REMARKS ON THE FINITE HEINE TRANSFORMATIONS

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Abstract: In this paper, we use the finite Heine ${}_2\Phi_1$ transformations given in [4] and some elementary simplifications to obtain several Rogers-Ramanujan type identities.

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1. Introduction

Heine ${}_2\Phi_1$ transformation formula play a fundamental role in the theory of q-series. There are many applications of this formula. Recently, G. E. Andrews [4] gave several finite Heine ${}_2\Phi_1$ transformations from the terminating Sears ${}_3\Phi_2$ transformation. Then he used them to give two finite Rogers-Ramanujan type identities. In this paper, using the method in [4], we get the identities

$$\sum_{k=0}^{M} \frac{q^{\binom{k}{2}}}{(-q;q)_k(q;q)_{M-k}} = \frac{2}{(q;q)_M(1+q^M)}, \tag{1.1}$$

$$\sum_{k=0}^{M} \frac{q^k}{(q^2;q^2)_k} = \frac{1}{(-q;q)_M} \sum_{k=0}^{M} \frac{q^{\binom{k+1}{2}}}{(q;q)_{M-k}}, \tag{1.2}$$

$$\sum_{k=0}^{M} \frac{q^k}{(cq;q)_k} = \frac{(q;q)_{M+1}}{(cq;q)_M} \sum_{k=0}^{M} \frac{(-c)^k q^{\binom{k+1}{2}}}{(q;q)_{k+1}(q;q)_{M-k}}, \tag{1.3}$$

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where

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1, 2, \cdots$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Clearly (1.1) and (1.2) converge to

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(-q;q)_k} = 2, \tag{1.4}$$

$$\sum_{k=0}^{\infty} q^{\binom{k+1}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.5}$$

as $M \to \infty$ respectively. Equation (1.4) dues to Andrews [1], equation (1.5) can be found in [2, 3, 5]. Letting $c = -1, M \to \infty$, the third identity dues to S. O. Warnaar [17]

$$\sum_{k=0}^{\infty} \frac{q^k}{(-q;q)_k} = 2 - \frac{1}{(-q;q)_\infty}. \tag{1.6}$$

Applying the limits of the finite Heine transformation formulas and some elementary techniques, we also get the following identities

$$\sum_{n=0}^{\infty} \frac{q^{2n+2}}{(q;q)_{2n+2}} = \frac{1}{2(q;q)_\infty} + \frac{1}{2(-q;q)_\infty} - 1, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n+1}} = \frac{1}{2(q;q)_\infty} - \frac{1}{2(-q;q)_\infty}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q;q)_{2n}} = \frac{(q;q)_\infty + (-q;q)_\infty}{2}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(q;q)_{2n+1}} = \frac{(-q;q)_\infty - (q;q)_\infty}{2q}. \tag{1.10}$$

The first two identities can be easily derived from q-binomial theorem, the last two are not included in the papers [1-7, 11, 14-17].

Throughout the paper, we take $0 < |q| < 1$. And we also use the following notations

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

$$\begin{bmatrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{bmatrix}_n = \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n},$$

and

$$
{}_r\Phi_s\left(\begin{matrix} a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix}; q, x\right)
= \sum_{n=0}^{\infty} \begin{bmatrix} a_1, & a_2, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{bmatrix}_n \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} x^n,
$$

and

$$
\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.
$$

2. Main Results

In [4], G. E. Andrews gave the following finite Heine transformation formulas

Lemma 2.1. We have

$$
{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & q^{1-M}/x, & c \end{matrix}; q, q\right)
= \frac{(ax, b; q)_M}{(x, c; q)_M}\, {}_3\Phi_2\left(\begin{matrix} q^{-M}, & x, & c/b \\ & cq^{1-M}/b, & ax \end{matrix}; q, q\right). \tag{2.1}
$$

Lemma 2.2. We have

$$
{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & q^{1-M}/x, & c \end{matrix}; q, q\right)
= \frac{(ax, c/a; q)_M}{(x, c; q)_M}\, {}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & abx/c \\ & aq^{1-M}/c, & ax \end{matrix}; q, q\right). \tag{2.2}
$$

Lemma 2.3. We have

$$
{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & q^{1-M}/x, & c \end{matrix}; q, q\right)
= \frac{(abx/c; q)_M}{(x; q)_M}\, {}_3\Phi_2\left(\begin{matrix} q^{-M}, & c/a, & c/b \\ & cq^{1-M}/abx, & c \end{matrix}; q, q\right). \tag{2.3}
$$

In [4], Andrews applied terminating Sears ${}_3\Phi_2$ transformation and some elementary techniques to arrive them. In fact, the identities above can be obtained from the identity

$$
{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & c, & d \end{matrix}; q, q\right)
= a^n \frac{(d/a, b; q)_M}{(c, d; q)_M}\, {}_3\Phi_2\left(\begin{matrix} q^{-M}, & c/b, & q^{1-M}/d \\ & aq^{1-M}/d, & q^{1-M}/b \end{matrix}; q, q\right) \tag{2.4}
$$

given by Gasper [10]. It is an extension of q-Chu-Vandermonde's identity. In [8, p. 1402], we gave a new proof and an application of it. To make our paper self-contained, we present the following proofs.

