Smallest generalized cuts and diameter-increasing sets of Johnson graphs*

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Abstract

For a vertex v in a graph G, a local cut at v is a set of size $d(v)$ consisting of the vertex x or the edge vx for each $x \in N(v)$. A set $U \subset V(G) \cup E(G)$ is a diameter-increasing set of G if the diameter of $G - U$ is greater than the diameter of G. In the present work, we first prove that every smallest generalized cutset of Johnson graph $J(n,k)$ is a local cut except for $J(4,2)$. Then we show that every smallest diameter-increasing set in $J(n,k)$ is a subset of a local cut except for $J(n,2)$ and $J(6,3)$.

Keywords: Johnson graph; Local cut; Generalized cut; Diameter

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1 Introduction

The Johnson graph $J(n,k)$ has as vertices the k-subsets of an n-element set Ω. Two vertices A, B are adjacent if and only if $|A \cap B| = k-1$. Hence $J(n,k)$ has $\binom{n}{k}$ vertices, and is $k(n-k)$-regular. Since $J(n,k) \cong J(n,n-k)$ [5], we always assume $n \geq 2k$ in this paper. Brouwer and Numata [2] and Numata [7] gave characterizations of $J(n,k)$. Some parameters of $J(n,k)$ have been discussed: $J(n,k)$ has connectivity $k(n-k)$ [3], diameter k, wide-diameter $k+1$ [6] and the chromatic number less than n [4].

Motivated by the reliability of a communication network, Yau [9] introduced generalized cutsets. A set $W \subset V(G) \cup E(G)$ is a generalized cutset

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of G if $G - W$ is disconnected or has only one vertex. It is known that the size of smallest generalized cutset is equal to the vertex connectivity [1], and the vertex connectivity is equal to the regular degree in $J(n,k)$ [3]. A local cut at v is a set of size $d(v)$ consisting of the vertex x or the edge vx for each $x \in N(v)$, where $N(v)$ is the set of neighbors of a vertex v. Clearly, every local cut is a generalized cutset and every local cut in $J(n,k)$ is a smallest generalized cutset. In Section 2, we show that every smallest generalized cutset in $J(n,k)$ is a local cut except for $J(4,2)$.

From [6], we know that the diameter of $J(n,k)$ is k. In Section 3, we study the smallest diameter-increasing set in $J(n,k)$. A set $U \subset V(G) \cup E(G)$ is a diameter-increasing set of G if the diameter of $G - U$ is greater than the diameter of G. Ramras [8] studied the smallest diameter -increasing set in hypercubes. We show that every smallest diameter-increasing set U in $J(n,k)$ is a subset of a local cut except for $J(n,2)$ and $J(6,3)$, and deleting U will increase the diameter by exactly 1. In addition, $|U| = k^2$ for $n > 2k$, and $|U| = k^2 - 1$ for $n = 2k$. Our analysis is based on the existence of pairwise internally disjoint short paths joining arbitrary pairs of vertices.

In this paper, by "element" we mean "element in Ω".

2 Smallest generalized sets

In Theorem 2 of [3], the authors gave the following Lemma which reveals the connectivity and minimum cutset of $J(n,k)$.

Lemma 1. [3] *$J(n,k)$ has connectivity $k(n-k)$. Furthermore, each minimum cutset in $J(n,k)$ is the set of vertices adjacent to a single vertex.* □

It is obvious that every minimum cutset in $J(n,k)$ is actually a local cut. In the following, we show that a smallest generalized cutset of $J(n,k)$ is also a local cut.

Before proving our main result, we give two useful lemmas. Define $N[v] = N(v) \cup \{v\}$. Let $\kappa(G)$ and $\kappa'(G)$ be the connectivity and edge-connectivity of a graph G respectively. Ramras characterized smallest generalized cutset of a graph G (not complete) with equal connectivity and edge-connectivity in the following lemma.

