Upper embeddable graphs via the degree-sum of adjacent vertices †

Guanghua Dong[1], Yanpei Liu[2] and Ning Wang[3]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300160, P.R. China. (E-mail: gh.dong@163.com)*

[2] *Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, P.R. China.*

[3] *Department of Information Science and Technology, Tianjin University of Finance and Economics, Tianjin, 300222, P.R. China.*

Abstract: A *semi-double graph* is such a connected multi-graph that each multi-edge consists of two edges. If there is at most one loop at each vertex of a semi-double graph then this graph is called a *single-petal graph*. In this paper we obtained that if G is a connected (resp. 2-edge-connected, 3-edge-connected) simple graph of order n, then G is upper embeddable if $d_G(u) + d_G(v) \geq \lceil \frac{2n-3}{2} \rceil$ (resp. $d_G(u) + d_G(v) \geq \lceil \frac{2n-2}{3} \rceil$, $d_G(u) + d_G(v) \geq \lceil \frac{2n-23}{2} \rceil$) for any two adjacent vertices u and v of G. In addition, by means of semi-double graph and single-petal graph, the upper embeddability of multi-graph and pseudograph are also discussed in this paper.

Key Words: maximum genus; upper embeddable graph; graph embedding; semi-double graph; single-petal graph

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1. Introduction

The idea of the maximum genus $\gamma_M(G)$ of a connected graph G was introduced by Nordhaus, Stewart and White [12] in 1971, and Ringeisen, who has studied the maximum genus extensively [13][14][15], gave the definition of upper embeddable graphs. From then on, many researchers have studied the upper embeddability of graphs, such as Kundu [8], Jaeger, Payan and Xuong [6], Jungerment [7], Škoviera [16], Huang and Liu [3] etc. In 1998, via the degree-sum of non-adjacent vertices of a graph, Huang and Liu [4] obtained the following result related to simple graph:

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Theorem 1.1 Let G be a 2-edge-connected (resp. 3-edge-connected) simple graph of order n. If $d_G(u) + d_G(v) \geq \frac{2(n-2)}{3}$ (resp. $d_G(u) + d_G(v) \geq \frac{n+1}{3}$) for any two non-adjacent vertices u and v of G, then G is upper embeddable. Furthermore, the bound is best possible.

Then naturally a question is raised that whether the upper embeddability of a graph can be shown by the degree-sum of adjacent vertices of the graph. The present paper offers an affirmative answer to the question . In addition, by means of semi-double graph and single-petal graph, the upper embeddability of multi-graph and pseudograph are also discussed in this paper.

A graph is denoted by $G = (V(G), E(G))$, and $V(G)$, $E(G)$ denotes its vertex set and edge set respectively. Between two distinct vertices, if there is only one edge joining them, this edge is called a *link*, and if there are more than one edge joining them, these edges are called *multi-edge* of the graph. A simple graph is a graph having neither loops nor multi-edges. A multi-graph is a graph which may have multi-edges but doesn't have a loop and a pseudograph is a graph allows loops and multi-edges. A connected multi-graph is called a *semi-double graph* if each multi-edge of this graph consists of two edges. If there is at most one loop at each vertex of a semi-double graph then this graph is called a *single-petal graph.* For example, in Figure 1, the graph G_1 is a semi-double graph, G_2 is a multi-graph but not a semi-double graph, G_3 is a single-petal graph, G_4 is a pseudograph but not a single-petal graph. The order of a graph G is the number of vertices in G. The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. For any set X, we use $|X|$ to denote the cardinality of X. For any real number x, $\lfloor x \rfloor$ denotes the floor of x, *i.e.*, the greatest integer which is less than or equal to x, and $\lceil x \rceil$ denotes the ceiling of x, *i.e.*, the smallest integer which is greater than or equal to x. Graphs considered here are permitted to have multi-edges and loops, and are all undirected, finite and connected unless the context requires otherwise. Terminologies and notations not explained here can be seen in [1] for general graph theory. It is assumed that the reader is somewhat familiar with topological graph theory. For general background, see Liu [9], Gross and Tucker [2] or White [17].

G_1 G_2 G_3 G_4

Fig.1.

Recall that the maximum genus $\gamma_M(G)$ of a connected graph G is the maximum integer k such that there exists an embedding of G into the orientable surface of genus k. Since any embedding must have at least one face, the Euler characteristic for one face leads to an upper bound on the maximum genus

$$\gamma_M(G) \leq \lfloor \frac{|E(G)| - |V(G)| + 1}{2} \rfloor,$$

where the number $|E(G)| - |V(G)| + 1$ is known as the *Betti number* (or *cycle rank*) of the connected graph G and is denoted by $\beta(G)$. A graph G is said to be upper embeddable if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$.

