Flag-transitive 2-$(v, k, 4)$ symmetric designs

Weidong Fang, Huili Dong, Shenglin Zhou*

Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China

Abstract Let $\mathcal{D}$ be a 2-$(v, k, 4)$ symmetric design, and G be a subgroup of the full automorphism group of $\mathcal{D}$. In this paper, we prove that if $G \leq Aut(\mathcal{D})$ is flag-transitive, point-primitive then G is of affine or almost simple type. We prove further that if a nontrivial 2-$(v, k, 4)$ symmetric design has a flag-transitive, point-primitive, almost simple automorphism group G, then $Soc(G)$ is not a sporadic simple group.

Keywords Symmetric design, automorphism group, flag-transitive, point-primitive, sporadic group

MR(2000) Subject Classification 05B05, 05B25, 20B25

1 Introduction

A *symmetric* $2-(v, k, \lambda)$ *design* is an incidence structure $\mathcal{D} = (P, \mathcal{B})$ where P is a set of v points and $\mathcal{B}$ is a set of b blocks with an incidence relation satisfying every block is incident with exactly k points, every 2-element subset of points is incident with exactly λ blocks, and $b = v$. The design is called *nontrivial* if $\lambda < k < v - 1$. We study nontrivial $2-(v, k, \lambda)$ symmetric designs here which denoted by (v, k, λ)-symmetric designs for simplicity. An *automorphism* of a design $\mathcal{D}$ is a permutation of the points which also permutes the blocks, preserving the incidence relation. The set of automorphisms of a design with the composition of functions is a group. A permutation group $G \leq Aut(\mathcal{D})$ is *transitive* on $\mathcal{D}$ if for any $\alpha, \beta \in P$, there exists an element $g \in G$ such that $\alpha^g = \beta$, and G is *point-primitive* on $\mathcal{D}$ if G is a primitive permutation group on the point set P, otherwise is called *point-imprimitive*. A *flag* in a block design is a point-block pair, such that the point is incident with the block. Thus to say that G is

*Corresponding author: slzhou@scut.edu.cn. The project was sponsored by SRF for ROCS, SEM and Research Fund for the Doctoral Program of Higher Education of China(No. 200805611097).

flag-transitive means that if α_1, α_2 are points, B_1, B_2 are blocks, and α_i is incident with B_i for $i = 1, 2$, then there is an automorphism of $\mathcal{D}$ taking (α_1, B_1) to (α_2, B_2).

Symmetric $2-(v, k, \lambda)$ designs with λ small are of interest. For example, those with $\lambda = 1$ are the *projective planes*, while those with $\lambda = 2$ are called *biplanes*. Flag-transitivity is just one of many conditions that can be imposed on the automorphism group G of a symmetric design $\mathcal{D}$. For the flag-transitive projective planes, Kantor [7] proved that either $\mathcal{D}$ is a Desarguesian projective plane and $PSL(3, n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, where n is even and $n^2 + n + 1$ is prime. In [12, 13, 14, 15], using the theorem of classification of finite simple groups, Regueiro reduced the classification of flag-transitive biplanes to the situation where the automorphism group is a one-dimensional affine group. After the work of Regueiro on biplanes, it is necessery to consider the classification of flag-transitive (v, k, λ)-symmetric designs with $\lambda \leq 4$. In 2009, Law, Praeger and Reichard also suggested the following problem.

Problem 1.1. (Law, Praeger, Reichard [8, Problem 1]) *Reduce the classification of flag-transitive 2-(v, k, λ) symmetric designs with $\lambda = 3$ or 4 to the case of one-dimensional affine automorphism groups.*

For the case $\lambda = 3$, a $(v, k, 3)$-symmetric design is called a *triplane*. Recently, in [17, 18, 19, 20] the authors reduced the classification of flag-transitive 2-$(v, k, 3)$ symmetric designs to the case that $G \leq Aut(\mathcal{D})$ is of affine type, and proved the following result.

Proposition 1.2. *If $\mathcal{D}$ is a nontrivial 2-$(v, k, 3)$ symmetric design with a flag-transitive automorphism group G, then one of the following holds:*

(i) $\mathcal{D}$ has parameters $(45, 12, 3)$,

(ii) $\mathcal{D}$ has parameters $(11, 6, 3)$,

(iii) $\mathcal{D}$ has parameters $(15, 7, 3)$,

(iv) G is of affine type.

