The Induced Matching Extendability of $C_{2n}(1, k)$

Xu Huafeng [a,b*], Bo Xianhui [c]

[a] *College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangshu 210016, P. R. China*

[b] *Henan University of Urban Construction, Pingdingshan, Henan 467001, P. R. China*

[c] *School of Accountancy, Central University of Finance and Economics, Beijing 100081, P. R. China*

Abstract: A simple graph G is induced matching extendable, shortly IM-extendable, if every induced matching of G is included in a perfect matching of G. The cyclic graph $C_{2n}(1, k)$ is the graph with $2n$ vertices $x_0, x_1, \cdots, x_{2n-1}$, such that $x_i x_j$ is an edge of $C_{2n}(1, k)$ if either $i - j \equiv \pm 1 \ (mod\ 2n)$ or $i - j \equiv \pm k \ (mod\ 2n)$. We show in this paper that the only IM-extendable graphs in $C_{2n}(1, k)$ are $C_{2n}(1, 3)$ for $n \geq 4$; $C_{2n}(1, n-1)$ for $n \geq 3$; $C_{2n}(1, n)$ for $n \geq 2$; $C_{2n}(1, \frac{n}{2})$ for $n \geq 4$; $C_{2n}(1, \frac{2n-1}{3})$ for $n \geq 5$; $C_{2n}(1, \frac{2n+1}{3})$ for $n \geq 4$; $C_{2n}(1, 4)$ for $3 \leq n \leq 8$ or $n = 10, 11$; $C_{2n}(1, \frac{2n+2}{3})$ for $n \leq 14$; $C_{2n}(1, \frac{2n-2}{3})$ for $n \leq 16$; $C_{2n}(1, 2)$ for $n \leq 4$; $C_{20}(1, 8)$; $C_{30}(1, 6)$; $C_{40}(1, 8)$; $C_{30}(1, 12)$ and $C_{32}(1, 10)$.

Keywords: induced matching, perfect matching, IM-extendable, cyclic graph.

1 Introduction

Graphs considered in this paper are finite and simple. For a graph G, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. For any vertex $v \in V(G)$, the neighbor set $N(v)$ of v is defined by

$$N(v) = \{u \in V(G) \setminus \{v\} : \text{ there is an edge } uv \in E(G)\}.$$

For $M \subseteq E(G)$, set

$$V(M) = \{u \in V(G) : \text{ there is a vertex } v \in V(G) \text{ such that } uv \in M\}.$$

*Corresponding author, email: xuhuafeng@hncj.edu.cn

A set of edges $M \subseteq E(G)$ is called a matching of G if no two of them share a common endpoint. A matching is perfect if it covers all vertices of G. A matching M is induced [2] if $E(V(M)) = M$. A graph G is induced matching extendable [11], shortly IM-extendable, if every induced matching M of G is included in a perfect matching of G. Researches on IM-extendable graphs can be found, for example, in [3-13].

We will denote by $C_{2n}(1,k)$ the cyclic graph with $2n$ vertices $x_0, x_1, \cdots, x_{2n-1}$ such that $x_i x_j$ is an edge of $C_{2n}(1,k)$ if either $i-j \equiv \pm 1 \ (mod\ 2n)$ or $i-j \equiv \pm k \ (mod\ 2n)$. For two graphs G and H, $G \times H$ is used to denote the product [1] of G and H.

Up to now, there are only few families of IM-extendable graphs have been characterized. By [11], the only 3-regular connected IM-extendable graphs are $C_n \times K_2$ for $n \geq 3$, and $C_{2n}(1,n)$ for $n \geq 2$. By [8], a connected K_4 minor free graph G is IM-extendable if and only if G is isomorphic to $T \times K_2$, where T is a tree, and the only connected IM-extendable outer planar graphs are ladders (i.e., $P_n \times K_2$). By [6], the only 4-regular claw-free connected IM-extendable graphs are C_6^2, C_8^2 and T_r, where T_r is the graph with $4r$ vertices u_i, v_i, x_i, y_i, $1 \leq i \leq r$, such that for each i with $1 \leq i \leq r$, $\{u_i, v_i, x_i, y_i\}$ is a clique of T_r and $x_i u_{i+1}, y_i v_{i+1} \in E(T_r) \ (mod\ r)$.