For a subset $A \subseteq E(G)$, $c(G\backslash A)$ denotes the number of all connected components of $G\backslash A$, and $b(G\backslash A)$ denotes the number of connected components of $G\backslash A$ with odd *Betti number*, where $G\backslash A$ means the subgraph obtained from G by deleting all the edges of A from G. Let T be a spanning tree of a connected graph G. Define the *deficiency* $\xi(G,T)$ of a spanning tree T in a graph G to be the number of components of $G\backslash E(T)$ which have an odd number of edges. The deficiency $\xi(G)$ of the graph G is defined to be the minimum value of $\xi(G,T)$ over all spanning tree T of G. Note that $\xi(G) \equiv \beta(G) \pmod 2$. Let $F_1, F_2, \cdots F_k$ be k ($k \geq 2$) distinct subgraphs of a graph G, then denotes by $E_G(F_1, F_2, \cdots, F_k)$ the edges of $E(G)$ whose one end vertex is in $V(F_i)$ and the other in $V(F_j)$ ($1 \leq i, j \leq k$, $i \neq j$), and denote by $E(F_i, G)$ the edges of $E(G)$ whose one end vertex is in $V(F_i)$ and the other not in $V(F_i)$ ($1 \leq i \leq k$). For a vertex $v \in V(F_i)$ ($1 \leq i \leq k$), we call v a non-contacting-vertex of $V(F_i)$ if v is not incident with any edge of $E(F_i, G)$, and call v a contacting-vertex of $V(F_i)$ if v is incident with at least one edge of $E(F_i, G)$, and v is called a m-contacting-vertex of $V(F_i)$ if v is incident with m ($m \geq 1$) edge(s) of $E(F_i, G)$.

2. Some lemmas

The following two lemmas, which are due to Liu [9][10], Xuong [18] and Nebeský[11] independently, give two combinatorial characterizations of the maximum genus of graphs.

Lemma 2.1 (Liu [9][10], Xuong [18]) Let G be a connected graph, then

1) G is upper embeddable if and only if $\xi(G) \leq 1$;

2) $\gamma_M(G) = \frac{\beta(G) - \xi(G)}{2}$.

Lemma 2.2 (Nebeský [11]) Let G be a connected graph, then

1) G is upper embeddable if and only if $c(G\backslash A) + b(G\backslash A) - 2 \leq |A|$ for any subset $A \subseteq E(G)$;

2) $\xi(G) = \max_{A \subseteq E(G)} \{c(G\backslash A) + b(G\backslash A) - |A| - 1\}$.

The following result, which is proved by Huang [5], provides a structural characterization for a non-upper embeddable graph.

Lemma 2.3 (Huang [5]) Let G be a graph. If $\xi(G) \geq 2$, namely G is not upper embeddable, then there exists a subset $A \subseteq E(G)$ such that the following properties are satisfied:

(i) $c(G\backslash A) = b(G\backslash A) \geq 2$;

(ii) F is a vertex-induced subgraph of G for each component F of $G\backslash A$;

(iii) for any k distinct components $F_1, F_2, \cdots, F_k$ of $G\backslash A$, $|E_G(F_1, F_2, \cdots F_k)| \leq 2k - 3$. Especially $|E_G(F, H)| \leq 1$ for any two distinct components F and H of $G\backslash A$;

(iv) $\xi(G) = 2c(G\backslash A) - |A| - 1$.

In the above lemma, for each component F of $G\backslash A$ we notice the following facts:

Fact 1 Property (i) implies that $\beta(F) \equiv 1 \pmod 2$. Therefore, there exists at least one cycle in F. Furthermore, it can be deduced that if G is a simple graph then $|V(F)| \geq 3$; if G is a multi-graph then $|V(F)| \geq 2$; and if G is a pseudograph then $|V(F)| \geq 1$.

Fact 2 If G is a 2-edge-connected graph then for each $F \in G\backslash A$ we have $|E(F, G)| \geq 2$ and $c(G\backslash A) = b(G\backslash A) \geq 3$.

If G is a 3-edge-connected graph then for each $F \in G\backslash A$ we have $|E(F, G)| \geq 3$ and $c(G\backslash A) = b(G\backslash A) \geq 4$.

Fact 3 $|A| = \frac{1}{2}\sum_{F} |E(F, G)|$, where F is taken over all the components of $G\backslash A$.

3. Main results related to simple graph

Since every 4-edge-connected graph is upper embeddable[8], we only need to discuss the graph with edge-connectivity less than 4.