In this paper we discuss nontrivial flag-transitive 2-$(v, k, 4)$ symmetric designs. Here are some elementary results on symmetric designs.

Lemma 1.3. ([1, Theorem 2.7]) *Let p be a prime divisor of the order of the automorphism group of a (v, k, λ)-symmetric design such that $1 < \lambda < p$ and $gcd(p, v) = 1$. Then $p \leq k$.*

Lemma 1.4. *Let $\mathcal{D}$ be a $(v, k, 4)$-symmetric design, and G be a flag-transitive primitive automorphism group of $\mathcal{D}$, then* *(i)* $k(k-1) = 4(v-1)$.

(ii) $16v - 15$ *is a square.*

(iii) $4v < k^2$, *and hence* $4|G| < |G_x|^3$, *where* x *is a point in* P.

(iv) $k \mid 4\gcd(v-1, |G_x|)$.

(v) $4(|G_x|_{v'})^2 > v$ *where* $|G_x|_{v'}$ *denotes the part of* $|G_x|$ *coprime with* v.

(vi) $k \mid 4d_i$, *where* d_i *is any subdegree of* G ([4]).

Proof. Part (i) is well-known. Part (ii) follows from (i). (iii) The equality $k(k-1) = 4(v-1)$ implies $k^2 = 4v - 4 + k$, so clearly $4v < k^2$. Since $v = |G : G_x|$, and $k \le |G_x|$, the result follows. (iv) Since $k \mid 4(v-1)$, $r \mid |G_x|$ and $k = r$, we have that $r \mid 4\gcd(|G_x|, v-1)$. (v) By (iv) we have that $r \le 4\gcd(|G_x|, v-1)$. Hence $4v < k^2 \le 16\gcd(|G_x|, v-1)^2 = 16\gcd(|G_x|_v|G_\alpha|_{v'}, v-1)^2$, and $v < 4\gcd(|G_x|_{v'}, v-1)^2$. Therefore $4(|G_x|_{v'})^2 > v$. (vi) Suppose that Γ is a nontrivial suborbit of G_x. Let $|\Gamma| = d_i$. Consider the action of G on the set $P \times P$. Let Δ be an orbital of G. Counting the number of triples (x, y, B) where $x \ne y \in B$, $B \in \mathcal{B}$ and $(x, y) \in \Delta$ in two ways, we find

$$4|\Delta| = vkt,$$

where vk is the number of flags (x, B), t the number of triples that contain the flag (x, B). Since G is flag-transitive, t is independent of the choice of the flag (x, B), and since $|\Delta| = vd_i$, then $4vd_i = vkt$. It follows that $4d_i = kt$, so $k \mid 4d_i$. □

In the next section we need the O'Nan-Scott theorem on primitive permutation groups. We restate it as follows.

Lemma 1.5. (O'Nan-Scott Theorem, [9]) *Let* $G \le Sym(\Omega)$ *be a primitive group of degree* $|\Omega| = n$. *Then one of the following holds.*

(1) Affine type: $G \le AGL(\Omega)$, *acting on a finite vector space* Ω *of prime power order* n;

(2) Diagonal type: $T^k \le G \le T^k(Out\,T \times S_k)$, *where* T *is a nonabelian simple group and* $k \ge 2$. *The stabiliser in* T^k *of a point of* Ω *is a diagonal subgroup of* T^k *(that is a subgroup conjugate in* $(Aut\,T)^k$ *to* $D = \{(t, t, \cdots, t) \mid t \in T\}$*) and* $n = |T^k : D| = |T|^{k-1}$.

(3) Almost simple type: $T \trianglelefteq G \le Aut\,T$ *for some nonabelian simple group* T;

(4) Product type: Here $\Omega = \Delta^\ell$ *for some set* Δ *and* $\ell > 1$, *and* $G \leq H \wr S_\ell$, *where* H *is a primitive permutation group on* Δ. $H \wr S_\ell$ *has the natural action on* Ω *and* G *projects onto a transitive subgroup of* S_ℓ. *Thus* $n = |\Delta|^\ell$.