In this paper we investigate the IM-extendability of cyclic graphs $C_{2n}(1,k)$. The main result of this paper is that: The only IM-extendable graphs in $C_{2n}(1,k)$ are $C_{2n}(1,3)$ for $n \geq 4$; $C_{2n}(1,n-1)$ for $n \geq 3$; $C_{2n}(1,n)$ for $n \geq 2$; $C_{2n}(1,\frac{n}{2})$ for $n \geq 4$; $C_{2n}(1,\frac{2n-1}{3})$ for $n \geq 5$; $C_{2n}(1,\frac{2n+1}{3})$ for $n \geq 4$; $C_{2n}(1,4)$ for $3 \leq n \leq 8$ or $n = 10, 11$; $C_{2n}(1,\frac{2n+2}{3})$ for $n \leq 14$; $C_{2n}(1,\frac{2n-2}{3})$ for $n \leq 16$; $C_{2n}(1,2)$ for $n \leq 4$; $C_{20}(1,8)$; $C_{30}(1,6)$; $C_{40}(1,8)$; $C_{30}(1,12)$ and $C_{32}(1,10)$.

2 Main results and proofs

The following two lemmas obtained in [11] will be used.

Lemma 1 [11]: $C_{2n}(1,n)$ $(n \geq 2)$ is IM-extendable.

Lemma 2 [11]: For every graph G, $G \times K_2$ is IM-extendable.

Furthermore, the following trivial lemma can be observed.

Lemma 3: $C_{2n}(1,2)$ is IM-extendable if and only if $n \leq 4$.

In the sequel, we always suppose that the $2n$ vertices of $C_{2n}(1,k)$ are $x_0, x_1, \cdots, x_{2n-1}$ and that $x_i x_j$ is an edge of $C_{2n}(1,k)$ if either $i-j \equiv \pm 1 \ (mod\ 2n)$ or $i-j \equiv \pm k \ (mod\ 2n)$. For the simplicity, G is used to denote the graph $C_{2n}(1,k)$, and the vertex set and edge set of $C_{2n}(1,k)$ are denoted by V and E, respectively. Furthermore, when we use the notation $\frac{k}{r}$, we always assume that $\frac{k}{r}$ is an integer, i.e., k be divisible by r.

Lemma 4: $C_{2n}(1,3)$ is IM-extendable for $n \geq 4$.

Proof Let M be an arbitrary induced matching M of $C_{2n}(1,3)$. Without loss of generality, we suppose that one of the two edges x_0x_1 and x_0x_3 is in M. We divide the edges of M into four sets:

$$E_1 = \{x_ix_j \in M : i \text{ is odd and } j-i=1\},$$
$$E_2 = \{x_ix_j \in M : i \text{ is even and } j-i=1\},$$
$$E_3 = \{x_ix_j \in M : i \text{ is odd and } j-i=3\},$$
$$E_4 = \{x_ix_j \in M : i \text{ is even and } j-i=3\}.$$

For $e \in E$, we define a subset $S(x_ix_j)$ of V by setting

$$S(x_{2i-1}x_{2i}) = \{x_{2i-1}, x_{2i}\},$$
$$S(x_{2i}x_{2i+1}) = \{x_{2i-1}, x_{2i}, x_{2i+1}, x_{2i+2}\},$$
$$S(x_{2i-1}x_{2i+2}) = \{x_{2i-1}, x_{2i}, x_{2i+1}, x_{2i+2}\},$$
$$S(x_{2i-2}x_{2i+1}) = \{x_{2i-3}, x_{2i-2}, x_{2i-1}, x_{2i}, x_{2i+1}, x_{2i+2}\}.$$

It can be observed that, for any edge $x_ix_j \in E$, we have $x_i, x_j \in S(x_ix_j)$. Furthermore, we can observe that $S(e)$ have the following properties.