Theorem 3.1 Let G be a connected simple graph of order n. If $d_G(u) + d_G(v) \geq \lceil \frac{2n-3}{2} \rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Proof Assume to the contrary that G is not upper embeddable. By Lemma 2.3, there exists a subset A of $E(G)$ such that the properties (i)-(iv) of Lemma 2.3 are satisfied. Let $\mathcal{R} = \{F_1, F_2, \cdots, F_l\}$($l = c(G\backslash A) = b(G\backslash A) \geq 2$) be all the connected components of $G\backslash A$, and x, y, and z be the number of such $F_i \in \mathcal{R}$ that $|E(F_i, G)| = 1$, 2, and 3 respectively. By means of **Fact**

3, it is obvious that $|A| = \frac{1}{2}\sum_{i=1}^{l}|E(F_i, G)| \geq \frac{x}{2} + y + \frac{3}{2}z + 2(l - x - y - z)$. From Lemma 2.3(iv), we have

$$\begin{aligned} 2 &\leq \xi(G) = 2l - |A| - 1 \\ &\leq 2l - (\frac{x}{2} + y + \frac{3}{2}z + 2(l - x - y - z)) - 1. \end{aligned}$$

It can be easily deduced that

$$x + y + z \geq 2.$$

From Fact 1 we have $|V(F)| \geq 3$ for each $F \in \mathcal{R}$. Noticing that for each $F \in \mathcal{R}$, if v is a non-contacting-vertex of $V(F)$ then $d_G(v) \leq |V(F)|-1$; if v is a 1-contacting-vertex of $V(F)$ then $d_G(v) \leq |V(F)|$; if v is a 2-contacting-vertex of $V(F)$ then $d_G(v) \leq |V(F)|+1$; and if v is a 3-contacting-vertex of $V(F)$ then $d_G(v) \leq |V(F)|+2$. we will consider two cases in the following.

Case 1 : $l = 2$.

Let F_1 and F_2 be the two components of $G\backslash A$. We first give the following claim.

Claim 3.1.1 In each F_i, there must exist two adjacent vertices v_{i1} and v_{i2} such that $d_G(v_{i1}) + d_G(v_{i2}) \leq 2|V(F_i)| - 2$ $(i = 1, 2)$.

From Fact 1 we can get that $|V(F_i)| \geq 3 (i = 1, 2)$. From Lemma 2.3 (iii) we can get that $|E_G(F_1, F_2)| = 1$. So the vertices in F_i(i=1,2) are all non-contacting-vertex except one 1-contacting-vertex. It is not a hard work to find out that there must exist two adjacent vertices v_{i1} and v_{i2} in F_i such that both of them are non-contacting-vertex. So we have $d_G(v_{i1}) + d_G(v_{i2}) \leq |V(F_i)| - 1 + |V(F_i)| - 1 = 2|V(F_i)| - 2$. Claim 3.1.1 is obtained.

From Claim 3.1.1 we have

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \leq\ & 2|V(F_1)| - 2 + 2|V(F_2)| - 2 \\ =\ & 2(|V(F_1)| + |V(F_2)|) - 4 \leq 2n - 4. \end{aligned}$$

On the other hand, from the condition required in Theorem 3.1 that $d_G(u) + d_G(v) \geq \lceil\frac{2n-3}{2}\rceil$ for any two adjacent vertices u and v of G, we have

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \geq\ & \lceil\frac{2n-3}{2}\rceil + \lceil\frac{2n-3}{2}\rceil \geq 2n - 3. \end{aligned}$$

Thus $2n-3 \leq d_G(v_{11})+d_G(v_{12})+d_G(v_{21})+d_G(v_{22}) \leq 2n-4$, a contradiction.

Case 2 : $l \geq 3$.

Because $x + y + z \geq 2$, without loss of generality, let F_1 and F_2 be any two such components of $G \backslash A$ that $1 \leq |E(F_i, G)| \leq 3 (i = 1, 2)$. We have the following claim.

Claim 3.1.2 In each F_i, there must exist two adjacent vertices v_{i1} and v_{i2} such that $d_G(v_{i1}) + d_G(v_{i2}) \leq 2|V(F_i)|$ $(i = 1, 2)$.