Lemma 2. [8] *If G is a simple graph with $\kappa(G) = \kappa'(G) < |V(G)| - 1$, then every smallest generalized cutset in G has size $\kappa(G)$ and consists of a subset of a minimum separating set and one edge incident to each remaining vertex of that separating set.* □

Lemma 3. *For each vertex v in $J(n,k)$, $J(n,k) - N[v]$ is connected.*

Proof. For $k = 1$, $J(n,1)$ is a complete graph, the result is obvious.

For $k \geq 2$, let $v = \{a_1, a_2, \cdots, a_k\}$. There exists a vertex $\bar{v} = \{a_{k+1}, a_{k+2}, \cdots, a_{2k}\}$ such that $|\bar{v} \cap v| = 0$ since $n \geq 2k$. Clearly, $\bar{v} \notin N[v]$. Now we show that there exists a path from v_0 to $\bar{v}$ for any vertex v_0 in $J(n,k) - N[v]$. Let $v_0 = \{b_1, b_2, \cdots, b_k\}$. Then $v_0, v_1, v_2, \cdots, v_{k-1}, \bar{v}$ is a path from v_0 to $\bar{v}$, where $v_i = \{a_{k+1}, a_{k+2}, \cdots, a_{k+i}, b_{i+1}, \cdots, b_k\}$, $i = 1, 2, \cdots, k-1$. Since v_0 is not adjacent to v, v_0 contains at least two elements different from the elements of v. Hence each v_i contains at least two elements different from the elements of v, which implies that $v_i \notin N[v]$. So $J(n,k) - N[v]$ is connected for each vertex v. □

Now, we give our main result of this section.

Theorem 4. *Every smallest generalized cutset in $J(n,k)$ is a local cut except for $J(4,2)$.*

Proof. Let W be a smallest generalized cutset in a graph and A, B be the sets of vertices and edges in W respectively.

For $J(n,1)$, since $J(n,1)$ is complete, we prove an even stronger assertion that the result holds for all complete graphs. Let G be a complete graph with $|V(G)| = n$ and $|A| = m$. Denote one component of $G - W$ by C_1, and $C_2 = G - W - C_1$. Suppose that $|V(C_1)| = x$. Then $|V(C_2)| = n - m - x$, and $|B| = x(n-m-x)$ since B is a smallest edge-cut of $G - A$. $\kappa'(G - A) = n - m - 1$ since $G - A$ is a complete graph. Hence $n - m - 1 = x(n - m - x)$, that is $(x-1)(x-(n-m-1)) = 0$. Therefore $x = 1$ or $x = (n-m-1)$, which implies that $|C_1| = 1$ or $|C_2| = 1$. Without loss of generality, suppose that $|C_1| = 1$ and v_1 is the vertex in C_1. Clearly, W is a local cut at v_1.

If $k \geq 2$, then $n \geq 2k \geq 4$ for $J(n,k)$. In the following, we first prove that for any vertex v in $J(n,k)$, each vertex in $N(v)$ has at least two neighbors outside $N[v]$. Let $v = \{a_1, a_2, a_3, \cdots, a_k\}$. Without loss of generality, denote any vertex v_2 in $N(v)$ by $\{a_{k+1}, a_2, a_3, \cdots, a_k\}$. Since $n \geq 2k$ and $k \geq 2$, it is easy to see that $n - k \geq 3$ except when $n = 4$ and $k = 2$. So there exist two elements $a_{k+2}, a_{k+3} \in \Omega$ such that they are different from $a_i, i = 1, 2, \cdots, k+1$. Then $v_2' = \{a_{k+1}, a_{k+2}, a_3, \cdots, a_k\}$ and $v_2'' = \{a_{k+1}, a_{k+3}, a_3, \cdots, a_k\}$ are two neighbors of v_2 outside $N[v]$.

By Lemma 1, $\kappa(J(n,k)) = k(n-k)$. Since $\kappa(J(n,k)) < |V(J(n,k))| - 1$ except for the complete graph $J(n,1)$ and the fact that $\kappa(H) \leq \kappa'(H) \leq \delta(H)$ for any simple graph H, we have

$$\kappa(J(n,k)) = \kappa'(J(n,k)) = k(n-k) < |V(J(n,k))| - 1.$$

By Lemmas 1 and 2, $A \subseteq N(v)$ for some vertex v and B consists of one edge incident to each remaining neighbor of v in $J(n,k)$.