Property 1: (1) For any $e \in E_1$, $G[S(e)] \cong K_2$.

(2) For any $e \in E_2 \bigcup E_3$, $G[S(e)] \cong C_4$.

(3) For an arbitrary edge $e \in E_4$, $G[S(e)] \cong K_{3,3}^-$, where $K_{3,3}^-$ is a graph obtained by $K_{3,3}$ deleted an arbitrary edge.

Property 2: For any $e \in E$, $G[S(e)]$ is IM-extendable.

Property 3: For distinct edges $e, f \in M$ with $e \neq f$, we have $S(e) \bigcap S(f) = \phi$.

Let $S(M) = \bigcup_{e \in M} S(e)$. From the definition of $S(e)$, we can see that $x_{2i-1} \in S(M)$ if and only if $x_{2i} \in S(M)$.

Now, for each edge $e \in M$, let $F(e)$ be a perfect matching of $G[S(e)]$ which includes the edge e. By Property 2, $F(e)$ must exist. Set

$$F(M) = \bigcup_{e \in M} F(e).$$

From Property 3, we have $V(F(M)) = S(M)$. Let

$$X = \{x_{2i-1} : x_{2i-1} \notin S(M)\},$$
$$N = \{x_{2i-1}x_{2i} : x_{2i-1} \in X\}.$$

Then we can see that $F(M)\bigcup N$ is a perfect matching of $C_{2n}(1,3)$ and $M\subseteq F(M)$. Thus $C_{2n}(1,3)$ is IM-extendable. □

Lemma 5: $C_{2n}(1,\frac{2n+1}{3})$ is IM-extendable for $n\geq 4$.
Proof Since $C_{2n}(1,\frac{2n+1}{3})$ is isomorphic to $C_{2n}(1,3)$, The result follows from Lemma 4. □

Lemma 6: $C_{2n}(1,\frac{2n-1}{3})$ is IM-extendable for $n\geq 5$.
Proof Since $C_{2n}(1,\frac{2n-1}{3})$ is isomorphic to $C_{2n}(1,3)$, The result follows from Lemma 4. □

Lemma 7: $C_{2n}(1,n-1)$ is IM-extendable for $n\geq 3$.
Proof Since, for each i with $0\leq i\leq n-1$, $N(x_i)=N(x_{i+n})$, we know that $C_{2n}(1,n-1)$ is isomorphic to the graph obtained by exchanging the indices of its two vertices x_i and x_{i+n}.

Let M be a given induced matching of G. If there is some vertex $x_k\in V(M)$ such that $n\leq k\leq 2n-1$, then $x_{k-n}\notin V(M)$. By exchanging the indices of the two vertices x_k and x_{k-n}, we pull x_{k-n} into $V(M)$ and push x_k out of $V(M)$. This exchanging does not change the structure of the graph. Hence, we can repeatedly take this exchanging until $V(M)\subseteq\{x_0,x_1,...,x_{n-1}\}$.

Without of generality, we can suppose that $x_0x_1\in M$. Set

$$N_1=\{x_{k+n-1}x_{k+n}: x_kx_{k+1}\in M\},$$
$$N_2=\{x_kx_{k+n-1}: 0\leq k\leq n-2, x_k\notin V(M)\}.$$

Then $M\bigcup N_1\bigcup N_2\bigcup\{x_{2n-2}x_{2n-1}\}$ is a perfect matching of $C_{2n}(1,n-1)$ including M. Thus $C_{2n}(1,n-1)$ is IM-extendable. □

Lemma 8: $C_{2n}(1,\frac{n}{2})$ is IM-extendable for $n\geq 4$.
Proof Let M be an arbitrary induced matching M of $C_{2n}(1,\frac{n}{2})$. For each edge $e\in E$, we define a subset $S(e)$ of V by setting

$$S(x_ix_{i+\frac{n}{2}})=\{x_i,x_{i+\frac{n}{2}},x_{i+n},x_{i+\frac{3}{2}n}\},$$
$$S(x_ix_{i+1})=\{x_i,x_{i+1},x_{i+\frac{n}{2}},x_{i+\frac{n}{2}+1},x_{i+n},x_{i+n+1},x_{i+\frac{3}{2}n},x_{i+\frac{3}{2}n+1}\}.$$