Because $1 \leq |E(F_i, G)| \leq 3$, there are at most three contacting-vertex in F_i, and each vertex in F_i is a non-contacting-vertex, or a 1-contacting-vertex, or a 2-contacting-vertex, or a 3-contacting-vertex of $V(F_i)$. Because there is at most one 3-contacting-vertex or at most one 2-contacting-vertex in F_i, and the 3-contacting-vertex and the 2-contacting-vertex can not appear in F_i at the same time, the vertices in F_i must belong to one of the following cases: (α) There is a 3-contacting-vertex in F_i. Then all the other vertices in F_i are all non-contacting-vertex of $V(F_i)$; (β) There is a 2-contacting-vertex in F_i. Then all the other vertices in F_i are all non-contacting-vertex except at most one 1-contacting-vertex of $V(F_i)$; (γ) Each vertex in F_i is either a non-contacting-vertex or a 1-contacting-vertex of $V(F_i)$. So there must exist two adjacent vertices v_{i1} and v_{i2} in F_i such that both of them are non-contacting-vertex of $V(F_i)$, or both of them are 1-contacting-vertex of $V(F_i)$, or one is a non-contacting-vertex and the other is a 1-contacting-vertex of $V(F_i)$. Anyway we have $d_G(v_{i1}) + d_G(v_{i2}) \leq 2|V(F_i)|$. Thus Claim 3.1.2 is obtained.

From Claim 3.1.2, $l \geq 3$, and $|V(F)| \geq 3$ for each $F \in \mathcal{R}$, we have that

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \leq & \ 2(|V(F_1)| + |V(F_2)|) \leq 2(n-3) = 2n - 6. \end{aligned}$$

On the other hand, according to the condition required in Theorem 3.1 that $d_G(u) + d_G(v) \geq \lceil \frac{2n-3}{2} \rceil$ for any two adjacent vertices u and v of G, we have

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \geq & \ \lceil \frac{2n-3}{2} \rceil + \lceil \frac{2n-3}{2} \rceil \geq 2n - 3. \end{aligned}$$

So $2n - 3 \leq d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \leq 2n - 6$, a contradiction.

From Case 1 and Case 2 we can achieve Theorem 3.1. Furthermore, the graph G_5(Fig.2.) shows that the lower bound can not be reduced to $\lceil \frac{2n-3}{2} \rceil - 1$. So the lower bound is best possible. (Although $d(u) + d(v) \geq 4 = \lceil \frac{2n-3}{2} \rceil - 1$ for any two adjacent vertices u and v of the graph G_5 depicted by Fig. 2, the graph is not upper embeddable.) □

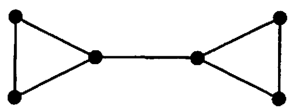

Fig.2. the graph G_5

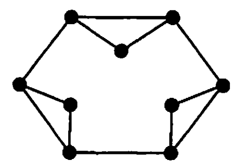

Fig.3. the graph G_6

From Theorem 3.1 we can easily get the following corollary.

Corollary 3.1 Let G be a connected simple graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{2n-3}{4} \rceil$ then G is upper embeddable.

Similarly, the following theorems related to 2-edge-connected and 3-edge-connected simple graphs can be easily obtained.

Theorem 3.2 Let G be a 2-edge-connected simple graph of order n. If $d_G(u)+d_G(v) \geq \lceil \frac{2n-2}{3} \rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Theorem 3.3 Let G be a 3-edge-connected simple graph of order n. If $d_G(u)+d_G(v) \geq \lceil \frac{2n-23}{2} \rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Furthermore, the graph G_6 which depicted by Fig.3. shows that the lower bound in Theorem 3.2 can not be reduced to $\lceil \frac{2n-2}{3} \rceil - 1$. So the lower bound is best possible (Although $d(u) + d(v) \geq 5= \lceil \frac{2n-2}{3} \rceil - 1$ for any two adjacent vertices u and v of the graph G_6, the graph is not upper embeddable). The graph G_7 depicted by Fig.4. shows that the lower bound in Theorem 3.3 can not be reduced to $\lceil \frac{2n-23}{2} \rceil - 1$. So the lower bound is best possible too (Although $d(u) + d(v) \geq 6= \lceil \frac{2n-23}{2} \rceil - 1$ for any two adjacent vertices u and v of the graph G_7, the graph is not upper embeddable).

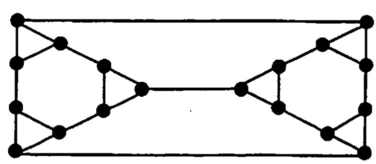

Fig.4. the graph G_7

According to Theorem 3.2, Theorem 3.3, and the fact that $\delta(G) \geq k'(G)$, where $k'(G)$ is the edge-connectivity of G, the following corollaries can be easily obtained.

Corollary 3.2 Let G be a 2-edge-connected simple graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{n-1}{3} \rceil$ then G is upper embeddable. In addition, any 2-edge-connected simple graph with order $n \leq 7$ is upper embeddable.

Corollary 3.3 Let G be a 3-edge-connected simple graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{2n-23}{4} \rceil$ then G is upper embeddable. In addition, any 3-edge-connected simple graph with order $n \leq 17$ is upper embeddable.

