On the Erdős-Sós Conjecture and graphs with large minimum degree

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Abstract

Suppose G is a simple graph with average vertex degree greater than $k-2$. Erdős and Sós conjectured that G contains every tree on k vertices. Sidorenko proved G contains every tree that has a vertex v with at least $\lceil \frac{k}{2} \rceil - 1$ leaf neighbors. We prove this is true if v has only $\lceil \frac{k}{2} \rceil - 2$ leaf neighbors. We generalize Sidorenko's result by proving that if G has minimum degree d, then G contains every tree that has a vertex with least $(k-1)-d$ leaf neighbors. We use these results to prove that if G has average degree greater than $k-2$ and minimum degree at least $k-4$, then G contains every tree on k vertices.

1 Terminology and definitions

We will use standard graph theory notation and will consider only simple graphs (finite and undirected with no loops and no multi-edges). The graph G has vertex set $V(G)$, edge set $E(G)$, order $|V(G)|$ and size $e(G)$. The degree of $v \in V(G)$ is the number of edges incident to v and is denoted $deg_G(v)$. The set of neighbors of v is denoted $N_G(v)$ (or $N(v)$), that is, $N(v) = \{w \in V(G) | vw \in E(G)\}$. The closed neighborhood of v is denoted $N_G[v]$ (or $N[v]$), that is, $N[v] = N(v) \cup \{v\}$. Note that $deg_G(v) = |N(v)|$.

If $uv \in E(G)$, we say u hits v and v hits u; if $uv \notin E(G)$ we say u misses v and v misses u. If a vertex in $A \subset V(G)$ hits a vertex in $B \subset V(G)$ we say that A hits B; otherwise we say A misses B. Specifically, if u hits any vertex in A, we say that u hits A. If vertex u hits every vertex in A, we say that u hits all of A. If u misses every vertex in A, we say that u misses all of A.

A leaf $w \in V(T)$ in a tree T is a vertex of degree one. If v has w (a leaf) as its neighbor, then w is referred to as a leaf neighbor of v. The set of leaf neighbors of v is denoted $L_G(v)$ (or simply $L(v)$), that is, $L(v) = \{w \in N(v) | w \text{ is a leaf}\}$.

The minimum degree among all vertices in $V(G)$ is denoted $\delta(G)$. The maximum degree among all vertices in $V(G)$ is denoted $\Delta(G)$.

For any graph H, we denote the average degree of H as $avedeg(H)$; that is, $avedeg(H) = 2e(H)/|V(H)|$.

P_j represents a path on j vertices; C_j represents a cycle on j vertices.

2 Background and Statement of Main Theorems

In 1959, Erdős and Gallai [1] proved the following (which we state as a theorem) for a fixed, positive integer k and for a graph G:

Theorem 1 *If $avedeg(G) > k-2$, then G contains a path on k vertices.*

In 1962, Erdős and Sós stated the following conjecture:

Conjecture 1 *If $avedeg(G) > k-2$, then G contains every tree on k vertices as a subgraph.*

Various specific cases of Conjecture 1 have already been proven. Each places limitations on the graph G or on the tree T. In 1989, Sidorenko [3] proved the following.

Theorem 2 *If $avedeg(G) > k-2$, then G contains every tree on k vertices that has a vertex with at least $\lceil \frac{k}{2} \rceil - 1$ leaf neighbors.*

In 2003, McLennan [2] proved the following theorem which proved very useful in the proofs of our main results.

Theorem 3 *If $avedeg(G) > k-2$, then every tree of order k whose diameter does not exceed 4 is contained in G as a subgraph.*

In particular, McLennan's theorem above and Theorem 1 imply that Conjecture 1 holds for $k \leq 6$. In this paper, we will prove the following theorems and corollaries. The first theorem is an improvement to Theorem 2.

Theorem 4 *If $avedeg(G) > k-2$, then G contains every tree on k vertices that has a vertex with at least $\lceil \frac{k}{2} \rceil - 2$ leaf neighbors.*

The following theorem generalizes Theorem 2.