For each pair of edges x_ix_{i+1} and $x_{i+n}x_{i+n+1}$, we define

$$S(x_ix_{i+1},x_{i+n}x_{i+n+1})$$
$$=\{x_i,x_{i+1},x_{i+\frac{n}{2}},x_{i+\frac{n}{2}+1},x_{i+n},x_{i+n+1},x_{i+\frac{3}{2}n},x_{i+\frac{3}{2}n+1}\},$$

and for each pair of edges x_ix_{i+1} and $x_{i+n+1}x_{i+n+2}$, we define

$$S(x_ix_{i+1}, x_{i+n+1}x_{i+n+2})$$
$$=\{x_i, x_{i+1}, x_{i+\frac{n}{2}}, x_{i+\frac{n}{2}+1}, x_{i+n}, x_{i+n+1}, x_{i+\frac{3}{2}n},$$
$$x_{i+\frac{3}{2}n+1}, x_{i+2}, x_{i+\frac{n}{2}+2}, x_{i+n+2}, x_{i+\frac{3}{2}n+2}\}.$$

It can be observed that, for any edge $x_ix_j \in E$, we have $x_i, x_j \in S(x_ix_j)$, and for every pair of edges $e_1, e_2 \in E$ with $S(e_1, e_2)$ having definition, we have $V(\{e, f\}) \in S(e, f)$. Furthermore, we can observe that $S(e)$ and $S(e_1, e_2)$ have the following properties.

Property 1: (1) For each $e \in E$, $G[S(e)] \cong C_4$ or $C_4 \times K_2$ is IM-extendable.

(2) For each pair of edges x_ix_{i+1} and $x_{i+n}x_{i+n+1}$ in E, $G[S(x_ix_{i+1}, x_{i+n}x_{i+n+1})] \cong C_4 \times K_2$ is IM-extendable.

(3) For each pair of edges x_ix_{i+1} and $x_{i+n+1}x_{i+n+2}$ in E, $G[S(x_ix_{i+1}, x_{i+n+1}x_{i+n+2})] \cong C_4 \times P_3 = P_3 \times K_2 \times K_2$ is IM-extendable.

Set

$$\mathcal{M}_1 = \{(x_ix_{i+1}, x_{i+n}x_{i+n+1}) : x_ix_{i+1}, x_{i+n}x_{i+n+1} \in M\},$$
$$\mathcal{M}_2 = \{(x_ix_{i+1}, x_{i+n+1}x_{i+n+2}) : x_ix_{i+1}, x_{i+n+1}x_{i+n+2} \in M\},$$
$$\mathcal{M}_3 = \{e \in M : e \text{ does not appear in } \mathcal{M}_1 \cup \mathcal{M}_2\}.$$

Then the following property can be observed.

Property 2: (1) $S(e) \bigcap S(f) = \emptyset$ for any two distinct edges $e, f \in \mathcal{M}_3$.

(2) $S(e) \bigcap S(e_1, e_2) = \emptyset$ for $e \in \mathcal{M}_3$ and $(e_1, e_2) \in \mathcal{M}_1 \cup \mathcal{M}_2$.

(3) $S(e_1, e_2) \bigcap S(f_1, f_2) = \emptyset$ for any two distinct pairs $(e_1, e_2), (f_1, f_2) \in \mathcal{M}_1 \cup \mathcal{M}_2$.

Let

$$S(M) = \cup_{e \in \mathcal{M}_3} S(e) \bigcup \cup_{(e_1,e_2) \in \mathcal{M}_1 \cup \mathcal{M}_2} S(e_1, e_2).$$

From the definition of $S(e)$ and $S(e_1, e_2)$, we have

Property 3: $x_i \in S(M)$ if and only if $x_{i+\frac{n}{2}}, x_{i+n}, x_{i+\frac{3}{2}n} \in S(M)$.

