On the Rank Polynomial and Whitney Numbers of Order Ideals of a Garland

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AMS Subject Classification: 05A15, 06A07.

Abstract

We prove explicit formulas for the rank polynomial and Whitney numbers of the distributive lattice of order ideals of the garland poset, ordered by inclusion.

1 Introduction and Preliminaries

Given a finite poset $(P, \leq)$, a very interesting and challenging computational and enumerative problem, see e.g. [2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23] and the references therein, is to study the distributive lattice of all its order ideals ordered by inclusion, and the corresponding Whitney numbers.
In particular, in [17] it is considered a specific class of posets having $2n$ elements, called *garlands* and denoted by $\mathcal{G}_n$, and it is determined the generating function of the sequence g_n, the number of all order ideals of $\mathcal{G}_n$.

Here we get a generalization of the results in [17], giving a closed formula for the rank polynomial of the lattice of order ideals of $\mathcal{G}_n$ ordered by inclusion, and the corresponding Whitney numbers $g_{n,k}$, the number of all order ideals of $\mathcal{G}_n$ having cardinality k.

*Fully supported by FWF Austrian Science Fund grant P17563–N13.

In the sequel we collect some definitions, notations and results that will be used in the following. For $x \in \mathbb{R}$ we let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$; for any $n, m \in \mathbb{N}$, $n \leq m$, we let $[n, m] = \{t \in \mathbb{N} : n \leq t \leq m\}$, and $[n] = [1, n]$, therefore $[0] = \emptyset$. The cardinality of a set $\mathcal{X}$ will be denoted by $\#\mathcal{X}$. For two sets $\mathcal{X}, \mathcal{Y}$ we denote with $\mathcal{X} \biguplus \mathcal{Y}$ the *disjoint union* of $\mathcal{X}$ and $\mathcal{Y}$, and with $\mathcal{X} \setminus \mathcal{Y} = \{x : x \in \mathcal{X},\ x \notin \mathcal{Y}\}$ the difference set.

We follow [1, 9] for poset notations and terminology, and we refer to [6, 8, 20] for comprehensive references about enumerative combinatorics.

We recall that a *ranked poset* is a poset $(P, \leq)$ with a function $\rho : P \longrightarrow \mathbb{N}$, called rank, such that $\rho(y) = \rho(z) + 1$ whenever z is covered by y in P and $\min\{\rho(z) : z \in P\} = 0$. The *rank polynomial* of a ranked finite poset P is the polynomial

$$\sum_{z \in P} X^{\rho(z)} = \sum_{j \geq 0} \omega_j X^j,$$

where $\omega_j = \#\{z \in P : \rho(z) = j\}$ are called *Whitney numbers* of P.

An *order ideal* of a poset $(P, \leq)$ is a subset $I \subseteq P$ such that if $y \in I$ and $z \leq y$, then $z \in I$; it is well known that the set of all order ideals of P ordered by inclusion is closed under unions and intersections, and hence forms a distributive lattice: we denote it by $\mathcal{J}(P)$, viz. $\mathcal{J}(P) = \{I \subseteq P : I \text{ is an order ideal}\}$. It is not hard to see that its rank function is the cardinality of order ideals.

Given a finite poset $(P, \leq)$, we denote with $W_P(k)$ the k–th Whitney numbers of the ranked poset of all order ideals of P, i.e. $W_P(k) = \#\{I \in \mathcal{J}(P) : \rho(I) = k\}$, where ρ is the rank function of $\mathcal{J}(P)$, and the rank polynomial of $\mathcal{J}(P)$ is denoted by $\mathcal{R}_P(X)$, i. e. $\mathcal{R}_P(X) = \sum_{k \geq 0} W_P(k) X^k$.