If $A = N(v)$ then $W = A$ and W is a local cut. So suppose that A is a proper subset of $N(v)$. As noted above, for each $w \in N(v) \setminus A$, w has (at least) two neighbors in $J(n,k) - N[v]$. By Lemma 2, at most one of the resulting edges belongs to B, so for some $z \notin N[v], wz \notin B$. By Lemma 3, $J(n,k) - N[v]$ is connected. Since $wz \notin B$, w is in the connected component of $J(n,k) - W$ containing $J(n,k) - N[v]$. Finally, if W is not a local cut, we can choose some $w_0 \in N(v) \setminus A$ such that $vw_0 \notin B$. Then for some $z_0 \notin N[v], v, w_0, z_0$ is a path of length two in $J(n,k) - W$. Thus $N[v] \setminus A$ is contained in the connected component of $J(n,k) - W$ containing $J(n,k) - N[v]$. Hence $J(n,k) - W$ is connected. Since this contradicts the fact that W is a generalized cutset, we conclude that W is indeed a generalized cutset.□

Now, we show that $J(4,2)$ is indeed a counterexample. Suppose that $\Omega=\{a_1, a_2, a_3, a_4\}$. Clearly, vertices $\{a_1, a_4\}$, $\{a_2, a_3\}$ and edges $\{\{a_2, a_4\}, \{a_3, a_4\}\}$, $\{\{a_1, a_2\}, \{a_1, a_3\}\}$ form a smallest generalized cutset of $J(4,2)$, but it is not a local cut.

3 Diameter-increasing sets

A set $U \subset V(G) \cup E(G)$ is a diameter-increasing set of G if the diameter of $G - U$ is greater than the diameter of G. In $J(n,k)$, destroying all the paths of length no more than k makes the resulting graph have larger diameter. We use this idea to get the smallest diameter-increasing set of $J(n,k)$. Denote the diameter of G by $diam(G)$. The distance between v_1 and v_2 is denoted by $d_G(v_1, v_2)$ in graph G. We first give an obvious Lemma.

Lemma 5. [8] *If v_1 and v_2 are two vertices in a graph G, and G has k pairwise internally disjoint v_1, v_2-paths of length at most s, then at least k vertices and edges must be deleted to make the distance between v_1 and v_2 larger than s.* □

Denote the set of pairwise internally disjoint paths of length t between vertices v_1 and v_2 by $P_t(v_1, v_2)$, and its size by $|P_t(v_1, v_2)|$. There exist $k(n-k)$ pairwise internally disjoint paths between any two vertices of $J(n,k)$ since $\kappa(J(n,k)) = k(n-k)$. The following lemma provides a classification of these paths.

Lemma 6. [6] *For $v_1, v_2 \in V(J(n,k))$ such that $|v_1 \cap v_2| = k - t$, where $1 \leq t \leq k$, $d_{J(n,k)}(v_1, v_2) = t$, and there exists a classification of $k(n-k)$*

pairwise internally disjoint v_1, v_2-paths such that

$$\begin{cases} |P_t(v_1,v_2)| = t^2, \\ |P_{t+1}(v_1,v_2)| = \begin{cases} nt - 2t^2, & \text{if} \quad t \neq k \\ nk - 2k^2, & \text{if} \quad t=k \end{cases} \\ |P_{t+2}(v_1,v_2)| = \begin{cases} nk - nt - k^2 + t^2, & \text{if} \quad t \neq k \\ 0, & \text{if} \quad t = k \end{cases} \end{cases} \square$$

From Lemma 6, we can obtain the following remark.

Remark. For $v_1, v_2 \in V(J(n,k))$ such that $|v_1 \cap v_2| = k - t$, where $1 \leq t \leq k$, v_1 has exactly t^2 neighbors such that each of them contains exactly $t-1$ elements different from the elements of v_2, while each of the remaining neighbors of v_1 contains at least t elements different from the elements of v_2.