(5) Twist wreath product type: G *is a twist wreath product* $T\, twr_\psi\, P$, *where* T *is a nonabelian simple group,* P *is a transitive permutation group on* $\{1, 2, \cdots, k\}$ *and* ψ *is a homomorphism from* P_1, *the stabiliser of 1 in* P, *to* $Aut\,T$ *such that the image of* ψ *contains* $Inn\,T$. *The stabiliser in* G *of a point of* Ω *is* P. *Thus* $n = |T|^k$.

2 Imprimitive Case

Let $\mathcal{D}$ be a $(v, k, 4)$-symmetric design. Suppose that $G \leq Aut(\mathcal{D})$ is flag-transitive, point-imprimitive. In [11], C. E. Praeger and S. L. Zhou discussed the flag-transitive imprimitive (v, k, λ)-symmetric designs. From [11, Theorem 1.1 and Table 1 of Corollary 1.3], we know that if a $(v, k, 4)$-symmetric design admits a flag-transitive imprimitive automorphism group, then it has parameters $(15, 8, 4)$ or $(96, 20, 4)$. There is a unique example with the first parameter set, see [11, Proposition 1.5], namely the points and hyperplane complements of the projective geometry $PG(3, 2)$ relative to a subgroup S_5 or $S_5.3$ of the full automorphism group $PSL(4, 2) \cong A_8$. Recently, in [8], M. Law, C. E. Praeger and S. Reichard proved that there are four non-isomorphic symmetric designs with parameters $(90, 20, 4)$, the structures and automorphism groups of these designs are given in [8, Section 4], and they are all point-imprimitive.

Proposition 2.1. *(i)* (Praeger, Zhou [11, Proposition 1.5]) *The design of points and hyperplane complements of the projective geometry PG(3, 2) is the unique design admitting a flag-transitive, point-imprimitive subgroup of automorphisms with parameters* $(15, 8, 4)$.

(ii) (Law, Praeger, Reichard [8, Theorem 1.1]) *There are up to isomorphism exactly four flag-transitive* $(96, 20, 4)$*-symmetric designs. Furthermore, they are all point-imprimitive.*

This, together with [10], gives the following

Proposition 2.2. (Law, Praeger, Reichard [9, Corollary 1.2]) *All 2-*(v, k, λ) *symmetric designs with* $\lambda \leq 4$ *admitting a flag-transitive, point-imprimitive subgroup of automorphisms are known.*

Corollary 2.3. *If* G *is a group acting flag-transitively on a 2-*$(v, k, 4)$ *symmetric design* $\mathcal{D}$ *other than* $(15, 8, 4)$ *and* $(96, 20, 4)$, *then* G *is point-primitive.*

The following lemma asserts that there exists a flag-transitive, point-primitive $(15, 8, 4)$-symmetric design.

Lemma 2.4. *If $\mathcal{D}$ is a flag-transitive, point-primitive* $(15, 8, 4)$*-symmetric design, then $\mathcal{D} = PG_2(3, 2)$, $G = A_6$, S_6, A_7 or A_8, and the stabiliser $G_x = S_4, S_4 \times Z_2$, $L_3(2)$ or $AGL_3(2)$ respectively.*

Proof. Let $\mathcal{D}$ be a $(15, 8, 4)$-symmetric design, and $G \leq Aut(\mathcal{D})$ be point-primitive. As a primitive permutation group of degree 15, we know that G is one of the following ([3, p. 597]): $A_6, S_6, A_7, A_8 \cong L_4(2), A_{15}, S_{15}$, and the stabiliser G_x is $S_4, S_4 \times Z_2, L_3(2), AGL_3(2), 14.3, 14.4$ respectively.

If $Soc(G) = A_6$, then $G_x = S_4$ or $S_4 \times Z_2$, and the subdegrees of G acting on the stabiliser G_x are 1, 6, 8, and so G is a rank 3 primitive group acting on points set (and blocks set) of a symmetric 2-$(15, 8, 4)$ design. By [5, Theorem] we know that this design exist and it is the symmetric $2-(15, 8, 4)$ design of points and complements of hyperplanes from $PG(3, 2)$. It is easy to see that $\mathcal{D}$ is flag-transitive.

If $Soc(G) = A_{15}$ with natural action, there exists a prime $p = 13$ satisfying the conditions of Lemma 1.3, but $p > k$, which is a contradiction.