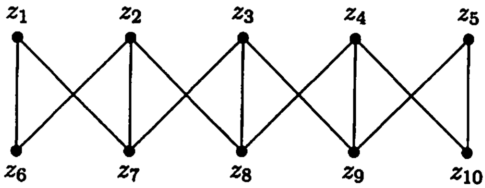
For any $n \in \mathbb{N}$, we denote by $\mathcal{G}_n$ the *garland* poset of order $2n$, viz. $\mathcal{G}_0 = \emptyset$, $\mathcal{G}_1$ is the chain with two element (i.e. $\mathcal{G}_1 = \{z_1, z_2\}$, with $z_1 < z_2$), and if $n \geq 2$ $\mathcal{G}_n$ is the poset $\{z_1, \ldots, z_{2n}\}$ in which the cover relations are the following:

- $z_{n+1} \lhd \{z_1, z_2\}$,
- $z_{n+j} \lhd \{z_{j-1}, z_j, z_{j+1}\}$ for any $j \in [2, n-1]$,
- $z_{2n} \lhd \{z_{n-1}, z_n\}$;

therefore

$$\rho(z_j) = \begin{cases} 1 & \text{if } j \in [n], \\ 0 & \text{if } j \in [n+1, 2n]. \end{cases}$$

For example, the Hasse diagram of $\mathcal{G}_5$ is depicted.



We also denote by $\mathcal{I}_n(k)$ the set of order ideals of $\mathcal{G}_n$ with cardinality k, and by $g_{n,k}$ the Whitney numbers of the poset of all order ideals of a garland of order $2n$, viz. $g_{n,k} = \#\mathcal{I}_n(k) = W_{\mathcal{G}_n}(k)$. Finally, we denote by $\mathcal{R}_n(X)$ the rank polynomial of the distributive lattice of all order ideals of the garland poset, ordered by inclusion, i.e. $\mathcal{R}_n(X) = \mathcal{R}_{\mathcal{G}_n}(X) = \sum_{k\geq 0} g_{n,k}X^k$.

2 Main Results

The organization of this section is as follows. In Theorem 2.1 we establish a recursion for $g_{n,k}$, which leads in Theorem 2.2 to an explicit formula for the generating function of the sequence of rank polynomials $\mathcal{R}_n(X)$. Using the latter result, we give explicit formulas for $\mathcal{R}_n(X)$ and $g_{n,k}$ in Theorems 2.4 and 2.5, respectively.

Theorem 2.1. *For all integers $n \in \mathbb{N}$ and $0 \leq k \leq 2n$,*

$$\begin{aligned} g_{n,k} &= \binom{n}{k} + \sum_{\alpha=1}^{n} \binom{\alpha-2}{2n+1-k-\alpha} + \sum_{\beta\geq 2} g_{n-\beta,k-2\beta+1} \\ &+ \sum_{\alpha\geq 2}\sum_{\beta>\alpha}\sum_{j\geq 0} \binom{\alpha-2}{j} g_{n-\beta,k-2\beta+2\alpha-2-j} \end{aligned}$$

holds, where we set $\binom{-1}{0} = 1$.

Proof. Write

$$\mathcal{I}_n(k) = \mathcal{Y}(0) \biguplus \left(\biguplus_{\alpha=1}^{n} \mathcal{Y}(\alpha) \right) \biguplus \left(\biguplus_{\alpha=1}^{n-1} \biguplus_{\beta=\alpha+1}^{n} \mathcal{Y}(\alpha,\beta) \right);$$

where

- $\mathcal{Y}(0) = \{I \in \mathcal{I}_n(k) : I \bigcap \{z_1, \ldots, z_n\} = \emptyset\}$,
- $\mathcal{Y}(\alpha) = \{I \in \mathcal{I}_n(k) : I \bigcap \left(\bigcup_{j=1}^{n}\{z_j\}\right) = \bigcup_{j=\alpha}^{n}\{z_j\}\}$,

- $\mathcal{Y}(\alpha,\beta) = \{I \in \mathcal{I}_n(k) : \min_{j\in[n-1]}\{z_j \in I\} = \alpha,\ \min_{h\in[\alpha+1,n]}\{z_h \notin I\} = \beta\}$.

Then we have

- $\#\mathcal{Y}(0) = \binom{n}{k}$,
- $\#\mathcal{Y}(\alpha) = \binom{\alpha-2}{2n+1-k-\alpha}$ where we set $\binom{-1}{0} = 1$,
- $\#\mathcal{Y}(1,\beta) = g_{n-\beta,k-2\beta+1}$,
- $\#\mathcal{Y}(\alpha,\beta) = \sum_{j=0}^{k-2\beta+2\alpha-2}\binom{\alpha-2}{j} g_{n-\beta,k-2\beta+2\alpha-2-j}$ for any $\alpha \in [2, n-1]$.