Proof. Without loss of generality, we may assume that $v_1 = \{1, 2, \cdots, k\}$ and $v_2 = \{1, 2, \cdots, k-t, k+1, \cdots, k+t\}$. We want to count the number of neighbors z of v_1 with the additional property that $|z \setminus v_2| = t - 1$. For z to be a neighbor of v_1, $|z \cap v_1| = k - 1$. Thus $z = (v_1 - \{j\}) \cup \{i\}$ for some $1 \leq j \leq k$ and some $i \geq k + 1$.

Case 1. $j \in v_2$, i.e. $1 \leq j \leq k - t$. Without loss of generality, we may assume that $j = k-t$. Then $z = \{1, 2 \cdots, k-t-1\} \cup \{k-t+1, \cdots, k\} \cup \{i\}$. But $|\{k-t+1, \cdots, k\}| = t$, and none of the elements of $\{k-t+1, \cdots, k\}$ are in v_2. So $|z \setminus v_2| \geq t > t - 1$. Thus in Case 1 the number of neighbors z with $|z \setminus v_2| = t - 1$ is 0.

Case 2. $j \notin v_2$. Without loss of generality we may assume that $j = k$. $z = v_1 - \{k\} \cup \{i\} = \{1, 2, \cdots, k-1\} \cup \{i\}$. So $z \setminus v_2 \supset \{1, 2, \cdots, k-1\} \setminus v_2 = \{k - t + 1, \cdots, k - 1\}$, a subset of cardinality $t-1$. Thus $|z \setminus v_2| = t-1$ if and only if $i \in v_2$. So $i \in v_2 \setminus v_1$, and therefore there are $|v_2 \setminus v_1| = k - (k-t) = t$ choices for i. Now we assumed, for simplicity, that $j = k$, but the same argument holds for any $1 \leq j \leq k$ with $j \notin v_2$. Since $v_2 = \{1, 2, \cdots, k-t, k+1, \cdots, k+t\}$, there are $k-(k-t) = t$ such j. Thus z is determined by t choices for i and t choices for j and so the number of z is t^2.$\square$

For $|v_1 \cap v_2| = 0$, we denote the set of the k^2 neighbors of v_1 (v_2) that each contains exactly $k-1$ elements different from the elements of v_2 (v_1) by $N_{v_2}(v_1)$ ($N_{v_1}(v_2)$). Note that $N_{v_2}(v_1)$ ($N_{v_1}(v_2)$) consists of precisely those neighbors of v_1 (v_2) whose distance from v_2 (v_1) is $k-1$.

Lemma 7. *Let $v_1', v_1'' \in N_{v_2}(v_1)$. $N_{v_1}(v_2)$ contains at least $k^2 - k$ vertices such that each of them lies on a shortest v_1, v_2-path with at least one of v_1' and v_1''.*

Proof. By Lemma 6, since $d(v_1', v_2) = k - 1$, $N_{v_1}(v_2)$ contains exactly $(k-1)^2$ vertices at distance $k-2$ from v_1'. We claim that there are at least $k-1$ other vertices in $N_{v_1}(v_2)$ at distance $k-2$ from v_1''.

We shall construct vertices $v_j', j = 2, 3, \cdots, k$ such that $v_j' \in N_{v_1}(v_2)$, $d(v_j', v_1'') = k - 2$ and $d(v_j', v_1') \geq k - 1$. Since $d(v_j', v_1') \neq k - 2, v_j'$ is not one of the $(k-1)^2$ vertices already found. Thus we will have at least $(k-1)^2 + (k-1) = k^2 - k$ vertices with the desired properties.