Since $S(e)$ with $e \in \mathcal{M}_3$ and $S(e_1, e_2)$ with $(e_1, e_2) \in \mathcal{M}_1 \cup \mathcal{M}_2$ are IM-extendable, there is a perfect matching F in $G[S(M)]$ such that $M \subseteq F$. Set

$$X = \{x_i : x_i \notin S(M)\},$$
$$N = \{x_ix_{i+\frac{n}{2}}, x_{i+n}x_{i+\frac{3}{2}n} : x_i \in X\}.$$

Then we can see that $F \bigcup N$ is a perfect matching of $C_{2n}(1, \frac{n}{2})$. Thus $C_{2n}(1, \frac{n}{2})$ is IM-extendable. □

Lemma 9: $C_{2n}(1,4)$ is IM-extendable if and only if $3 \le n \le 8$ or $n = 10, 11$.

Proof For $n \le 11$, the result can be verified by enumerating. For $n \ge 12$, we consider the induced matching M of $C_{2n}(1,4)$ of the form

$$M = \{x_1x_5, x_7x_8, x_{10}x_{14}, x_{16}x_{17}, x_{19}x_{23}\}.$$

Then $C_{2n}(1,4) - V(M)$ has a component G_1 with vertex set

$$V(G_1) = \{x_9, x_{11}, x_{12}, x_{13}, x_{15}\}.$$

That is G_1 is an odd component of $C_{2n}(1,4) - V(M)$. Thus M cannot be included in any perfect matching. We conclude that $C_{2n}(1,4)$ is not IM-extendable for $n \ge 12$. □

Lemma 10: $C_{2n}(1, \frac{2}{3}n)$ is IM-extendable if and only if $n \le 6$.

Proof For $n \le 6$, the result can be verified by enumerating. For $n \ge 9$, we consider the induced matching M of $C_{2n}(1, \frac{2}{3}n)$ of the form

$$M = \{x_{\frac{2}{3}n+1}x_{\frac{2}{3}n+2}, x_3x_{\frac{4}{3}n+3}, x_{\frac{2}{3}n+4}x_{\frac{2}{3}n+5}\}.$$

Then $C_{2n}(1, \frac{2}{3}n) - V(M)$ has an isolated vertex $x_{\frac{2}{3}n+3}$. Thus M cannot be included in any perfect matching. We conclude that $C_{2n}(1, \frac{2}{3}n)$ is not IM-extendable for $n \ge 9$. □

Lemma 11: $C_{2n}(1, \frac{2}{5}n)$ is IM-extendable if and only if $n = 10, 15, 20$.

Proof For $n \le 20$, the result can be verified by enumerating. For $n \ge 25$, we consider the induced matching M of $C_{2n}(1, \frac{2}{5}n)$ of the form

$$M = \{x_{\frac{2}{5}n+4}x_{\frac{2}{5}n+5}, x_{\frac{2}{5}n+7}x_{\frac{2}{5}n+8}, x_{\frac{4}{5}n+2}x_{\frac{4}{5}n+3}, x_{\frac{4}{5}n+9}x_{\frac{4}{5}n+10}, x_{\frac{6}{5}n+4}x_{\frac{6}{5}n+5}, x_{\frac{6}{5}n+7}x_{\frac{6}{5}n+8}, x_6x_{\frac{8}{5}n+6}\}.$$

Then $C_{2n}(1, \frac{2}{5}n) - V(M)$ has a component G_1 with vertex set

$$V(G_1) = \{x_{\frac{2}{5}n+6}, x_{\frac{4}{5}n+4}, x_{\frac{4}{5}n+5}, x_{\frac{4}{5}n+6}, x_{\frac{4}{5}n+7}, x_{\frac{4}{5}n+8}, x_{\frac{6}{5}n+6}\}.$$

That is G_1 is an odd component of $C_{2n}(1, \frac{2}{5}n) - V(M)$. Thus M cannot be included in any perfect matching. Consequently, $C_{2n}(1, \frac{2}{5}n)$ is not IM-extendable for $n \ge 25$. □

Lemma 12: $C_{2n}(1, \frac{4}{5}n)$ is IM-extendable if and only if $n \le 15$.