We explain how to calculate $\#\mathcal{Y}(\alpha,\beta)$ for any $\alpha \in [2, n-1]$; the other cases are similar and simpler.
By definition, $\mathcal{Y}(\alpha,\beta)$ is the set of all $I \in \mathcal{I}_n(k)$ such that

$z_j \notin I$ for all $j \in [\alpha - 1]$ and $z_\beta \notin I$,

$z_k \in I$ for all $k \in [\alpha, \beta - 1]$, thus $z_t \in I$ for all $t \in [n + \alpha - 1, n + \beta]$.

Hence $2\beta - 2\alpha + 2$ elements of I are fixed and the others can be chosen inside the subset $\{z_j : j \in [n+1, n+\alpha-2]\} \biguplus \{z_j, z_{n+j} : j \in [\beta+1, n]\}$.

Noticing that $\{z_j, z_{n+j} : j \in [\beta+1, n]\} \simeq \mathcal{G}_{n-\beta}$, we get the formula for $\#\mathcal{Y}(\alpha,\beta)$.

Therefore

$$\begin{aligned} g_{n,k} &= \binom{n}{k} + \sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha} + \sum_{\beta=2}^{n} g_{n-\beta,k-2\beta+1} \\ &+ \sum_{\alpha=2}^{n-1}\sum_{\beta=\alpha+1}^{n}\sum_{j=0}^{k-2\beta+2\alpha-2}\binom{\alpha-2}{j} g_{n-\beta,k-2\beta+2\alpha-2-j}, \end{aligned}$$

and the desired result follows. □

Theorem 2.2. *Let $\mathbf{H}(X,Y) = \sum_{n\geq 0}\mathcal{R}_n(X)Y^n = \sum_{\substack{n\geq 0\\ k\geq 0}} g_{n,k}X^kY^n$ be the generating function of the sequence of rank polynomials $\mathcal{R}_n(X)$; then*

$$\begin{aligned} \mathbf{H}(X,Y) &= \left(\frac{1}{1-Y(1+X)} + \sum_{\substack{n\geq 0\\ k\geq 0}}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^kY^n\right) \\ &\quad \left(1 - \frac{X^3Y^2(1-Y)}{(1-X^2Y)(1-Y(1+X))}\right)^{-1}, \end{aligned}$$

where we set $\binom{-1}{0} = 1$.

Proof. Taking in account Theorem 2.1 with the initial values conditions $g_{n,k} = 0$ if $n < 0$ or $k \notin [0, 2k]$, we get

$$\mathcal{R}_n(X) = \sum_{k\geq 0} g_{n,k}X^k = \sum_{k\geq 0}\binom{n}{k}X^k + \sum_{k\geq 0}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^k$$
$$+\sum_{\beta\geq 2}X^{2\beta-1}\left(\sum_{k\geq 0} g_{n-\beta,k-2\beta+1}X^{k-2\beta+1}\right)$$
$$+\sum_{\alpha\geq 2}\sum_{\beta>\alpha}\sum_{j\geq 0}\binom{\alpha-2}{j}X^{2\beta-2\alpha+2+j}\left(\sum_{k\geq 0} g_{n-\beta,k-2\beta+2\alpha-2-j}X^{k-2\beta+2\alpha-2-j}\right)$$
$$=\sum_{k\geq 0}\binom{n}{k}X^k + \sum_{k\geq 0}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^k$$
$$+\sum_{\beta\geq 2}X^{2\beta-1}\mathcal{R}_{n-\beta}(X) + \sum_{\alpha\geq 2}\sum_{\beta>\alpha}\sum_{j\geq 0}\binom{\alpha-2}{j}X^{2\beta-2\alpha+2+j}\mathcal{R}_{n-\beta}(X).$$