Let $v_1 = \{a_1, a_2, \cdots, a_k\}$, $v_2 = \{b_1, b_2, \cdots, b_k\}$ and $v_1' = \{b_1, a_2, a_3, \cdots, a_k\}$. If $b_1 \in v_1''$, then without loss of generality we may assume that $v_1'' = \{a_1, b_1, a_3, \cdots, a_k\}$. For $j = 2, 3, \cdots, k$, let $v_j' = \{a_1, b_1, b_2, \cdots, b_k\} \setminus \{b_j\}$. It is easy to verify that $d(v_j', v_1) = k - 1, d(v_j', v_1'') = k - 2$, and $d(v_j', v_1') = k - 1$. If, on the other hand, $b_1 \notin v_1''$, without loss of generality we may assume that $v_1'' = \{a_1, b_q, a_3, \cdots, a_k\}$ for some $q \in \{2, 3, \cdots, k\}$. Then for $p \neq 2, v_p' = \{a_p, b_2, b_3, \cdots, b_k\}$ are $k-1$ vertices satisfying $d(v_p', v_1) = k-1$, $d(v_p', v_1'') = k - 2$, and $d(v_p', v_1') \geq k - 1$. □

Now, we prove an important Lemma which reveals the smallest set $U \subset V(J(n,k)) \cup E(J(n,k))$ whose deletion makes two disjoint vertices in $J(n,k)$ have larger distance in the resulting graph.

Lemma 8. *For any $v_1, v_2 \in V(J(n,k)), k \neq 2$, and $|v_1 \cap v_2| = 0$, let $U \subseteq V(J(n,k)) \cup E(J(n,k)) - \{v_1, v_2\}$ be a smallest set such that $d_{J(n,k)-U}(v_1, v_2) > k$. Then U is a subset of a local cut at v_1 or v_2.*

Proof. If $k = 1$, $J(n,1)$ is a complete graph and edge $e = v_1v_2$ is the unique desired set U, which is a subset of a local cut at v_1 or v_2.

Now we consider the case $k \geq 3$. Using Lemma 6 and the Remark, for $1 \leq t \leq k$ it is easy to prove by induction on t that if $d(w, v_2) = t$ then the number of shortest w, v_2-paths is $(t!)^2$. Hence, taking $t = k$, the number of shortest v_1, v_2-paths is $(k!)^2$. Thus for $1 \leq t \leq k - 1$ and any vertex v, the number of shortest v_1, v_2-paths such that $d(v, v_2) = t$ and $d(v_1, v) = k - t$ is $(t!)^2((k-t)!)^2 = [t!(k-t)!]^2$. This is $\leq [(k-1)!]^2$ since for $1 \leq t \leq k-1$, $\binom{k}{t} \geq k$, and equality holds if and only if $t = 1$ or $k - 1$. Similarly, for an edge $e = xy$ on any shortest v_1, v_2-path such that $d(x, v_2) = t + 1$ and $d(y, v_2) = t, 0 \leq t \leq k-1, e$ lies on $[t!(k-1-t)!]^2$ shortest v_1, v_2-paths, and again equality holds if and only if $t = 0$ or $k - 1$. Thus deleting one vertex or edge destroys at most $[(k-1)!]^2$ shortest v_1, v_2-paths. So deleting a set of k^2 vertices and edges destroys at most $[(k-1)!]^2 \cdot k^2 = (k!)^2$ shortest

v_1, v_2-paths, with exactly $(k!)^2$ if and only if no two vertices or edges in U lie on same shortest v_1, v_2-paths and U is a subset of the union of a local cut at v_1 and a local cut at v_2. Furthermore, $|U| = k^2$.

In the following, we prove that U is a subset of either the local cut at v_1 or the local cut at v_2. Suppose to the contrary that some objects of U belong to the local cut at v_1 and the others belong to the local cut at v_2.

(i) If only one object $w \in U$ belongs to the local cut at v_1 or v_2, say at v_1, then each of the remaining $k^2 - 1$ objects of U must lie on different shortest v_1, v_2-paths not containing w. By the Remark after Lemma 6, $N_{v_1}(v_2)$ has exactly k^2 vertices on shortest v_1, v_2-paths. Thus there are exactly $k^2 - (k-1)^2 = 2k - 1$ vertices in $N_{v_1}(v_2)$ such that each lies on a different v_1, v_2-path not containing w. Let $U' = U \backslash \{w\}$. Then $|U'| = k^2 - 1$, so for $k \geq 3$, $|U'| > 2k - 1$. Thus there must be two vertices of U' that lie on the same shortest v_1, v_2-path not containing w. This contradicts the fact that no path contains two different objects of U.