4. Main results related to non-simple graph

In section 3, the upper embeddability of simple graphs have been investigated. However, it looks more complicated to determine the upper embeddability of non-simple graphs than that of simple graphs. *Even-deletion* is such an edge deleting operation on a graph G that the following requirements are satisfied: (i) the edges deleted from G may be links, multi-edges, and loops; (ii) the remainder of the graph is connected; (iii) the number of edges deleted from G should be an even number, and the subgraph induced by the deleted edges should be connected. An *even-ancestry* of a non-simple graph G is such a simple graph, or a semi-double graph, or a single-petal graph that is obtained from G by a sequence of even-deletions. For convenience, these definitions are illustrated by Fig.10, where both G_{14} and G_{15} are even-ancestries of G_{16}. It is obvious that a non-simple graph may have more than one even-ancestry. Furthermore, according to Theorem 4.0, whose proof will be given in Section 5, we can study the upper embeddability of non-simple graphs through that of simple graphs, or semi-double graphs, or single-petal graphs.

Theorem 4.0 A non-simple graph G is upper embeddable if and only if one of its even-ancestries G' is upper embeddable.

In this section we will focus on such field as the upper embeddability of semi-double graphs and single-petal graphs.

Theorem 4.1 Let G be a connected semi-double graph of order n. If $d_G(u) + d_G(v) \geq \lceil \frac{4n-5}{2} \rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Proof Assume to the contrary that G is not upper embeddable. By Lemma 2.3, there exists a subset A of $E(G)$ such that the properties (i)-(iv) of Lemma 2.3 are satisfied. Let $\mathcal{R} = \{F_1, F_2, \cdots, F_l\}(l = c(G\backslash A) = b(G\backslash A) \geq 2)$ be all the connected components of $G\backslash A$, and x, y, and z be the number of such $F_i \in \mathcal{R}$ that $|E(F_i, G)| = 1$, 2, and 3 respectively. By means of Fact 3, it is obvious that $|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, G)| \geq \frac{x}{2} + y + \frac{3}{2}z + 2(l - x - y - z)$. From Lemma 2.3(iv), we have

$$\begin{aligned} 2 &\leq \xi(G) = 2l - |A| - 1 \\ &\leq 2l - (\frac{x}{2} + y + \frac{3}{2}z + 2(l - x - y - z)) - 1. \end{aligned}$$

It can be easily deduced that

$$x + y + z \geq 2.$$

From Fact 1 we have $|V(F)| \geq 2$ for each $F \in \mathcal{R}$. Noticing that for each $F \in \mathcal{R}$, if v is a non-contacting-vertex of F then $d_G(v) \leq 2|V(F)| - 2$; if v is a 1-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| - 1$; if v is a 2-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)|$; and if v is a 3-contacting-vertex of $V(F)$ then $d_G(v) \leq 2|V(F)| + 1$. we will consider two cases in the following.

Case 1 : $l = 2$.

Let F_1, F_2 be the two components of $G\backslash A$. We have the following claim.

Claim 4.1.1 In each F_i, there must exist two adjacent vertices v_{i1} and v_{i2} such that $d_G(v_{i1}) + d_G(v_{i2}) \leq 4|V(F_i)| - 3$ $(i = 1, 2)$.

It can be get from Fact 1 that $|V(F_i)| \geq 2 (i = 1, 2)$. From Lemma 2.3 (iii) we have $|E_G(F_1, F_2)| = 1$. So the vertices in F_i(i=1,2) are all non-contacting-vertex except one 1-contacting-vertex. It is not difficult to find out that there must exist two adjacent vertices v_{i1} and v_{i2} in F_i such that either both of them are non-contacting-vertex or one of them is a non-contacting-vertex and the other is a 1-contacting-vertex. Anyway we have $d_G(v_{i1}) + d_G(v_{i2}) \leq 2|V(F_i)| - 2 + 2|V(F_i)| - 1 = 4|V(F_i)| - 3$. Thus Claim 4.1.1 is obtained.

From Claim 4.1.1 we have

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \leq\ & 4|V(F_1)| - 3 + 4|V(F_2)| - 3 \\ =\ & 4(|V(F_1)| + |V(F_2)|) - 6 \leq 4n - 6. \end{aligned}$$

On the other hand, according to the condition required in Theorem 4.1 that $d_G(u) + d_G(v) \geq \lceil\frac{4n-5}{2}\rceil$ for any two adjacent vertices u and v of G, we have

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \geq\ & \lceil\frac{4n-5}{2}\rceil + \lceil\frac{4n-5}{2}\rceil \geq 4n - 5. \end{aligned}$$

So we have $4n-5 \le d_G(v_{11})+d_G(v_{12})+d_G(v_{21})+d_G(v_{22}) \le 4n-6$. It is a contradiction.