Proofs of Lemmas 2.1-2.3

For the left hand side of (2.4) is a symmetrical about the variable c and d, we have

$$
\begin{aligned}
&{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & c, & d \end{matrix}; q, q\right) \\
&= a^M \frac{(c/a, b; q)_M}{(c, d; q)_M} {}_3\Phi_2\left(\begin{matrix} q^{-M}, & d/b, & q^{1-M}/c \\ & aq^{1-M}/c, & q^{1-M}/b \end{matrix}; q, q\right).
\end{aligned}
\tag{2.5}
$$

Iterating (2.5), we have (cf.[9, p. 61, Eq. (3.2.3)])

$$
\begin{aligned}
&{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & c, & d \end{matrix}; q, q\right) \\
&= \left(\frac{ab}{c}\right)^M \frac{(cd/ab; q)_M}{(d; q)_M} {}_3\Phi_2\left(\begin{matrix} q^{-M}, & c/b, & c/a \\ & cd/ab, & c \end{matrix}; q, q\right).
\end{aligned}
\tag{2.6}
$$

Again iterating (2.6), we have

$$
\begin{aligned}
&{}_3\Phi_2\left(\begin{matrix} q^{-M}, & a, & b \\ & c, & d \end{matrix}; q, q\right) \\
&= a^M \frac{(c/a, d/a; q)_M}{(c, d; q)_M} {}_3\Phi_2\left(\begin{matrix} q^{-M}, & abq^{1-M}/cd, & a \\ & aq^{1-M}/d, & aq^{1-M}/c \end{matrix}; q, q\right)
\end{aligned}
\tag{2.7}
$$

Letting $d = q^{1-M}/x$, equation (2.4) turns to (2.1), equation (2.6) reduces to (2.3), equation (2.7) come to (2.2). Now we complete the proofs. ■

Setting $x = q$, then letting $b \to 0$, Lemma 2.1 comes to

Theorem 2.1. We have

$$
\sum_{k=0}^{M} \frac{(a;q)_k q^k}{(q,c;q)_k} = \frac{(aq;q)_M}{(c;q)_M} \sum_{k=0}^{M} \frac{(-c)^k q^{\binom{k}{2}}}{(aq;q)_k (q;q)_{M-k}}.
\tag{2.8}
$$

Taking $x = q$, then letting $c \to 0$, Lemma 2.2 reduces to

Theorem 2.2. We have

$$
\sum_{k=0}^{M} \frac{(a,b;q)_k q^k}{(q;q)_k} = \frac{(a;q)_{M+1}}{(q;q)_M} \sum_{k=0}^{M} \begin{bmatrix} M \\ k \end{bmatrix} \frac{(-bq)^k q^{\binom{k}{2}}}{1 - aq^k}.
\tag{2.9}
$$

Putting $x = q$, then letting $a \to 0$ in Lemma 2.3, we have

Theorem 2.3. We have

$$\sum_{k=0}^{M} \frac{(b;q)_k q^k}{(q,c;q)_k} = \sum_{k=0}^{M} \frac{(-bq)^k q^{\binom{k}{2}} (c/b;q)_k}{(q,c;q)_k (q;q)_{M-k}}. \tag{2.10}$$

In (2.10), taking $q \to 1/q$, then replacing (b, c) by $(1/b, 1/c)$ respectively, we get

Theorem 2.4. We have

$$\sum_{k=0}^{M} \frac{(b;q)_k (-c/b)^k q^{\binom{k}{2}}}{(q,c;q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k+1}{2}} (c/b;q)_{M-k}}{(q;q)_k (q,c;q)_{M-k}}. \tag{2.11}$$

3. Some special cases

Theorem 3.1. Identity (1.1) is valid.

Proof. In (2.8), setting $a = c = -1$ in (2.8), we complete the proof. ■

Theorem 3.2. Identity (1.2) is valid.

Proof. In (2.8), setting $a = 0, c = -q$, we complete the proof. ■

Theorem 3.3. Identity (1.3) is valid.