Using the identity $\sum_{k\geq 0}\binom{n}{k}X^k = (1+X)^n$ and the closed form of the geometric serie, we have

$$\mathbf{H}(X,Y) = \sum_{\substack{n\geq 0\\ k\geq 0}}\binom{n}{k}X^kY^n + \sum_{\substack{n\geq 0\\ k\geq 0}}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^kY^n$$
$$+\sum_{\beta\geq 2}X^{2\beta-1}Y^{\beta}\left(\sum_{n\geq 0}\mathcal{R}_{n-\beta}(X)Y^{n-\beta}\right)$$
$$+\sum_{\alpha\geq 2}\sum_{\beta>\alpha}\sum_{j\geq 0}\binom{\alpha-2}{j}X^{2\beta-2\alpha+2+j}Y^{\beta}\left(\sum_{n\geq 0}\mathcal{R}_{n-\beta}(X)Y^{n-\beta}\right)$$
$$=\frac{1}{1-Y(1+X)} + \sum_{\substack{n\geq 0\\ k\geq 0}}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^kY^n$$
$$+\mathbf{H}(X,Y)\left(\frac{(X^2Y)^2}{X(1-X^2Y)} + \sum_{\alpha\geq 2}\sum_{j\geq 0}\binom{\alpha-2}{j}X^{2-2\alpha+j}\frac{(X^2Y)^{\alpha+1}}{1-X^2Y}\right)$$
$$=\frac{1}{1-Y(1+X)} + \sum_{\substack{n\geq 0\\ k\geq 0}}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^kY^n$$

$$+\mathbf{H}(X,Y)\frac{(X^2Y)^2}{1-X^2Y}\left(\frac{1}{X}+Y\sum_{\substack{j\geq 0\\ t\geq 0}}\binom{t}{j}X^jY^t\right)$$

$$=\frac{1}{1-Y(1+X)}+\sum_{\substack{n\geq 0\\ k\geq 0}}\sum_{\alpha=1}^{n}\binom{\alpha-2}{2n+1-k-\alpha}X^kY^n$$

$$+\mathbf{H}(X,Y)\frac{X^3Y^2(1-Y)}{(1-X^2Y)(1-Y(1+X))},$$

and the desired result follows. □

We define by $\mathbb{N}^*$ the free monoid on $\mathbb{N}$, viz. the set of all words with only finitely many non–zero letters using $\mathbb{N}$ as alphabet, and for any multi–index $\alpha=(\alpha_0,\alpha_1,\alpha_2,\ldots)\in\mathbb{N}^*$, we set $\|\alpha\|=\sum_{j\geq 0}\alpha_j$ and $\Omega(\alpha)=\sum_{j\geq 0}j\cdot\alpha_j$. For multi–indices $\alpha=(\alpha_j)_j,\beta=(\beta_k)_k\in\mathbb{N}^*$, we set $\alpha+\beta=(\alpha_j+\beta_j)_j\in\mathbb{N}^*$.

The following result is well–known.

Lemma 2.3. *For any $r\in\mathbb{N}$ and any sequence $(z_0,z_1,z_2,\ldots)$ of real numbers,*

$$\left(\sum_{j\geq 0}z_j\right)^r=\sum_{\substack{\alpha\in\mathbb{N}^*\\ \|\alpha\|=r}}\frac{r!}{\prod_{k\geq 1}(\alpha_k!)}\left(\prod_{j\geq 0}z_j^{\alpha_j}\right).$$

□

Theorem 2.4. *For all $n\in\mathbb{N}$,*

$$\mathcal{R}_n(X)=\sum_{j=0}^{n}\left[\left[(1+X)^j+\sum_{k\geq 0}\sum_{\alpha=1}^{j}\binom{\alpha-2}{2j+1-k-\alpha}X^k\right]\right.$$

$$\cdot\left[\sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}^*\\ \|\alpha\|=\|\beta\|=\|\gamma\|\\ \Phi(\alpha,\beta,\gamma)=n-j}}\left(X^{3\|\alpha\|+2\Omega(\alpha)}(1+X)^{\Omega(\beta)}\right.\right.$$

$$\left.\left.\left.\cdot\frac{(\|\alpha\|!)^3}{\prod_{z\geq 0}(\alpha_z!)(\beta_z!)(\gamma_z!)}(-1)^{\Omega(\gamma)}\prod_{r\geq 0}\binom{\|\gamma\|}{r}^{\gamma_r}\right)\right]\right]$$

holds, where we set $\binom{-1}{0}=1$ and $\Phi(\alpha,\beta,\gamma)=2\|\alpha\|+\Omega(\alpha+\beta+\gamma)$ for all $\alpha,\beta,\gamma\in\mathbb{N}^*$.

