □

Lemma 2.3. *The number of terms in the sum formula*

$$\sum_{\substack{1=i_1<i_2<\cdots<i_r=r+t\\ i_{j+1}-i_j\le 2,\ j=1,2,\cdots,r-1}} i_1i_2\cdots i_r$$

is $\binom{r-1}{t}$ $(t=0,1,2,\cdots;\ r=1,2,\cdots)$.

Proof. For $1 = i_1 < i_2 < \cdots < i_r = r+t$ and $i_{j+1} - i_j \leq 2,\ j = 1, 2, \cdots, r-1$, let

$$x_1 = i_2 - i_1, x_2 = i_3 - i_2, \cdots, x_{r-1} = i_r - i_{r-1},$$

then

$$x_1 + x_2 + \cdots + x_{r-1} = r+t-1,\ 1 \leq x_j \leq 2\ (j = 1, 2, \cdots, r-1). \tag{3}$$

The generating function of the indefinite equation (3) is

$$(y+y^2)^{r-1} = y^{r-1} + \binom{r-1}{1} y^r + \cdots + \binom{r-1}{t} y^{r+t-1} + \cdots + \binom{r-1}{r-1} y^{2r-2}.$$

The coefficient $\binom{r-1}{t}$ of y^{r+t-1} is the number of integer solutions of the indefinite equation (3). Thus the proposition holds. □

Remark 2.4. *The explicit expression* (2) *and* (1) *have the same number of terms.*

In fact, from Lemma 2.3 we know that the number of terms in (2) is $\binom{n-k-1}{k-1}$. In addition, the number of terms in (1) is the number of integer solutions of the indefinite equation

$$i_1 + i_2 + \cdots + i_k = n,\ \ i_j \geq 2\ (j = 1, 2, \cdots, k). \tag{4}$$

The equation (4) is equivalent to the indefinite equation

$$x_1 + x_2 + \cdots + x_k = n - 2k,\ \ x_j \geq 0\ (j = 1, 2, \cdots, k). \tag{5}$$

The number of the non-negative integer solution of (5) is $\binom{n-2k+k-1}{n-2k} = \binom{n-k-1}{k-1}$.

Remark 2.5. *Although the explicit expression* (2) *and* (1) *have the same number of terms, there is an essential distinction between them.*

For example, from the explicit expression (2) we have

$$\begin{aligned} d(8,3) &= \sum_{\substack{1=i_1<i_2<i_3<i_4<i_5=7 \\ i_{j+1}-i_j\leq 2,\ j=1,2,3,4}} i_1 i_2 i_3 i_4 i_5 \\ &= 1\cdot2\cdot3\cdot5\cdot7 + 1\cdot2\cdot4\cdot5\cdot7 + 1\cdot2\cdot4\cdot6\cdot7 + 1\cdot3\cdot4\cdot5\cdot7 \\ &\quad + 1\cdot3\cdot4\cdot6\cdot7 + 1\cdot3\cdot5\cdot6\cdot7 \\ &= 210 + 280 + 336 + 420 + 504 + 630 \\ &= 2380. \end{aligned}$$

From the explicit expression (1) we have

$$d(8,3) = \frac{8!}{3!} \sum_{\substack{i_1+i_2+i_3=8 \\ i_j\geq 2,\ j=1,2,3}} \frac{1}{i_1 i_2 i_3}$$

$$
\begin{aligned}
&= \frac{8!}{3!}\left(\frac{1}{2\cdot 2\cdot 4}+\frac{1}{2\cdot 4\cdot 2}+\frac{1}{4\cdot 2\cdot 2}+\frac{1}{2\cdot 3\cdot 3}+\frac{1}{3\cdot 2\cdot 3}+\frac{1}{3\cdot 3\cdot 2}\right)\\
&= 420+420+420+\frac{1120}{3}+\frac{1120}{3}+\frac{1120}{3}\\
&= 2380.
\end{aligned}
$$

From Theorem 2.1 we may obtain several boundary values.

Corollary 2.6.
(1) $d(2k,k) = 1\cdot 3\cdots(2k-1) = (2k-1)!!$;
(2) $d(2k+1,k) = \frac{(2k+1)!}{(k-1)!}\cdot\frac{1}{3\cdot 2^{k-1}}$;
(3) $d(2k+2,k) = \frac{(2k+2)!}{(k-1)!}\cdot\frac{4k+5}{9\cdot 2^{k+1}}$.

Corollary 2.7.
(1) $d(n,1) = (n-1)!$;
(2) $d(n,2) = (n-1)!(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-2})(n\geq 4)$;
(3) $d(n,3) = (n-1)![(\frac{1}{2\cdot 4}+\frac{1}{2\cdot 5}+\cdots+\frac{1}{2(n-2)})+(\frac{1}{3\cdot 5}+\frac{1}{3\cdot 6}+\cdots+\frac{1}{3(n-2)})$
$+\cdots+\frac{1}{(n-4)(n-2)}](n\geq 6)$.

3 The Regular Binary Tree Representation for d(n,k)

We construct a regular binary tree T as follows.

Layer 1: The root is 1;
Layer 2: The left son of 1 is $1\cdot 2$, with right son $1\cdot 3$;
Layer 3: The left son of $1\cdot 2$ is $1\cdot 2\cdot 3$, with right son $1\cdot 2\cdot 4$; the left son of $1\cdot 3$ is $1\cdot 3\cdot 4$, with right son $1\cdot 3\cdot 5$;
......
Usually, we construct the layer $(k+1)$ from the layer k as follows: any branch node $i_1\cdot i_2\cdots i_k$ ($i_1<i_2<\cdots<i_k$) of the layer k has left son $i_1\cdot i_2\cdots i_k\cdot(i_k+1)$ and right son $i_1\cdot i_2\cdots i_k\cdot(i_k+2)$.
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The binary tree T has the following properties.