Proof. In (2.8), setting $a = q, c \to cq$, we complete the proof. ■

In (2.8), setting $a = -1, c = q$, we have

Theorem 3.4. We have

$$\sum_{k=0}^{M} \frac{(-1;q)_k q^k}{(q;q)_k^2} = \frac{(-q;q)_M}{(q;q)_M} \sum_{k=0}^{M} \frac{(-q)^k q^{\binom{k}{2}}}{(-q;q)_k (q;q)_{M-k}}. \tag{3.1}$$

In (2.9), setting $a = q$, we have

Theorem 3.5. We have

$$\sum_{k=0}^{M} (b;q)_k q^k = (1 - q^{M+1}) \sum_{k=0}^{M} \begin{bmatrix} M \\ k \end{bmatrix} \frac{(-bq)^k q^{\binom{k}{2}}}{1 - q^{k+1}}. \tag{3.2}$$

In (2.10), letting $b \to 0, q \to q^2$, we have

Theorem 3.6. We have

$$\sum_{k=0}^{M} \frac{q^{2k}}{(q^2,c;q^2)_k} = \sum_{k=0}^{M} \frac{q^{2k^2}c^k}{(q,c;q^2)_k(q^2;q^2)_{M-k}}. \tag{3.3}$$

In (2.11), taking $c = q, b = -1$, we get

Theorem 3.7. We have

$$\sum_{k=0}^{M} \frac{q^{\binom{k+1}{2}}(-1;q)_k}{(q;q)_k^2} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k+1}{2}}(-q;q)_{M-k}}{(q;q)_k(q;q)_{M-k}^2}. \tag{3.4}$$

In (1.3), setting $M \to \infty$, we have

Theorem 3.8. For $0 < |c| < 1$, we have

$$\sum_{k=0}^{\infty} \frac{q^k}{(cq;q)_k} = 1 - \frac{1}{c} + \frac{1}{c(cq;q)_\infty}. \tag{3.5}$$

If $c \to -1$, equation (3.5) reduces to the identity (1.6).

Theorem 3.9. Identity (1.7) is valid.

Proof. Setting $b = q, c = -q, M \to \infty$ in (2.10), combining with (1.6), we have

$$\sum_{k=1}^{\infty} \frac{(-1)^k q^{2k} q^{\binom{k}{2}}}{(q;q)_{k-1}(1-q^{2k})} = (q;q)_\infty - \frac{(q;q)_\infty(q;q^2)_\infty}{2} - \frac{1}{2}. \tag{3.6}$$

For

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k q^{2k} q^{\binom{k}{2}}}{(q;q)_{k-1}(1-q^{2k})} &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} q^{2k+2} q^{\binom{k+1}{2}}}{(q;q)_k} \sum_{n=0}^{\infty} q^{(2m+2)n} \\ &= -\sum_{n=0}^{\infty} q^{2n+2} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(2n+3)k} q^{\binom{k}{2}}}{(q;q)_k} = -(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n+2}}{(q;q)_{2n+2}}.\end{aligned}$$

Substituting the identity above into (3.6), from $(q;q^2)_\infty(-q;q)_\infty = 1$, we complete the proof. ■

Corollary 3.1. Identity (1.8) is valid.

Proof. Applying (1.7), then combining with the q-binomial theorem, we immediately complete the proof. ■

Remark. The two identities (1.7) and (1.8) can be easily derived from the q-binomial theorem. For example, from

$$\sum_{k=0}^{\infty}\frac{q^{2k}}{(q;q)_{2k}}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(1+(-1)^k)q^k}{(q;q)_k}$$
$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^kq^k}{(q;q)_k}$$
$$=\frac{1}{2(q;q)_\infty}+\frac{1}{2(-q;q)_\infty},$$

we get the identity (1.7). So we conclude that

Corollary 3.2. If $\omega^n=1,\omega\neq 1,n=1,2,\cdots$, then

$$\sum_{k=0}^{\infty}\frac{q^{nk}}{(q;q)_{nk}}=\frac{1}{n}\sum_{s=0}^{n-1}\frac{1}{(q\omega^s;q)_\infty}.$$

Corollary 3.3. Identity (1.9) is valid.

Proof. Setting $c=q, M\to\infty$ in (3.3), then applying (1.7), we complete the proof. ■

Corollary 3.4. Identity (1.10) is valid.

Proof. Setting $c=q^3, M\to\infty$ in (3.3), then using (1.8), we complete the proof. ■

As $M\to\infty$, (3.1), (3.2) and (3.4) tend to

$$\sum_{k=0}^{\infty}\frac{(-1;q)_kq^k}{(q;q)_k^2}=\frac{(-q;q)_\infty}{(q;q)_\infty^2}\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{k+1}{2}}}{(-q;q)_k}, \tag{3.7}$$

$$\sum_{k=0}^{\infty}(b;q)_kq^k=\frac{1}{b}-\frac{(b;q)_\infty}{b}, \text{ where } 0<|b|<1, \tag{3.8}$$

$$\sum_{k=0}^{\infty}\frac{q^{\binom{k+1}{2}}(-1;q)_k}{(q;q)_k^2}=\frac{(-q;q)_\infty}{(q;q)_\infty} \tag{3.9}$$

respectively. The identity (3.8) can be found in G. E. Andrews, J. Jiménez-Urroz and K. Ono's paper [6].

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