Proof For $n \le 15$, the result can be verified by enumerating. For $n \ge 20$, we consider the induced matching M of $C_{2n}(1, \frac{4}{5}n)$ of the form

$$M = \{x_{\frac{6}{5}n+3}x_{\frac{6}{5}n+4}, x_{\frac{6}{5}n+6}x_{\frac{6}{5}n+7}, x_{\frac{2}{5}n+1}x_{\frac{2}{5}n+2}, x_{\frac{2}{5}n+8}x_{\frac{2}{5}n+9}, x_{\frac{8}{5}n+3}x_{\frac{8}{5}n+4}, x_{\frac{8}{5}n+6}x_{\frac{8}{5}n+7}, x_5x_{\frac{4}{5}n+5}\}.$$

Then $C_{2n}(1, \frac{4}{5}n) - V(M)$ has a component G_1 with vertex set

$$V(G_1) = \{x_{\frac{6}{5}n+5}, x_{\frac{2}{5}n+3}, x_{\frac{2}{5}n+4}, x_{\frac{2}{5}n+5}, x_{\frac{2}{5}n+6}, x_{\frac{2}{5}n+7}, x_{\frac{8}{5}n+5}\}.$$

That is such that G_1 is an odd component of $C_{2n}(1, \frac{4}{5}n) - V(M)$. Thus M can not be included in any perfect matching. Consequently, $C_{2n}(1, \frac{4}{5}n)$ is not IM-extendable for $n \geq 20$. □

Lemma 13: $C_{2n}(1, \frac{2n+2}{3})$ is IM-extendable if and only if $n \leq 14$.

Proof For $n \leq 14$, the result can be verified by enumerating. For $n \geq 17$, we consider the induced matching M of $C_{2n}(1, \frac{2n+2}{3})$ of the form

$$M = \{x_i x_{i+1}, x_{i+5}x_{i+6}, x_{i+\frac{2n+2}{3}-2}x_{i+\frac{2n+2}{3}-1}, x_{i+\frac{2n+2}{3}+7}x_{i+\frac{2n+2}{3}+8},$$
$$x_{i+\frac{4n+4}{3}}x_{i+\frac{4n+4}{3}+1}, x_{i+\frac{4n+4}{3}+5}x_{i+\frac{4n+4}{3}+6}\}.$$

Then $C_{2n}(1, \frac{2n+2}{3}) - V(M)$ has a component G_1 with vertex set

$$\{x_{i+2}, x_{i+3}, x_{i+4}, x_{i+\frac{2n+2}{3}}, x_{i+\frac{2n+2}{3}+1}, x_{i+\frac{2n+2}{3}+2}, x_{i+\frac{2n+2}{3}+3},$$
$$x_{i+\frac{2n+2}{3}+4}, x_{i+\frac{2n+2}{3}+5}, x_{i+\frac{2n+2}{3}+6}, x_{i+\frac{4n+4}{3}+2}, x_{i+\frac{4n+4}{3}+3}x_{i+\frac{4n+4}{3}+4}\}.$$

That is G_1 is an odd component of $C_{2n}(1, \frac{2n+2}{3}) - V(M)$. Thus M cannot be included in any perfect matching. We conclude that $C_{2n}(1, \frac{2n+2}{3})$ is not IM-extendable for $n \geq 17$. □

Lemma 14: $C_{2n}(1, \frac{2n-2}{3})$ is IM-extendable if and only if $n \leq 16$.