Proof. With an eye on Theorem 2.2, observe that from Lemma 2.3 we get

$$\left(1 - \frac{X^3Y^2(1-Y)}{(1-X^2Y)(1-Y(1+X))}\right)^{-1}$$

$$= \sum_{j\geq 0}\left[\left(X^3Y^2(1-Y)\right)^j\left(\sum_{t\geq 0}X^{2t}Y^t\right)^j\left(\sum_{n\geq 0}Y^n(1+X)^n\right)^j\right]$$

$$= \sum_{j\geq 0}\left[X^{3j}Y^{2j}\left(\sum_r(-1)^r\binom{j}{r}Y^r\right)^j\left(\sum_{\substack{\alpha\in\mathbf{N}^*\\ \|\alpha\|=j}}\frac{j!}{\prod_{v\geq 0}(\alpha_v!)}(X^2Y)^{\Omega(\alpha)}\right)\right.$$

$$\left.\cdot\left(\sum_{\substack{\beta\in\mathbf{N}^*\\ \|\beta\|=j}}\frac{j!}{\prod_{z\geq 0}(\beta_z!)}Y^{\Omega(\beta)}(1+X)^{\Omega(\beta)}\right)\right]$$

$$= \sum_{j\geq 0}\left[X^{3j}Y^{2j}(j!)^2\left(\sum_{\substack{\alpha,\beta\in\mathbf{N}^*\\ \|\alpha\|=\|\beta\|=j}}Y^{\Omega(\alpha+\beta)}\frac{X^{2\Omega(\alpha)}(1+X)^{\Omega(\beta)}}{\prod_{z\geq 0}(\alpha_z!)(\beta_z!)}\right)\right.$$

$$\left.\cdot\sum_{\substack{\gamma\in\mathbf{N}^*\\ \|\gamma\|=j}}\frac{j!}{\prod_{v\geq 0}(\gamma_v!)}(-1)^{\Omega(\gamma)}Y^{\Omega(\gamma)}\prod_{r\geq 0}\binom{j}{r}^{\gamma_r}\right]$$

$$= \sum_{j\geq 0}\left[X^{3j}Y^{2j}(j!)^3\right.$$

$$\left.\cdot\left(\sum_{\substack{\alpha,\beta,\gamma\in\mathbf{N}^*\\ \|\alpha\|=\|\beta\|=\|\gamma\|=j}}Y^{\Omega(\alpha+\beta+\gamma)}\frac{X^{2\Omega(\alpha)}(1+X)^{\Omega(\beta)}}{\prod_{z\geq 0}(\alpha_z!)(\beta_z!)(\gamma_z!)}(-1)^{\Omega(\gamma)}\prod_{r\geq 0}\binom{j}{r}^{\gamma_r}\right)\right]$$

$$= \sum_{\substack{\alpha,\beta,\gamma\in\mathbf{N}^*\\ \|\alpha\|=\|\beta\|=\|\gamma\|}}\left[Y^{2\|\alpha\|+\Omega(\alpha+\beta+\gamma)}X^{3\|\alpha\|+2\Omega(\alpha)}(1+X)^{\Omega(\beta)}\right.$$

$$\left.\cdot\frac{(\|\alpha\|!)^3}{\prod_{z\geq 0}(\alpha_z!)(\beta_z!)(\gamma_z!)}(-1)^{\Omega(\gamma)}\prod_{r\geq 0}\binom{\|\gamma\|}{r}^{\gamma_r}\right]$$

$$= \sum_{n\geq 0} Y^n \left[\sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}^* \\ \|\alpha\|=\|\beta\|=\|\gamma\| \\ \Phi(\alpha,\beta,\gamma)=n}} \left(X^{3\|\alpha\|+2\Omega(\alpha)} (1+X)^{\Omega(\beta)} \right.\right.$$

$$\left.\left. \cdot \frac{(\|\alpha\|!)^3}{\prod_{z\geq 0} (\alpha_z!)(\beta_z!)(\gamma_z!)} (-1)^{\Omega(\gamma)} \prod_{r\geq 0} \binom{\|\gamma\|}{r}^{\gamma_r} \right) \right].$$

The desired result follows from Theorem 2.2. □

The following Theorem follows from Theorem 2.4 using the same techniques.