Case 2 : $l \ge 3$.

Because $x+y+z \ge 2$, without loss of generality, let F_1 and F_2 be any two such components of $G \backslash A$ that $1 \le |E(F_i, G)| \le 3 (i=1,2)$. We have the following claim.

Claim 4.1.2 In each F_i, there must exist two adjacent vertices v_{i1} and v_{i2} such that $d_G(v_{i1}) + d_G(v_{i2}) \le 4|V(F_i)| - 1$ $(i=1,2)$.

As $1 \le |E(F_i, G)| \le 3$, there are at most three contacting-vertex in F_i, and each vertex in F_i is a non-contacting-vertex, or a 1-contacting-vertex, or a 2-contacting-vertex, or a 3-contacting-vertex of $V(F_i)$. Because there is at most one 3-contacting-vertex or at most one 2-contacting-vertex in F_i, and the 3-contacting-vertex and the 2-contacting-vertex can not appear in F_i at the same time, the vertices in F_i must belong to one of the following cases: (α) There is a 3-contacting-vertex in F_i. Then all the other vertices in F_i are all non-contacting-vertex of $V(F_i)$; (β) There is a 2-contacting-vertex in F_i. Then all the other vertices in F_i are all non-contacting-vertex except at most one 1-contacting-vertex of $V(F_i)$; (γ) Each vertex in F_i is either a non-contacting-vertex or a 1-contacting-vertex of $V(F_i)$. So there must exist two adjacent vertices v_{i1} and v_{i2} in F_i such that both of them are non-contacting-vertex of $V(F_i)$, or both of them are 1-contacting-vertex of $V(F_i)$, or one of them is a non-contacting-vertex and the other is a 1-contacting-vertex of $V(F_i)$, or one of them is a non-contacting-vertex and the other is a 2-contacting-vertex of $V(F_i)$, or one of them is a non-contacting-vertex and the other is a 3-contacting-vertex of $V(F_i)$. Anyway we have $d_G(v_{i1}) + d_G(v_{i2}) \le 2|V(F_i)| - 2 + 2|V(F_i)| + 1 = 4|V(F_i)| - 1$. Thus Claim 4.1.2 is obtained.

From Claim 4.1.2, $l \ge 3$, and $|V(F)| \ge 2$ for each $F \in \mathcal{R}$, we have that

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \le \; & 4(|V(F_1)| + |V(F_2)|) - 2 \le 4(n-2) - 2 = 4n - 10. \end{aligned}$$

On the other hand, from the condition required in Theorem 4.1 that $d_G(u) + d_G(v) \ge \lceil \frac{4n-5}{2} \rceil$ for any two adjacent vertices u and v of G, we have

$$\begin{aligned} & d_G(v_{11}) + d_G(v_{12}) + d_G(v_{21}) + d_G(v_{22}) \\ \ge \; & \lceil \frac{4n-5}{2} \rceil + \lceil \frac{4n-5}{2} \rceil \ge 4n - 5. \end{aligned}$$

So $4n-5 \le d_G(v_{11})+d_G(v_{12})+d_G(v_{21})+d_G(v_{22}) \le 4n-10$. It is a contradiction.

From Case 1 and Case 2 we can achieve Theorem 4.1. Furthermore, the graph G_8(Fig.5.) shows that the lower bound can not be reduced to

$\lceil\frac{4n-5}{2}\rceil - 1$. So the lower bound is best possible. (Although $d(u)+d(v) \geq 5= \lceil\frac{4n-5}{2}\rceil - 1$ for any two adjacent vertices u and v of the graph G_8 depicted by Fig.5, the graph is not upper embeddable.) □

Fig.5. the graph G_8

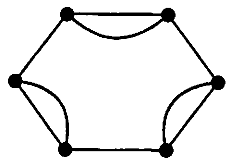

Fig.6. the graph G_9

From Theorem 4.1 we can easily get the following corollary.

Corollary 4.1 Let G be a connected semi-double graph of order n. If the minimum degree $\delta(G) \geq \lceil\frac{4n-5}{4}\rceil$ then G is upper embeddable.

Similarly, the following theorems related to 2-edge-connected and 3-edge-connected semi-double graphs can be easily obtained.