Theorem 5 *If $avedeg(G) > k-2$ and $\delta(G) \geq d$ ($d \in \mathbb{N}$), then G contains every tree on k vertices that has a vertex with at least $(k-1)-d$ leaf neighbors.*

We use Theorem 5 to prove Theorem 6 which has, as corollaries, two special cases of Conjecture 1.

Theorem 6 *If $k \geq 8$, $\delta(G) \geq k-4$ and G has a vertex of degree at least $k-1$, then G contains every tree on k vertices that has diameter at least 5.*

Corollary 1 *If $avedeg(G) > k-2$ and $\delta(G) \geq k-4$, then G contains every tree on k vertices.*

Corollary 2 *If $avedeg(G) > k-2$, $k \leq 8$, then G contains every tree on k vertices.*

In sections 3, 4 and 5, we provide necessary lemmas and background information as well as proofs for Theorems 4, 5 and 6, respectively.

3 Preliminaries and proof of Theorem 4

Before proving Theorem 4, we define some terms and prove some useful lemmas.

3.1 Embedding trees into graphs

Let T be a tree on k vertices and let $g : V(T) \to V(G)$ be an isomorphism from $V(T)$ to a k-subset of $V(G)$. If g preserves edges, that is, if $g(u)g(v) \in E(G)$ for every $uv \in E(T)$, then we call g an embedding of T into G. If such an embedding exists, then G contains a copy of T as a subgraph. Or, we say G contains T or simply $T \subseteq G$.

Let $T' \subsetneq T$ be a proper subtree of T and let g' be an embedding of T' into G. If there exists an embedding $g : V(T) \to V(G)$ such that $g(v) = g'(v)$ for all $v \in V(T')$, we say that g' can be extended to an embedding of T or simply, that g' is T-extensible.

Definition 1 *Suppose G is a graph, T is a tree and $p, q \in V(T)$. If $f : V(T) \to V(G)$ is an embedding, we construct a new function (not necessarily an embedding) $f_{p \leftrightarrow q} : V(T) \to V(G)$ from f by switching the values of $f(p)$ and $f(q)$. That is, $f_{p \leftrightarrow q}(p) = f(q)$, $f_{p \leftrightarrow q}(q) = f(p)$ and $f_{p \leftrightarrow q}(t) = f(t)$ for all $t \in V(T) - \{p, q\}$.*

Suppose G is a graph, H is a tree and $h : V(H) \to V(G)$ is an embedding of H. Let $u, v \in V(H)$. If $h(u)$ hits all of $h(N(v) - \{u\})$ and $h(v)$ hits all of $h(N(u) - \{v\})$, then $h_{u \leftrightarrow v} : V(H) \to V(G)$ is an embedding of H.

If G is a graph with $\delta(G) = d$ and T is any tree on $d+1$ vertices, then one can easily find an embedding of T in G. Moreover, McLennan [2] provides a proof of the following lemma which will be used in the proofs of our lemmas and theorems.

Lemma 1 *If $\delta(G) \geq d$ and T is a tree on at most $d+1$ vertices, then any embedding of a subtree of T in G can be extended to an embedding of T.*

3.2 Average degree of subgraphs of G

Suppose G is a graph with $avedeg(G) > d$ for some $d \in \mathbb{N}$. It is well-known that removing a vertex v with $deg_G(v) \leq \frac{d}{2}$ results in a subgraph of G whose average degree still exceeds d. If we iteratively remove every such vertex, the result is a subgraph $G' \subseteq G$ with $avedeg(G') > d$ and $\delta(G') > \frac{d}{2}$. After defining new terms, we extend this notion in Lemma 2.