(ii) At least $\lceil \frac{k^2}{2} \rceil$ objects in U belong to the local cut at v_1 or v_2, say at v_2. We consider the case that at least two objects in U belong to the local cut at v_1. Let $w_1, w_2 \in U$ belong to the local cut at v_1. Then each of the $\lceil \frac{k^2}{2} \rceil$ objects must lie on different shortest v_1, v_2-paths with w_1 and w_2 respectively. By lemma 7, there are at most $k^2 - (k^2 - k) = k$ vertices in $N_{v_1}(v_2)$ such that each of them lies on different shortest v_1, v_2-path with w_1 and w_2. Since for $k \geq 3$, $\lceil \frac{k^2}{2} \rceil \geq \frac{k^2}{2} > k$, this contradicts the fact that no two objects in U lie on the same shortest v_1, v_2-path.

Hence, for $k \geq 3$, U is a subset of a local cut at v_1 or v_2. □

Now, we show that $J(n, 2)$ is indeed a counterexample. Since $N_{v_2}(v_1) = N_{v_1}(v_2)$ and $|N_{v_2}(v_1)| = 4$, it is easy to see that two edges from v_1 to any two vertices in $N_{v_2}(v_1)$ and two edges from v_2 to the remaining vertices in $N_{v_2}(v_1)$ form a diameter-increasing set U, which is not a subset of a local cut at v_1 or v_2.

The following theorem is the main result of this section. Let $J[X]$ be the subgraph induced by the vertex set X.

Theorem 9. *For $(n, k) \neq (n, 2), (6, 3)$, let U be a smallest diameter-increasing set in $J(n, k)$. Then U is a subset of a local cut, $|U| = k^2$ for $n > 2k$ and $|U| = k^2 - 1$ for $n = 2k$. In addition, diam$(J(n, k) - U) = k + 1$.*

Proof. Let U be a smallest diameter-increasing set of $J(n, k)$. For some $v_1, v_2 \in V(J(n, k))$, $d_{J(n,k)-U}(v_1, v_2) > k$. If $d_{J(n,k)}(v_1, v_2) \leq k-2$, then by Lemmas 5 and 6, $|U| \geq k(n-k)$. Letting $t = k-1$, if $d_{J(n,k)}(v_1, v_2) = k-1$, then $|U| \geq t^2 + (nt - 2t^2) = t(n-t)$. If $d_{J(n,k)}(v_1, v_2) = k$, then $|U| \geq k^2$.

It is easy to verify that $k(n-k)$ is the largest among $k(n-k), t(n-t)$

and k^2. Also, except for $(n,k)=(5,2)$, if $n>2k$ then $t(n-t)>k^2$. On the other hand, for $n=2k, t(n-t)<k^2$.

Thus for $n>2k$, $|U|\geq k^2$. For $n=2k$, $|U|\geq t(n-t)=(k-1)(2k-k+1)=k^2-1$. Now we distinguish the following two cases:

Case 1. $n>2k$. By the definition of $N_{v_2}(v_1)$, $d_{J(n,k)-N_{v_2}(v_1)}(v_1,v_2)>d(v_1,v_2)=k$, so $N_{v_2}(v_1)$ is a diameter-increasing subset of $V(J(n,k))$. Hence $|U|\leq|N_{v_2}(v_1)|\leq k^2$. Since we already know that $|U|\geq k^2$, we have $|U|=|N_{v_2}(v_1)|=k^2$. By Lemma 8, for $k\neq 2, U$ is a subset of a local cut at v_1 or at v_2.