If $G = A_7$ or A_8, the subdegrees are 1, 14, then the action of G is 2-transitive, this has been done by Kantor. From [6, Theorem] we know that there is a 2-transitive symmetric design, i.e. a projective space $PG_2(3, 2)$, with parameters $(15, 8, 4)$ which is the complement of a $(15, 7, 3)$-symmetric design. It is easily know that $\mathcal{D}$ is also flag-transitive. So Lemma 2.4 holds. □

3 Primitive Case

From [8] we know that the automorphism group of flag-transitive $(96, 20, 4)$-symmetric design cannot be point-primitive. Therefore, by Corollary 2.3 and Lemma 2.4, if a $(v, k, 4)$-symmetric design other than $(15, 8, 4)$ admits a flag-transitive automorphism group G, then it must be primitive. So we just need to analysis the primitive case. Here, we will investigate the case in which $\mathcal{D}$ admits a primitive, flag-transitive automorphism group. The O'Nan-Scott Theorem (see Lemma 1.5) classifies primitive groups into the following five types:

(1) Affine type.

(2) Almost simple type.

(3) Simple diagonal type.

(4) Product type.

(5) Twisted wreath product type.

In this section we will prove that cases (3), (4) and (5) cannot occur. Here is the main theorem.

Theorem 3.1. *If $\mathcal{D}$ is a $(v,k,4)$-symmetric design admitting a flag-transitive primitive automorphism group G, then G is of affine, or almost simple type.*

We will prove the theorem by a series of lemmas. Suppose that G has a product action on the set P of points. Then there is a group H acting primitively on a set Γ ($|\Gamma| \geq 5$) with almost simple or diagonal action, where

$$P = \Gamma^{\ell}, \ell \geq 2, \text{ and } G \leq H^{\ell} \rtimes S_{\ell} = H \wr S_{\ell}.$$

We first give the following general lemma on a flag-transitive, point-primitive 2-(v,k,λ) symmetric design which will be very useful for the proof of Theorem 3.1.

Lemma 3.2. ([12, Lemma 4]) *If G is a primitive permutation group acting flag-transitively on a (v,k,λ)-symmetric design $\mathcal{D}$, with a product action on P, then $v = |\Gamma|^{\ell} \leq \lambda\ell^2(|\Gamma|-1)^2$, and $\ell = 2$ forces $\lambda > 4$.*

Lemma 3.3. *If D is a $(v,k,4)$-symmetric design admitting a flag-transitive, primitive automorphism group G, then G does not have a nontrivial product action or twisted wreath action on the points of D.*

Proof. Let $|\Gamma| = m$. By Lemma 3.2, we have $m^{\ell} \leq 4\ell^2(m-1)^2$ which implies $\ell < 5$. If $\ell = 4$ then $m^4 \leq 64(m-1)^2$. It follows that $m = 5$ or 6, and $v = 5^4$ or 6^4 respectively, but then in every case $16v - 15$ is not a square, contrary to Lemma 1.4(*ii*).

Therefore $\ell = 3$, and $m^3 \leq 36(m-1)^2 < 36m^2$ which implies $5 \leq m < 36$. On the one hand, it is easily known that the only value of m such that $16v - 15 = 16m^3 - 15$ is a square is $m = 34$.

On the other hand, by Lemma 1.4(*i*), $k \mid 4(v-1) = 4(m^3-1)$, also from the proof of [12, Lemma 4] we know that $k \mid \lambda\ell(m-1)$, so $k \mid \gcd(4(m^3-1), 12(m-1)) = 4(m-1)\gcd(3, m^3+m+1)$. Hence k divides $12(m-1)$. Suppose that $k = \frac{12(m-1)}{n}$ for some positive integer n, then from the equality $k(k-1) = 4(v-1)$ we have

$$\frac{12(m-1)}{n}\Big(\frac{12(m-1)}{n} - 1\Big) = 4(m^3-1),$$

so

$$3[12(m-1) - n] = (m^2+m+1)n^2.$$

If $m = 34$, then $397n^2 + n - 396 = 0$, it follows that $n = \frac{396}{397}$ or $n = -1$, a contradiction.