We denote the sum of the degrees (in G) of the vertices in $W \subset V(G)$ as $sumdeg_G(W)$; that is, $sumdeg_G(W) = \sum_{v \in W} deg_G(v)$. We denote the number of edges that are incident to at least one vertex in W as $e^*_G(W)$. Notice that $e^*_G(W) = sumdeg_G(W) - e(G[W])$ where $G[W]$ is the subgraph of G induced by W.

Lemma 2 *Let G be a graph with $avedeg(G) > d$ for some $d \in \mathbb{N}$, let $W \subsetneq V(G)$ and let $G' = G - W$. If $2e^*_G(W) \leq d|W|$, then $avedeg(G') > d$.*

Proof. Let $n = |V(G)|$ and observe that:

$$e^*_G(W) \leq \frac{d|W|}{2} \quad \Rightarrow \quad e^*_G(W) \leq \left\lfloor \frac{dn}{2} \right\rfloor - \left\lfloor \frac{d(n-|W|)}{2} \right\rfloor$$

$$\Rightarrow \quad 1 + \left\lfloor \frac{d(n-|W|)}{2} \right\rfloor \leq 1 + \left\lfloor \frac{dn}{2} \right\rfloor - e^*_G(W) \leq e(G')$$

Therefore, $avedeg(G') > d$. ■

For a fixed integer $j \leq |V(G)|$, suppose every j-subset of $V(G)$ that satisfies the hypothesis of Lemma 2 is iteratively removed from G. Then the resulting subgraph $G' \subseteq G$ has $avedeg(G') > d$ and for all j-subsets $W \subseteq V(G)$, $\frac{e^*_{G'}(W)}{|W|} > \frac{d}{2}$. We characterize this below.

Definition 2 *Suppose G is a graph with $avedeg(G) > d$, $d \in \mathbb{N}$. For a fixed $c \in \mathbb{N}$, we say G has **Property A(d,c)** if $2e^*_G(W) > d|W|$ for every subset $W \subseteq V(G)$ with $|W| \leq c$.*

The following proposition is a consequence of Lemma 2 and will be used in the proof of Theorem 4.

Proposition 1 *If $d \in \mathbb{N}$ and graph G has property $A(d,3)$, then:*

(i) $deg_G(v) \geq \lfloor \frac{d+2}{2} \rfloor$ for all $v \in V(G)$

(ii) if d is odd and $uw \in E(G)$, then one of $\{u, w\}$ has degree at least $\frac{d+3}{2}$.

(iii) if d is even and three vertices $x, y, z \in V(G)$ induce a K_3 in G, then one of $\{x, y, z\}$ has degree at least $\frac{d+4}{2}$.

Suppose $G' \subseteq G$ a subgraph of G having Property $A(d, 3)$. Since an embedding into G' is also an embedding into G, to prove our results there is no loss of generality in assuming that G has Property $A(d, 3)$.

3.3 Proof of Theorem 4

Without loss of generality, assume G has Property $A(k-2,3)$. So, by Proposition 1, $\delta(G) \geq m = \lfloor \frac{k}{2} \rfloor$. If T has a vertex with $\lceil \frac{k}{2} \rceil - 1$ leaf neighbors then $T \subseteq G$ by Theorem 2, so assume $a \in V(T)$ has $|L(a)| = \lceil \frac{k}{2} \rceil - 2$ leaf neighbors. We will prove that $T \subseteq G$.

If the diameter of T is at most 4, then $T \subseteq G$ by Theorem 3, so assume this is not the case and that $P = a_0, a_1, \ldots, a_r$, $r \geq 5$, is a longest path in T. Since a must be distance at least 3 from either a_0 or a_r, assume $dist(a, a_0) \geq 3$; so, in particular, assume $a \notin \{a_0, a_1, a_2\}$. Let $n_1 = |L(a_1)|$.

Let $H = T - (L(a) \cup L(a_1))$; so, $|V(H)| = m + 2 - n_1$. Map $a \in V(H)$ to $z \in V(G)$ where $deg_G(z) \geq k-1$ and extend this to an embedding f of H into G (possible by Lemma 1). Let $X = V(G) - f(V(H))$.