Case 2. $n=2k$. Let $v,w\in V(J(n,k))$ such that $d_{J(n,k)}(v,w)=k-1$. Then $|v\cap w|=1$. Suppose that $\Omega=\{a_i,b_i: i=1,2,\cdots,k\}$. We may assume that $v=\{a_1,a_2,a_3,\cdots,a_k\}$ and $w=\{a_1,b_2,b_3,\cdots,b_k\}$. $N(w)=\{u_{ij},u_i,u_1,u'_i: i,j=2,3,\cdots,k\}$, where $u_{ij}=\{a_1,a_i,b_2,b_3,\cdots,b_k\}\backslash\{b_j\}$, $u_1=\{b_1,b_2,b_3,\cdots,b_k\}, u_i=\{a_i,b_2,\cdots,b_k\}, u'_i=\{a_1,b_1,b_2,\cdots,b_k\}\backslash\{b_i\}$. Let U' consist of exactly one of $\{u_{ij},u_{ij}w\}$ for each u_{ij}, exactly one of $\{u_i,u_iw\}$ for each u_i, and exactly one of $\{u'_i,u'_iw\}$ for each u'_i. Then in $J(n,k)-U'$, the only neighbor of w is u_1. Since $|v\cap u_1|=0$, $d_{J(n,k)-U'}(v,u_1)=k$ and therefore $d_{J(n,k)-U'}(v,w)=k+1$. Thus U' is a diameter-increasing set. Since U is a smallest such set, $|U|\leq|U'|=k^2-1$. We already have $|U|\geq k^2-1$, so $|U|=k^2-1$.

It remains to show that U is a subset of some local cut. Since $n=2k$, deleting U must make some two vertices with distance $k-1$ have larger distance. Without loss of generality, we let these two vertices be v and w. Partition the vertex set of $J(2k,k)$ into two parts X and Y, where

$$X=\{\text{the vertices in } J(2k,k) \text{ containing } a_1\},$$
$$Y=\{\text{the vertices in } J(2k,k) \text{ not containing } a_1\}.$$

Then

$$J[X]\cong J(2k-1,k-1),$$
$$J[Y]\cong J(2k-1,k).$$

Clearly, $v,w\in V(J[X])$, $diam(J[X])=k-1$. By Lemma 6, we know that there are k^2 pairwise internally disjoint paths between v and w in $J(2k,k)$, where $(k-1)^2$ of which have length $k-1$, $2(k-1)$ of which have length k, and the last one of which has length $k+1$. Since $d_{J(2k,k)}(v,w)=k-1$, all the $(k-1)^2$ pairwise internally disjoint paths of length $k-1$ contain a_1, and then these paths also lie in $J[X]$. Deleting U must break all the paths of length $k-1$ between v and w. Thus in particular deleting U must break all the paths of length $k-1$ between v and w in $J[X]$. Now in $J[X]$ by Lemma 8 and the minimality of U, we get that U contains a subset of a local cut at v or w, which implies that U contains exactly one of $\{u_{ij}, u_{ij}w\}$ for each u_{ij}, or exactly one of $\{u'_{ij}, u'_{ij}v\}$ for each u'_{ij} if $k-1 \geq 3$, that is $k \geq 4$, where $u'_{ij} = \{a_1, b_i, a_2, a_3, \cdots, a_k\} \backslash \{a_j\}, i, j = 2, 3, \cdots, k$. Without loss of generality let U contain exactly one of $\{u_{ij}, u_{ij}w\}$ for each u_{ij}. For simplicity we set $U_1 = \{\{u_{ij}, u_{ij}w\} \cap U : i, j = 2, 3, \cdots, k\}$. So $|U_1| = (k-1)^2$.