Theorem 4.2 Let G be a 2-edge-connected semi-double graph of order n. If $d_G(u)+d_G(v) \geq \lceil\frac{4n-5}{3}\rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Theorem 4.3 Let G be a 3-edge-connected semi-double graph of order n. If $d_G(u)+d_G(v) \geq \lceil\frac{4n-33}{2}\rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Furthermore, the graph G_9 which depicted by Fig.6. shows that the lower bound in Theorem 4.2 can not be reduced to $\lceil\frac{4n-5}{3}\rceil - 1$. So the lower bound is best possible (Although $d(u) + d(v) \geq 6= \lceil\frac{4n-5}{3}\rceil - 1$ for any two adjacent vertices u and v of the graph G_9, the graph is not upper embeddable). The graph G_{10} depicted by Fig.7. shows that the lower bound in Theorem 4.3 can not ba reduced to $\lceil\frac{4n-33}{2}\rceil - 1$. So the lower bound is best possible too (Although $d(u) + d(v) \geq 7= \lceil\frac{4n-33}{2}\rceil - 1$ for any two adjacent vertices u and v of the graph G_{10}, the graph is not upper embeddable).

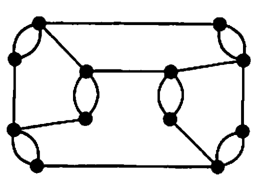

Fig.7. the graph G_{10}

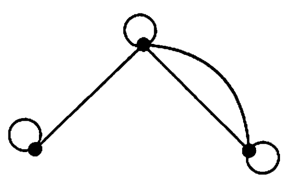

Fig.8. G_{11}: connected

According to Theorem 4.2, Theorem 4.3, and the fact that $\delta(G) \geq k'(G)$, where $k'(G)$ is the edge-connectivity of G, the following corollaries can be easily obtained.

Corollary 4.2 Let G be a 2-edge-connected semi-double graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{4n-5}{6} \rceil$ then G is upper embeddable. In addition, any 2-edge-connected semi-double graph with order $n \leq 4$ is upper embeddable.

Corollary 4.3 Let G be a 3-edge-connected semi-double graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{4n-33}{4} \rceil$ then G is upper embeddable. In addition, any 3-edge-connected semi-double graph with order $n \leq 11$ is upper embeddable.

As for single-petal graph we have the following results.

Theorem 4.4 Let G be a connected single-petal graph of order n. If $d_G(u) + d_G(v) \geq 2n + 3$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Theorem 4.5 Let G be a 2-edge-connected single-petal graph of order n. If $d_G(u) + d_G(v) \geq \lceil \frac{4n+13}{3} \rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Theorem 4.6 Let G be a 3-edge-connected single-petal graph of order n. If $d_G(u) + d_G(v) \geq \lceil \frac{4n+7}{3} \rceil$ for any two adjacent vertices u and v of G, then G is upper embeddable.

Proof (of Theorem 4.4, 4.5, 4.6) From a deduction similar to that of Theorem 4.1, the theorems can be obtained noticing the fact that $|V(F_i)| \geq 1$ for each $F_i \in \mathcal{R}$, and that if v is a non-contacting-vertex of $F_i (\in \mathcal{R})$ then $d_G(v) \leq 2|V(F_i)|$; if v is a 1-contacting-vertex then $d_G(v) \leq 2|V(F_i)| + 1$; if v is a 2-contacting-vertex then $d_G(v) \leq 2|V(F_i)| + 2$; and if v is a 3-contacting-vertex then $d_G(v) \leq 2|V(F_i)| + 3$. Furthermore, the graph G_{11}(Fig.8), G_{12}(Fig.9), and G_{13}(Fig.9) shows that the lower bound $2n+3$, $\lceil \frac{4n+13}{3} \rceil$, and $\lceil \frac{4n+7}{3} \rceil$ can not be reduced to $2n+2$, $\lceil \frac{4n+13}{3} \rceil - 1$, and $\lceil \frac{4n+7}{3} \rceil - 1$ respectively. □

G_{12}: 2-edge-connected

G_{13}: 3-edge-connected

Fig.9.

The following corollary can be obtained easily.

Corollary 4.4 Let G be a connected(resp. 2-edge-connected, 3-edge-connected) single-petal graph of order n. If the minimum degree $\delta(G) \geq \lceil \frac{2n+3}{2} \rceil$(resp. $\lceil \frac{4n+13}{6} \rceil$, $\lceil \frac{4n+7}{6} \rceil$) then G is upper embeddable.

5. Conclusions

Let G be a simple graph, or a semi-double graph, or a single-petal graph. *Even-addition* on G is such an edge-adding operation on G which meets the following requirements: (i) the edges added to G may be links, multi-edges and loops; (ii) the number of edges added to G should be an even number; (iii) the subgraph induced by the edges added to G should be connected. The graph G^* obtained from G by a sequence of even-additions is called an *even-posterity* of G. For convenience, these definitions are illustrated by Fig.10, where the graph G_{14} is a single-petal graph, both G_{15} and G_{16} are even-posterities of G_{14}.