Since $deg_G(f(a)) \geq k-1$, f is $T-extensible$ if $f(a_1)$ hits n_1 vertices in X; so we may assume this is not the case. Thus, $f(a_1)$ hits all of $V(f(H - a_1))$ as well as $n_1 - 1$ vertices in X, in particular, $deg_G(f(a_1)) = m$.

If $f(a_2)$ hits n_1 vertices in X, then $f_{a_1 \leftrightarrow a_2}$ is T-extensible. Since this must be the case if k is odd (by Proposition 1), assume k is even and $f(a_2)$ does not hit n_1 vertices in X. Thus, $f(a_2)$ hits all of $V(f(H - a_2))$ as well as $n_1 - 1$ vertices in X, in particular, $deg_G(f(a_2)) = m$.

Let $a' \in N_H(a)$. Since $\{f(a_1), f(a_2), f(a')\}$ induce a K_3 in G, it must be that $deg_G(f(a')) \geq m+1$ (Proposition 1) and that $f(a')$ hits at least n_1 vertices in X. Since $f(a_1)$ hits all of $f(N(a'))$, $f_{a_1 \leftrightarrow a'}$ is T-extensible. ∎

4 Lemmas and Proof of Theorem 5.

We begin this section with two lemmas, the first of which proves that Conjecture 1 holds when $k = 7$.

Lemma 3 *Conjecture 1 holds true for* $k = 7$.

Proof of Lemma 3. Assume G is Property $A(5,3)$ and let T be a tree on 7 vertices. If T is a path, if T has diameter at most 4, or if if T has a vertex with at least 2 leaf neighbors, then $T \subseteq G$ by Theorem 1, 3, or 4, respectively. So, assume T is a non-path tree on 7 vertices that has diameter 5 and no vertex with 2 leaf neighbors. It must be that T is isomorphic to $v_0, v_1, \ldots, v_5 + v_2v_6$.

Let $P = w_0, w_1, \ldots, w_r$, $r \geq 6$, be a maximal path in G (so P is not a subgraph of a longer path in G). Such a path exists by Theorem 1.

If w_0 hits w_3, then $w_1, w_2, \ldots, w_6 + w_3w_0$ is a copy of T in G; so assume w_0 misses w_3. If w_0 hits w_4, then $w_1, w_2, \ldots, w_6 + w_4w_0$ is a copy of T in G; so assume w_0 misses w_4. If w_0 hits w_5, then $w_3, w_4, w_5, w_0, w_1, w_2, +w_5w_6$ is a copy of T in G; so assume w_0 misses w_5.

Since $\delta(G) \geq 3$, w_0 hits $\{w_6, w_7, \ldots, w_r\}$. Let $j \geq 6$ be the smallest index such that w_0 hits w_j. If w_0 hits w_2, then $w_5, w_4, w_3, w_2, w_0, w_j, +w_2w_1$ is a copy of T in G; so assume w_0 misses w_2. Since $6 \leq j \leq r-1$, $w_{j-2}, w_{j-1}, w_j, w_0, w_1, w_2 + w_jw_{j+1}$ is a copy of T in G. ■

McLennan [2] showed that the hypothesis of Theorem 2 can be weakened. Specifically, the condition "$avedeg(G) > k-2$" can be replaced with "$\delta(G) \geq \lfloor \frac{k}{2} \rfloor$ and $\Delta(G) \geq k-1$". We generalize this in a lemma which will be used in the proof of Theorem 5.

Lemma 4 *Suppose a graph G is such that $\Delta(G) \geq k-1$ and $\delta(G) \geq d$ for some $0 \leq d \leq k-2$. Then G contains every tree T on k vertices that has a vertex t with at least $(k-1)-d$ leaf neighbors. Moreover, there is an embedding of T into G that maps t to $v \in V(G)$ where $deg_G(v) = \Delta(G)$.*

Proof. Let G be a graph with $\delta(G) \geq d$ and let $v \in V(G)$ be such that $deg_G(v) \geq k-1$. Let T be a tree on k vertices and let $a \in V(T)$ be such that $|L(a)| \geq (k-1)-d$. We will prove that $T \subseteq G$.