In the following, we prove that U also contains U_2, where U_2 consists of exactly one of $\{u_i, u_i w\}$ for each u_i, and exactly one of $\{u'_i, u'_i w\}$ for each u'_i, for $2 \leq i \leq k$. Thus $|U_2| = 2(k-1)$. Suppose to the contrary that there exists some vertex $p \in \{u_i, u'_i : i = 2, 3, \cdots, k\}$ such that p, $pw \notin U$. Since $d_{J(n,k)-U}(w, v) > k$, deleting U must break all the paths of length $k-1$ between p and v. Since $d(p, v) = k-1$ (whether $p = u_i$ or u'_i), by lemma 6 there are $(k-1)^2$ pairwise internally disjoint paths of length $k-1$ between p and v. It is easy to verify that u_i has exactly $k-1$ neighbors contained in $\{u_{ij} : j = 2, 3, \cdots, k\}$, while u'_i has exactly one. Hence p has at most $k-1$ neighbors in $\{u_{ij} : j = 2, 3, \cdots, k\}$. Thus at most $k-1$ neighbors of p belong to U_1. Hence U must contain U_1 and an additional $(k-1)^2-(k-1)$ vertices. Therefore $|U| \geq |U_1|+(k-1)^2-(k-1)$.

Subcase 2.1. $k \geq 5$. Now we get that $(k-1)^2 - (k-1) > 2(k-1)$. Then $|U| > |U_1| + 2(k-1) = k^2 - 1$, which contradicts the fact that $|U| = k^2 - 1$. Hence U consists of U_1 and U_2, which is a subset of a local cut at w.

Subcase 2.2. $k = 4$. $(k-1)^2 - (k-1) = 2(k-1)$, $n = 8$, $v = \{a_1, a_2, a_3, a_4\}$ and $w = \{a_1, b_2, b_3, b_4\}$. If $|(U_2 \setminus \{p, pw\}) \cap U| > 0$, $|U| = |U_1| + (k-1)^2 - (k-1) + 1 > k^2 - 1$. Hence we suppose that $|U_2 \cap U| = 0$ and also we assume that p is any vertex of u_i. There exists a path $\{a_1, b_2, b_3, b_4\}, \{a_1, b_1, b_3, b_4\}, \{a_1, b_1, a_3, b_4\}, \{a_1, b_1, a_3, a_4\}, \{a_1, a_2, a_3, a_4\}$ from w to v such that the internal vertices on this path contain b_1. Therefore these vertices don't lie on any path of length 3 between v and p. Then $|U| \geq |U_1| + ((k-1)^2 - (k-1)) + 1 > k^2 - 1$, which contradicts the fact that $|U| = k^2 - 1$. Hence U consists of U_1 and U_2, which is a subset of a local cut at w. □

For $J(5,2)$, let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$. Then edges $\{\{a_1, a_2\}, \{a_2, a_3\}\}$, $\{\{a_1, a_2\}, \{a_1, a_3\}\}$, $\{\{a_2, a_3\}, \{a_2, a_4\}\}$ and $\{\{a_2, a_3\}, \{a_2, a_5\}\}$ form a smallest diameter-increasing set U such that $diam(J(5,2)-U) = d_{J(5,2)-U}(\{a_1, a_2\}, \{a_2, a_5\}) = 3$. However U is not a subset of a local cut.

For $J(4,2)$, let $\Omega = \{a_1, a_2, a_3, a_4\}$. Then edges $\{\{a_1, a_2\}, \{a_1, a_3\}\}$, $\{\{a_1, a_2\}, \{a_1, a_4\}\}$ and $\{\{a_1, a_3\}, \{a_2, a_3\}\}$ form a smallest diameter-increasing set U such that $diam(J(4,2)-U) = d_{J(4,2)-U}(\{a_1, a_2\}, \{a_1, a_3\}) = 3$. However U is not a subset of a local cut.

For $J(6,3)$, let $\Omega = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Then vertices $\{a_1, a_3, a_4\}$, $\{a_1, a_3, a_5\}$, $\{a_2, a_3, a_4\}$, $\{a_2, a_3, a_5\}$, $\{a_2, a_3, a_6\}$, $\{a_1, a_3, a_6\}$ and edges $\{\{a_3, a_4, a_5\}, \{a_1, a_4, a_5\}\}$, $\{\{a_3, a_4, a_5\}, \{a_2, a_4, a_5\}\}$ form a smallest diameter-increasing set U such that $diam(J(6,3) - U) = d_{J(6,3)-U}(\{a_1, a_2, a_3\}, \{a_3, a_4, a_5\}) = 4$. However U is not a subset of a local cut.

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