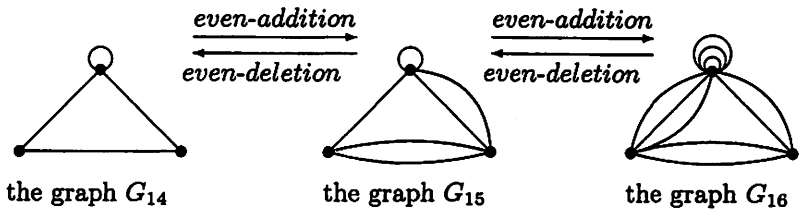

Fig.10.

Theorem 5.1 Let G be a simple graph, or a semi-double graph, or a single-petal graph, and G^* be an even-posterity of G. If G is upper embeddable then G^* is upper embeddable.

Proof According to the definition of the even-posterity of G, the edges added to G each time are an even number of edges, and the subgraph induced by the edges added to G each time is a connected subgraph of G^*, so the deficiency of G^* is no more than that of G. By Lemma 2.1 we can get that if G is upper embeddable then G^* is upper embeddable. □

(The proof of Theorem 4.0) According to Lemma 2.1, if the non-simple graph G is upper embeddable, then there must exist a spanning tree T of G such that the deficiency $\xi(G)$ of G is at most one. Performing some times of even-deletions on G with respect to T, G', which is upper embeddable and an even-ancestry of G, would be obtained.

Conversely, if one of the even-ancestries G' of G is upper embeddable, then G is upper embeddable according to Theorem 5.1. □

Remark 1 Since k-vertex-connectivity implies k-edge-connectivity, the condition required in the theorems obtained in this paper that G is a k-edge-connected graph can be replaced by that G is a k-vertex-connected graph ($k = 1, 2, 3$).

Remark 2 Theorem 5.1 provides a sufficient condition but not a sufficient and necessary condition, *i.e.*, if the even-posterity G^* of G is upper embeddable, G may be not upper embeddable. For example, in Fig.11, the graph G_{18}, which is an even-posterity of G_{17}, is upper embeddable, but the graph G_{17} is not upper embeddable.

the graph G_{17} the graph G_{18}

Fig.11.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications. Macmillan, London, 1976.

[2] J.L. Gross, T.W. Tucker, Topological Graph Theory. Wiley-Interscience, New York, 1987.

[3] Y. Huang, Y. Liu, Face size and the maximum genus of a graph, J Combin. Theory Ser B. 80 (2000) 356–370.

[4] Y. Huang, Y. Liu, The degree sum of nonadjacent vertices and upper embeddability of graphs, Chin Ann of Math. 5 (19A) (1998) 651−656 (in Chinese).

[5] Y. Huang, Maximum genus and chromatic number of graphs, Discrete Math. 271 (2003) 117−127.

[6] F. Jaeger, C. Payan, N.H. Xuong, A classes of upper embeddable graphs, J. Graph Theory. 3 (1979) 387−391.

[7] M. Jungerment, A characterization of upper embeddable graphs, Trans. Math. Soc. 241 (1979) 401−406.

[8] S. Kundu, Bounds on number of disjoint spanning trees, J. Combin. Theory Ser. B. 17 (1974) 199−203.

[9] Y. Liu, Embeddability in Graphs. Kluwer Academic, Dordrecht, Boston, London, 1995. 255−265.

[10] Y. Liu, The maximum orientable genus of a graph. *Scientia Sinical (Special Issue)*, (II)(1979) 41–55.

[11] L. Nebeský, A new characterization of the maximum genus of a graph. Czechoslova Math. J. 31(106) (1981) 604–613.

[12] E.A. Nordhause, B.M. Stewart, A.T. White, On the maximum genus of a graph, J. Combin. Theory. 11 (1971) 258–267.

[13] R. Ringeisen, The maximum genus of a graph, Ph. D. Thesis. Michigan State Univ., 1970.

[14] R. Ringeisen, Upper and lower embeddable graphs, Graph Theory and Applications. Lecture Notes in Math., Springer-Verlag, Berlin and New York, 303 (1972).

[15] R. Ringeisen, Determining all compact orientable 2-manifolds upon which $K_{m,n}$ has 2-cell imbeddings, J. Combin. Theory Ser. B. 12 (1974) 101–104.

[16] M. Škoviera, The maximum genus of graphs diameter two, Discrete Math. 87 (1991) 175–180.

[17] A.T. White, Graphs, Groups, and Surfaces. Amsterdam, North-Holland, 1984.

[18] N.H. Xuong, How to determine the maximum genus of a graph, J. Combin. Theory Ser. B. 26 (1979) 217–225.