Groups with a twisted wreath action are contained in twisted wreath groups $H\ twr_\psi\ S_\ell$ with a product action and H is of diagonal type. Here we have also considered subgroups of G, thereby also ruling out groups with a twisted wreath action. This completes the proof of Lemma 3.3. □

Lemma 3.4. *If $\mathcal{D}$ is a $(v, k, 4)$-symmetric design which admits a flag-transitive, primitive automorphism group G, then G is not of simple diagonal type.*

Proof. Suppose that G is of simple diagonal type. Then

$$Soc(G) = N = T^m,\ m \geq 2,$$

for some nonabelian simple group T, where $T \cong N_x \lhd G_x \leq Aut(T) \wr S_m$. Here $v = |T|^{m-1} = |N_x|^{m-1}$.

The fact that G is flag-transitive implies that G_x is transitive on the k blocks through x, so $N_x \lhd G_x$ implies that all the orbits of N_x on the set of k blocks through x have the same size, say, ℓ. Therefore ℓ divides k, so it divides $4(v-1)$, and also divides $|T|$, that is, ℓ divides $\gcd(|T|, 4(|T|^{m-1} - 1)) = \gcd(|T|, 4)$. So $\ell = 1, 2$ or 4.

If $\ell = 2$ or 4 then, for a block $B \in \mathcal{B}$, we have $|N_x : N_{xB}| = 2$ or 4. This is impossible since N_x is a nonabelian simple group.

Thus $\ell = 1$, and N_x fixes all the k blocks through x. Since N_x is a nonabelian simple group, then there exists an odd prime p such that $p \mid |G|$ and $p \geq 5$. Otherwise, $|N_x| = 2^a3^b$, by p^aq^b-Theorem, N_x is solvable, and so it is not simple, a contradiction. Choose an element $t \in N_x$, $o(t) = p \geq 5$. There is a point y which is not fixed by t. The point-pair $\{x, y\}$ is incident with exactly 4 blocks. Since y is in each of these blocks, the points in the t-orbit O of y, together with x, must be also incident with each of these blocks as these blocks are fixed by N_x. Since $|O| \geq 5$, then every pair of these 4 blocks is incident with at least $|O| + 1 \geq 6$ points, this contradicts the fact that there are exactly four points incident with two blocks. □

Proof of Theorem 3.1. Now Theorem 3.1 follows from Lemmas 3.3, 3.4. □

4 The case where the socle is a sporadic group

Theorem 4.1. *Let $\mathcal{D}$ be a nontrivial $(v, k, 4)$-symmetric design with a flag-transitive, point-primitive, almost simple automorphism group G, then $Soc(G)$ can not be a sporadic simple group.*

Proof. We suppose there is a nontrivial $(v, k, 4)$-symmetric design $\mathcal{D}$ which has a flag-transitive, point-primitive, almost simple automorphism group G with socle X, where X is a sporadic simple group, and arrive at a contradiction. We follow the same procedure as in [13, 17] for biplanes and triplanes.

Assume that the automorphism group G of $\mathcal{D}$ is almost simple, such that $X \trianglelefteq G \leq Aut(X)$ with X a sporadic group. Then $G = X$, or $G = Aut(X)$, since for all sporadic groups X either $Aut(X) = X$ or $Aut(X) = X.2$. We know that $v = |G : G_x|$, where $x \in P$ and G_x is a maximal subgroup of G. The lists of maximal subgroups of X and $Aut(X)$ appear in [2]. Note that they are complete except for the 2-local subgroups of the Monster group M, and any possible maximal subgroup of the Monster M which is not listed in [2] has socle isomorphic to one of the following simple groups: $L_2(13), L_2(27), Sz(8), U_3(4), U_3(8)$.

For each sporadic group (and its automorphism group), we rule out the maximal subgroups whose index $|G : G_x| = v$ such that $16v - 15$ is not a square. In the remaining cases, for those $v > 2$, we rule out the maximal subgroups whose order is too small to satisfy $4|G| < |G_x|^3$.