Lemma 3.1. *Suppose that T is the regular binary tree constructed as above, then the branch nodes in the layer k of T have the following forms:*

$$i_1\cdot i_2\cdots i_k,$$

where $1=i_1<i_2<\cdots<i_k\leq 2k-1$ *and* $i_{j+1}-i_j\leq 2$ $(j=1,2,\cdots,k-1)$.

Proof. It follows from the construction of T. □

Definition 3.2. *For the regular binary tree T constructed as above, the maximum value i_k of the set $\{i_1,i_2,\cdots,i_k\}$ is called* **the last number** *of* *branch nodes*

$$i_1 \cdot i_2 \cdots i_k \; (1 = i_1 < i_2 < \cdots < i_k \leq 2k-1).$$

For example, the last number of branch node $1 \cdot 3 \cdot 5 \cdot 6 \cdot 8$ is 8.

Lemma 3.3. *For the regular binary tree T constructed as above, we denote the set of branch nodes with last number t in the layer k by $A(k,t)$, then*

$$A(k,t) = \{i_1 \cdot i_2 \cdots i_k \mid 1 = i_1 < i_2 < \cdots < i_k = t, \; i_{j+1} - i_j \leq 2 \; (j = 1, 2, \cdots, k-1)\},$$

and $|A(k,t)| = \binom{k-1}{t-k}$.

Proof. We use induction on k.

It is clear that the proposition holds for $k = 1$.

Assume that the proposition holds for k. We study $A(k+1,t)$, the set of branch nodes with last number t in the layer $k+1$. From the construction of T, we know that there are two kinds of the branch nodes in $A(k+1,t)$.

One kind is the right son of all the branch nodes with last number $t-2$ in the layer k of T, that is

$$A(k,t-2) \cdot t :=$$

$$\{i_1 \cdot i_2 \cdots i_k \cdot t \mid 1 = i_1 < i_2 < \cdots < i_k = t-2, \; i_{j+1} - i_j \leq 2 \; (j = 1, 2, \cdots, k-1)\}.$$

It follows from inductive assumption that $|A(k,t-2) \cdot t| = |A(k,t-2)| = \binom{k-1}{t-2-k}$.

The other kind is the left son of all the branch nodes with last number $t-1$ in the layer k of T, that is

$$A(k,t-1) \cdot t :=$$

$$\{i_1 \cdot i_2 \cdots i_k \cdot t \mid 1 = i_1 < i_2 < \cdots < i_k = t-1, \; i_{j+1} - i_j \leq 2 \; (j = 1, 2, \cdots, k-1)\}.$$

It follows from inductive assumption that

$$|A(k,t-1) \cdot t| = |A(k,t-1)| = \binom{k-1}{t-1-k}.$$

Thus

$$A(k+1,t) = A(k,t-2) \cdot t \cup A(k,t-1) \cdot t$$

$$= \{i_1 \cdot i_2 \cdots i_k \cdot i_{k+1} \mid \; 1 = i_1 < i_2 < \cdots < i_k < i_{k+1} = t, \; i_{j+1} - i_j \leq 2 \; (j = 1, 2, \cdots, k)\}.$$

Since $A(k,t-2) \cdot t \cap A(k,t-1) \cdot t = \emptyset$, we have

$$|A(k+1,t)| = |A(k,t-2) \cdot t| + |A(k,t-1) \cdot t|$$

$$= \binom{k-1}{t-2-k} + \binom{k-1}{t-1-k} = \binom{k}{t-1-k} = \binom{(k+1)-1}{t-(k+1)}.$$

Then the proposition holds for $k+1$. This completes the proof. □

From Theorem 2.1 and Lemma 3.3, we give the regular binary tree representations for $d(n,k)$.

Theorem 3.4. *The numbers of disarrangements with k cycles of the set $[n] = \{1, 2, \cdots, n\}$ has a regular binary tree representation:*
$d(n,k) =$ the sum of branch nodes with last number $n-1$ in layer $n-k$ of T.

Proof. From Theorem 2.1 and Lemma 3.3 it follows that

$$\begin{aligned} d(n,k) &= \sum_{\substack{1=i_1<i_2<\cdots<i_{n-k}=n-1 \\ i_{j+1}-i_j\leq 2,\ j=1,2,\cdots,n-k-1}} i_1 \cdot i_2 \cdots i_{n-k} \\ &= \sum_{i_1\cdot i_2\cdots i_{n-k}\in A(n-k,n-1)} i_1 \cdot i_2 \cdots i_{n-k} \end{aligned}$$

$=$ the sum of branch nodes with last number $n-1$ in layer $n-k$ of T. □

For example,

$$\begin{aligned} d(9,4) &= \text{the sum of branch nodes with last number 8 in layer 5 of T} \\ &= 1\cdot 2\cdot 4\cdot 6\cdot 8 + 1\cdot 3\cdot 4\cdot 6\cdot 8 + 1\cdot 3\cdot 5\cdot 6\cdot 8 + 1\cdot 3\cdot 5\cdot 7\cdot 8 \\ &= 2520. \end{aligned}$$

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