Let $T' = T - L(a)$; so T' is a tree on $d+1$ vertices. Map $a \in V(T')$ to $v \in V(G)$ and extend this to an embedding f of T' into G (possible by Lemma 1).

Since $deg_G(f(a)) \geq k-1$, clearly f is T-extensible. ■

Proof of Theorem 5. Assume G has Property $A(k-2, 1)$; so $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$. Since $avedeg(G) > k-2$, there must exist a vertex $v \in V(G)$ such that $deg_G(v) \geq k-1$. The rest of the proof follows from Lemma 4. ■

5 Lemmas and Proof of Theorem 6.

A *cut-vertex* in a graph G is a vertex whose removal disconnects the graph, and a *block* of G is a maximal connected subgraph of G that has no cut-vertex.

Lemma 5 *Suppose $k \geq 8$ and G is a graph with $\delta(G) \geq k-4$. If $B \subseteq G$ is a block with $\delta(B) \geq k-4$ and such that B contains exactly one cut-vertex $b \in V(G)$, then G contains every tree on k vertices that has diameter at least 5.*

Proof. Notice that $G' = G - V(B)$ is such that $\delta(G') \geq k-5$. Let $b' \in V(G')$ be such that $bb' \in E(G)$.

Let T be a tree on k vertices that has diameter at least 5 and let $P = a_0, a_1, \ldots, a_r$, $r \geq 5$, be a longest path in T. Without loss of generality, assume $sumdeg_T(\{a_1, a_2\}) \leq sumdeg_T(\{a_{r-2}, a_{r-1}\})$. So, the number of vertices in $V(T) - V(P)$ that hit $\{a_1, a_2\}$ is at most $\left\lfloor \frac{k-(r+1)}{2} \right\rfloor \leq \lfloor \frac{k}{2} \rfloor - 3 \leq k-7$.

Consider the two subtrees of T that remain after removing edge a_2a_3 from T. Let T_1 be the subtree that contains a_2 and let T_2 be the subtree that contains a_3. Notice that $|V(T_2)| \leq k-3$ and $|V(T_1)| \leq k-4$.

Map $a_3 \in V(T_2)$ to $b \in V(B)$ and extend this to an embedding of T_2 into B (possible by Lemma 1). Map $a_2 \in V(T_2)$ to $b' \in V(G')$ and extend this to an embedding T_2 into G' (possible by Lemma 1). Since $bb' \in E(G)$, we have an embedding of T into G. ■

Lemma 6 *Suppose T is a tree and vertices $v, v_1, v_2, w, w_1 \in V(T)$ (v may equal w) are such that v is the neighbor of leaves v_1 and v_2, w is the neighbor of leaf w_1 and f is an embedding of T into a graph G. If there is a matching from $f(\{v_1, v_2\})$ into $f(\{w, w_1\})$, then G also contains a copy of the tree $T' = T - vv_1 + v_1v_2$.*

Proof. Suppose there is a matching from $f(\{v_1, v_2\})$ into $f(\{w, w_1\})$. If $\{f(v_1)f(w), f(v_2)f(w_1)\} \subset E(G)$, then $f_{v_1 \leftrightarrow w_1}$ is an embedding of T' into G. If $\{f(v_1)f(w_1), f(v_2)f(w)\} \subset E(G)$, then set $g = f$, set $g(v_1) = f(w_1)$, $g(v_2) = f(v_1)$, $g(w_1) = f(v_2)$ and g is an embedding of T' into G. ■

Proof of Theorem 6.

Let T be a tree on k vertices with diameter at least 5. Let $P = a_0, a_1, \ldots, a_r$, $r \geq 5$, be a longest path in T. If any vertex in T has 3 leaf neighbors, then $T \subseteq G$ by Lemma 4, so assume this is not the case.