To illustrate this procedure, suppose $X = J_2$, the sporadic Hall-Janko group. Then $G = J_2$ or $J_2.2$, since $|Out(J_2)| = 2$. The maximal subgroups H of J_2 and $J_2.2$, with their orders, indices and $16v - 15$ are listed as follows([2, p.42]):

Order	Index	X	$X.2$	$16v - 15$
6048	100	$U_3(3)$	$: U_3(3) : 2$	5.317
2160	280	$3^{\cdot}PGL_2(9)$	$: 3^{\cdot}A_6{}^{\cdot}2^2$	5.19.47
1920	315	$2_-^{1+4} : A_5$	$: 2_-^{1+4}.S_5$	$3.5^2.67$
1152	525	$2^{2+4} : (3 \times S_3)$	$: H.2$	3.5.13.43
720	840	$A_4 \times A_5$	$: (A_4 \times A_5) : 2$	$3.5^2.179$
600	1008	$A_5 \times D_{10}$	$: (A_5 \times D_{10}){}^{\cdot}2$	3.41.131
336	1800	$L_3(2) : 2$	$: L_3(2) : 2 \times 2$	3.5.19.101
300	2016	$5^2 : D_{12})$	$: U_3(3) : 2$	3.11.977
60	10080	A_5	$: S_5$	3.5.13.827

Table 1: Maximal subgroups of groups J_2 and $J_2.2$

From the last column of Table 1 we know that in every case $16v - 15$ is not a square, a contradiction. Hence $X \neq J_2$.

For the Monster group M, there are 43 classes of maximal subgroups known so far. These 43 classes of maximal subgroups G_x can be ruled out by the fact that $|G_x|^3 < 4|G|$ and $16v - 15$ is not a square. Any other possible maximal subgroup N of M which is not listed in [2] satisfies $S \trianglelefteq N \leq Aut(S)$ where S is isomorphic to one of the following simple groups: $L_2(13), L_2(27), U_3(4), U_3(8), Sz(8)$. However, for any possible maximal subgroup N, its order is also too small to satisfy $4|M| < |N|^3$, a contradiction. □

References

[1] M. Aschbacher, On collineation groups of symmetric block designs, *J. Combin. Theory Ser. A* **11**(1971), 272-381.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of Finite Groups, London: Oxford University Press, 1985.

[3] C. J. Colbourn, J. H. Dinitz, The CRC Handbook of Combinatorial Designs, Boca Raton, FL: CRC press, 1996.

[4] H. Davies, Flag-transitivity and primitivity, *Discrete Math.* **63**(1987), 91-93.

[5] U. Dempwolff, Primitive rank 3 groups on symmetric designs, *Des. Codes and Cryptogr.* **22**(2001), 191-207.

[6] W. M. Kantor, Classification of 2-transitive symmetric designs, *Graphs Comb.* **1** (1985), 165-166.

[7] W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1)(1987), 15-45.

[8] M. Law, C. E. Praeger, S. Reichard, Flag-transitive symmetric 2-(96, 20, 4)-design, *J. Combin. Theory Ser. A* **116**(2009), 1009-1022.

[9] M. W. Liebeck, C. E. Praeger, J. Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, *J. Aust. Math. Soc. Ser. A* **44**(1988), 389-396.

[10] C. E. Praeger, The flag-transitive symmetric designs with 45 points, blocks of size 12, and 3 blocks on every point pair, *Des. Codes and Cryptogr.* **44**(2007), 115-132.

[11] C. E. Praeger, S. L. Zhou, Imprimitive flag-transitive symmetric designs, *J. Combin. Theory Ser. A* **113**(7)(2006), 1381-1395.

[12] E. O'Reilly Regueiro, On primitivity and reduction for flag-transitive symmetric designs, *J. Combin. Theory Ser. A* **109** (2005), 135-148.

[13] E. O'Reilly Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with alternating or sporadic socle, *European J. Combin.* **26**(2005), 577-584.

[14] E. O'Reilly Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with classical socle, *J. Algebraic Combin.* **26**(2007), 529-552.

[15] E. O'Reilly Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with exceptional socle of Lie type, *J. Algebraic Combin.* **27**(2008), 479-491.

[16] H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.

[17] S. L. Zhou, H. L. Dong, Sporadic groups and flag-transitive triplanes, *Science in China Ser. A*, **52**(2)(2009), 394-400.

[18] S. L. Zhou, H. L. Dong, Exceptional groups and flag-transitive triplanes, *Science in China Ser. A*, **52**(2009), DOI: 10.1007/s11425-009-0051-5.

[19] S. L. Zhou, H. L. Dong, W. D. Fang, Finite classical groups and flag-transitive triplanes, *Discrete Math.*, **309**(16)(2009), 5183-5195.

[20] S. L. Zhou, H. L. Dong, Alternating groups and flag-transitive triplanes, submitted, January, 2009.