Proof For $n \leq 16$, the result can be verified by enumerating. For $n \geq 19$, we consider the induced matching M of $C_{2n}(1, \frac{2n-2}{3})$ of the form

$$M = \{x_i x_{i+1}, x_{i+5}x_{i+6}, x_{i+\frac{2n-2}{3}-2}x_{i+\frac{2n-2}{3}-1}, x_{i+\frac{2n-2}{3}+7}x_{i+\frac{2n-2}{3}+8},$$
$$x_{i+\frac{4n-4}{3}}x_{i+\frac{4n-4}{3}+1}, x_{i+\frac{4n-4}{3}+5}x_{i+\frac{4n-4}{3}+6}\}.$$

Then $C_{2n}(1, \frac{2n-2}{3}) - V(M)$ has a component G_1 with vertex set

$$\{x_{i+2}, x_{i+3}, x_{i+4}, x_{i+\frac{2n-2}{3}}, x_{i+\frac{2n-2}{3}+1}, x_{i+\frac{2n-2}{3}+2}, x_{i+\frac{2n-2}{3}+3}, x_{i+\frac{2n-2}{3}+4},$$
$$x_{i+\frac{2n-2}{3}+5}, x_{i+\frac{2n-2}{3}+6}, x_{i+\frac{4n-4}{3}+2}, x_{i+\frac{4n-4}{3}+3}x_{i+\frac{4n-4}{3}+4}\}.$$

It can be seen that such that $|V(G_1)|$ is odd. Thus M can not be included in any perfect matching. We conclude that $C_{2n}(1, \frac{2n-2}{3})$ is not IM-extendable for $n \geq 19$. □

Lemma 15: $C_{2n}(1, k)$ is not IM-extendable, where $1 \leq k \leq n$ and

$$k \neq 2, 3, 4, n-1, n, \frac{2}{3}n, \frac{2}{5}n, \frac{4}{5}n, \frac{n}{2}, \frac{2n-2}{3}, \frac{2n+2}{3}, \frac{2n-1}{3}, \frac{2n+1}{3}.$$

Proof We can easily see the following properties:

Property 1: the vertex x_{i+1} is adjacent with x_{i-1} if and only if $k = 2$.

Property 2: the vertex x_{i+2} is adjacent with x_{i-2} if and only if $k = 4$.

Property 3: the vertex x_{i+1} is adjacent with x_{i-2} if and only if $k = 3$.

Property 4: the vertex x_{i+2} is adjacent with x_{i-1} if and only if $k = 3$.

Property 5: the vertex x_{i+k} is adjacent with x_{i-k} if and only if $k = \frac{2n}{3}$.

Property 6: the vertex x_{i+2k} is adjacent with x_{i-2k} if and only if $k = \frac{2n}{3}, \frac{2}{5}n, \frac{4}{5}n$.

Property 7: the vertex x_{i+k} is adjacent with x_{i-2k} if and only if $k = n$, or $k = \frac{n}{2}$.

Property 8: the vertex x_{i+2k} is adjacent with x_{i-k} if and only if $k = n$, or $k = \frac{n}{2}$.

Property 9: the vertex x_{i-2k} is adjacent with x_{i+2} if and only if $k = \frac{2n-2}{3}$.

Property 10: the vertex x_{i+2k} is adjacent with x_{i-2} if and only if $k = \frac{2n-2}{3}$.

Property 11: the vertex x_{i-2k} is adjacent with x_{i-2} if and only if $k = \frac{2n+2}{3}$.

Property 12: the vertex x_{i+2k} is adjacent with x_{i+2} if and only if $k = \frac{2n+2}{3}$.

Property 13: the vertex x_{i-k} is adjacent with x_{i+2} if and only if $k = n - 1$.

Property 14: the vertex x_{i+k} is adjacent with x_{i-2} if and only if $k = n - 1$.

Property 15: the vertex x_{i-2k} is adjacent with x_{i+1} if and only if $k = \frac{2n-1}{3}$.

Property 16: the vertex x_{i+2k} is adjacent with x_{i-1} if and only if $k = \frac{2n-1}{3}$.

Property 17: the vertex x_{i-2k} is adjacent with x_{i-1} if and only if $k = \frac{2n+1}{3}$.

Property 18: the vertex x_{i+2k} is adjacent with x_{i+1} if and only if $k = \frac{2n+1}{3}$.

Property 19: x_{i-k} and x_{i+1}, x_{i-k} and x_{i-1}, x_{i+k} and x_{i+1}, x_{i+k} and x_{i-1}, x_{i-k} and x_{i-2}, x_{i+k} and x_{i+2} can not be adjacent in any cases.