Case 1 *A vertex $a \in T$ has 2 leaf neighbors.*

The distance from a to one of $\{a_0, a_r\}$ is at least 3; assume $dist(a, a_0) \geq 3$. Consider the subtree of T defined as $H = T - L_T(a)$. Let $v \in V(G)$ be such that $deg_G(v) \geq k - 1$. To prove $T \subseteq G$, it suffices to show there is an embedding of H into G where a is mapped to v.

Let $H' = H - a_0$. Map $a \in V(H')$ to $v \in V(G)$ and extend this to an embedding g of H' into G (possible by Lemma 1). Let $X = V(G) - g(V(H'))$.

So, $V(H') = k-3$ and every vertex $z \in g(V(H'))$ is such that either $N_G[z] = g(V(H'))$ or z hits X. If $g(V(H' - a))$ misses X, then since $g(a)$ hits X, $g(a)$ is a cut-vertex of G, $g[V(H')]$ is K_{k-3} in G, and G contains every tree on k vertices by Lemma 5; so assume one of $g(V(H' - a))$ hits X.

If $g(a_1)$ hits X, then clearly g is H-extensible (and therefore, T-extensible); so assume this is not the case; thus $N[g(a_1)] = V(H')$.

Case 1.1 $|L_H(a_1)| = 1$

If $g(a_2)$ hits X, then $g_{a_1 \leftrightarrow a_2}$ is H-extensible; so assume this is not the case; thus $N[g(a_2)] = V(H')$. Let $a' \in V(H') - \{a, a_1, a_2\}$ be such that $g(a')$ hits X. Then $g_{a_1 \leftrightarrow a'}$ is H-extensible.

Case 1.2 $|L_H(a_1)| = 2$

Let $u \in V(H' - a_1)$ be the leaf neighbor of a_1.

Suppose $g(a_2)$ hits $x \in X$. If x hits two vertices in $X \cup \{g(u)\}$, then set $f = g$ and $f(a_1) = x$ and f is H-extensible. Suppose x hits at most one vertex $x_1 \in X \cup \{g(u)\}$. Then, let $a' = \{a_3, a_4\} \backslash \{a\}$ and set $f = g$, $f(a') = x$, $f(a_0) = g(a')$ and f is an embedding of H into G. So, assume $g(a_2)$ misses X. Thus, $N[g(a_2)] = V(H')$.

If $g(u)$ hits X, then $g_{a_1 \leftrightarrow u}$ is H-extensible, so assume $g(u)$ misses X; thus, $N[g(u)] = V(H')$. Let $a'' \in g(V(H' - \{a, a_1, a_2, u\}))$ be such that a'' hits a $z \in X$. Then set $f = g_{a_1 \leftrightarrow a''}$ and $f(a_0) = z$ and f is an embedding of H into G.

Case 2 *No vertex in T has 2 leaf neighbors.*

Since $T' = T - a_0a_1 + a_0a_2$ has two leaf neighbors, there is an embedding g of T' into G. Let $X = V(G) - g(V(T'))$.

If $g(a_0)$ hits $g(a_1)$ or if $g(\{a_0, a_1\})$ hits X, then clearly $T \subseteq G$; so assume this is not the case. Thus, each of $g(\{a_0, a_1\})$ misses at most two vertices in $g(V(T' - \{a_0, a_1\}))$.

A caterpillar is a tree in which a single path (the spine) is incident to (or contains) every edge.

Case 2.1 *T is a caterpillar.*

First, suppose T is a P_k. Since $k \geq 8$ and $deg_G(g(a_0)) \geq k - 4$, it must be that $g(a_0)$ hits two adjacent vertices $g(a_i)$ and $g(a_{i+1})$ for some $2 \leq i \leq 6$. Thus, $g(a_1), g(a_2), \ldots, g(a_i), g(a_0), g(a_{i+1}), g(a_{i+2}), \ldots, g(a_{k-1})$ is a P_k in G. So, assume that T is not a P_k.