Theorem 2.5. *For all integers $n \in \mathbb{N}$ and $0 \leq h \leq 2n$,*

$$g_{n,h} = \sum_{j=0}^{n} \sum_{\substack{t,k\geq 0 \\ \alpha,\beta,\gamma\in\mathbb{N}^* \\ \|\alpha\|=\|\beta\|=\|\gamma\| \\ \Phi(\alpha,\beta,\gamma)=n-j \\ \Psi(\alpha,t,k)=h}} \left[\binom{j}{k} + \sum_{\alpha=1}^{j} \binom{\alpha-2}{2j+1-k-\alpha} \right]$$

$$\cdot \binom{\Omega(\beta)}{t} \frac{(\|\alpha\|!)^3}{\prod_{z\geq 0} (\alpha_z!)(\beta_z!)(\gamma_z!)} (-1)^{\Omega(\gamma)} \prod_{r\geq 0} \binom{\|\gamma\|}{r}^{\gamma_r}$$

holds, where we set $\binom{-1}{0} = 1$, $\Phi(\alpha,\beta,\gamma) = 2\|\alpha\| + \Omega(\alpha+\beta+\gamma)$, and $\Psi(\alpha,t,k) = 3\|\alpha\| + 2\Omega(\alpha) + t + k$ for all $\alpha,\beta,\gamma \in \mathbb{N}^*$ and $t,k \in \mathbb{N}$.

□

Using the previous results, we have a purely combinatorial proof of the following remarkably identity.

Corollary 2.6. *For all integers $n \in \mathbb{N}$ and $0 \leq h \leq 2n$,*

$$\sum_{j=0}^{n} \sum_{\substack{t,k\geq 0\\ \alpha,\beta,\gamma\in\mathbb{N}^*\\ \|\alpha\|=\|\beta\|=\|\gamma\|\\ \Phi(\alpha,\beta,\gamma)=n-j\\ \Psi(\alpha,t,k)=h}} \left[\binom{j}{k} + \sum_{\alpha=1}^{j} \binom{\alpha-2}{2j+1-k-\alpha}\right]$$

$$\cdot \binom{\Omega(\beta)}{t} \frac{(\|\alpha\|!)^3}{\prod_{z\geq 0}(\alpha_z!)(\beta_z!)(\gamma_z!)} (-1)^{\Omega(\gamma)} \prod_{r\geq 0} \binom{\|\gamma\|}{r}^{\gamma_r}$$

$$= \sum_{j=0}^{n} \sum_{\substack{t,k\geq 0\\ \alpha,\beta,\gamma\in\mathbb{N}^*\\ \|\alpha\|=\|\beta\|=\|\gamma\|\\ \Phi(\alpha,\beta,\gamma)=n-j\\ \Psi(\alpha,t,k)=2n-h}} \left[\binom{j}{k} + \sum_{\alpha=1}^{j} \binom{\alpha-2}{2j+1-k-\alpha}\right]$$

$$\cdot \binom{\Omega(\beta)}{t} \frac{(\|\alpha\|!)^3}{\prod_{z\geq 0}(\alpha_z!)(\beta_z!)(\gamma_z!)} (-1)^{\Omega(\gamma)} \prod_{r\geq 0} \binom{\|\gamma\|}{r}^{\gamma_r}$$

holds, where we set $\binom{-1}{0} = 1$, $\Phi(\alpha,\beta,\gamma) = 2\|\alpha\| + \Omega(\alpha+\beta+\gamma)$, *and* $\Psi(\alpha,t,k) = 3\|\alpha\| + 2\Omega(\alpha) + t + k$ *for all* $\alpha,\beta,\gamma \in \mathbb{N}^*$ *and* $t,k \in \mathbb{N}$.

Proof. This follows immediately from Theorem 2.5 noticing that $\mathcal{G}_n$ is a self–dual poset, and therefore $g_{n,h} = g_{n,2n-h}$. □

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