Suppose that

$$k \neq 2, 3, 4, n-1, n, \frac{2}{3}n, \frac{2}{5}n, \frac{4}{5}n, \frac{n}{2}, \frac{2n-2}{3}, \frac{2n+2}{3}, \frac{2n-1}{3}, \frac{2n+1}{3}.$$

We consider the induced matching M of $C_{2n}(1,k)$ of the form

$$M = \{x_{i+1}x_{i+2}, x_{i-1}x_{i-2}, x_{i-k}x_{i-2k}, x_{i+k}x_{i+2k}\}.$$

From the discussions above, it can be observed that $C_{2n}(1,k) - V(M)$ has an isolated vertex x_i. Hence, $C_{2n}(1,k)$ is not IM-extendable. The result follows. □

Summing the above lemmas up, we deduce the following main result of this paper.

Theorem The only IM-extendable graphs in $C_{2n}(1,k)$ are $C_{2n}(1,3)$ for $n \geq 4$; $C_{2n}(1,n-1)$ for $n \geq 3$; $C_{2n}(1,n)$ for $n \geq 2$; $C_{2n}(1,\frac{n}{2})$ for $n \geq 4$; $C_{2n}(1,\frac{2n-1}{3})$ for $n \geq 5$; $C_{2n}(1,\frac{2n+1}{3})$ for $n \geq 4$; $C_{2n}(1,4)$ for $3 \leq n \leq 8$ or $n = 10, 11$; $C_{2n}(1,\frac{2n+2}{3})$ for $n \leq 14$; $C_{2n}(1,\frac{2n-2}{3})$ for $n \leq 16$; $C_{2n}(1,2)$ for $n \leq 4$; $C_{20}(1,8)$; $C_{30}(1,6)$; $C_{40}(1,8)$; $C_{30}(1,12)$ and $C_{32}(1,10)$.

Acknowledgment: This project was supported by National Natural Science Foundation of China (No.70971064) and "211 Project (the 3rd phase)" from Central University of Finance and Economics.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, London, Macmillan Press Ltd, 1976.

[2] K. Cameron, Induced Matchings. *Discrete Applied Mathematics*, 1989, **24** 97-102.

[3] Y. Liu, J.J. Yuan and S.Y. Wang, Degree conditions of IM-extendable graphs, *Applied Mathematics-JCU*, **15B:1** (2000), 1-6.

[4] Q. Wang and J.J. Yuan, Degree sum conditions of induced matching extendable graphs, *Journal of Zhengzhou University*, **32:1** (2000), 19-21.

[5] Q. Wang and J.J. Yuan, Maximal induced matching extendable graphs, *Discrete Mathematics*, **240** (2001), 295-298.

[6] Q. Wang and J.J. Yuan, 4-regular claw-free IM-extendable graphs, *Discrete Mathematics*, **294** (2005), 303 309

[7] L.S. Wu, Q. Wang and J.J. Yuan, Local Operation of Induced Matching Extendable Graphs, *Journal of Mathematical Study*, **35:2**(2002), 16-20.

[8] F. Yang and J.J. Yuan, Induced matching extendable graph without K_4 minor, *in: Advances in Operations Research and Systems Engineering* (Edited by J.F. Gu, G.H. Fan, S.Y. Wang and B. Wei), 1998, 142-145.

[9] F. Yang and J.J. Yuan, NP-completeness of induced matching problem and co-NP-completeness of induced matching extendable problem, *OR Transactions*, **4:1** (2000), 76-80.

[10] F. Yang and J.J. Yuan, IM-extendable Claw-free Graphs, *Journal of Mathematical study*, **32:1** (1999), 34-37.

[11] J.J. Yuan, Induced Matching Extendable Graph, *Journal of Graph Theory*, **28**(1998), 203-213.

[12] J.J. Yuan and Y. Liu, Degree conditions of induced matching extendable bipartite graphs, *Henan Science*, **16:1**(1999).

[13] J.J. Yuan and Q. Wang, Induced matching extendability of G^3, *Graph Theory Notes of New York*, **XLIII**(2002), 16-19.