Case 2.1.1 *T has at least 2 leaves besides a_0 and a_r*

Let $2 \leq i < j \leq r - 2$ be such that a_i and a_j have leaf neighbors l_i and l_j, respectively.

If there is a matching from $g(\{a_0, a_1\})$ to one of the three pairs $g(\{a_i, l_i\})$, $g(\{a_j, l_j\})$ or $g(\{a_{r-1}, a_r\})$, then $T \subseteq G$ by Lemma 6. Notice both $g(a_0)$ and $g(a_1)$ hit at least 4 of the 6 vertices in the three pairs; so $g(a_0)$ hits both vertices in one of the pairs. Assume $g(a_0)$ hits both of $g(\{a_i, l_i\})$ (the proofs for the other two pairs are similar). If $g(a_1)$ hits $g(\{a_i, l_i\})$ we have a matching so assume this is not the case. So $g(a_1)$ hits all 4 vertices in $g(\{a_j, l_j\}) \cup g(\{a_{r-1}, a_r\})$ and $g(a_0)$ hits at least two of them. Therefore, $T \subseteq G$.

Case 2.1.2 *T has exactly 1 leaf l besides a_0 and a_r*

Let $2 \leq i \leq r - 2$ be such that a_i hits l. If $g(a_0)$ hits $g(a_3)$, then set $f = g$, $f(a_0) = g(a_1)$, $f(a_1) = g(a_2)$, $f(a_2) = g(a_0)$ and f is an embedding of T into G. If $g(a_1)$ hits $g(a_3)$, we reach a similar result; so assume $g(\{a_0, a_1\})$

misses $g(a_3)$. Thus, each of $g(\{a_0, a_1\})$ misses at most one vertex in $g(V(T' - \{a_0, a_1, a_3\}))$.

If $4 \leq i \leq r-4$ then there is a matching from $g(\{a_0, a_1\})$ into either $g(\{a_i, l\})$ or $g(\{a_r, a_{r-1}\})$; so $T \subseteq G$ by Lemma 6; so assume $i \in \{2, 3, r-2, r-3\}$.

Case 2.1.2.1 $i \in \{2, r-2\}$

We may assume $i = r - 2$. Since each of $g(\{a_1, a_2\})$ hits 3 vertices in $g(\{a_i, l\}) \cup g(\{a_{r-1}, a_r\})$, there must be a matching from $g(\{a_1, a_2\})$ into either $g(\{a_i, l\})$ or $g(\{a_{r-1}, a_r\})$. By Lemma 6, $T \subseteq G$.

Case 2.1.2.2 $i \in \{3, r-3\}$

We may assume $i = 3$. If there is a matching from $g(\{a_0, a_1\})$ into $g(\{a_{r-1}, a_r\})$ then $T \subseteq G$ by Lemma 6; so assume this isn't the case. So, each of $g(\{a_0, a_1\})$ misses exactly one vertex in $g(\{a_{r-1}, a_r\})$ and each of $g(\{a_0, a_1\})$ hits $g(l)$.

If $g(a_0)$ hits $g(a_{r-1})$, then $C = (g(a_0), g(a_2), g(a_1), g(l), g(a_3), g(a_4), g(a_5), \ldots, g(a_{r-1}))$ is a C_{k-1} in G and $C + a_{r-1}a_r$ contains T; so assume $g(a_1)$ hits $g(a_r)$. Then $C' = (g(a_0), g(a_2), g(a_1), g(l), g(a_3), g(a_4), g(a_5), \ldots, g(a_r))$ is a C_k in G. If $z \in V(C')$ hits $x \notin V(C')$ then $C' + xz$ contains T, so assume $V(C') = V(G)$. Thus, one of the vertices in $V(C')$ has degree $k-1$; assume $deg_G(g(a_r)) = k - 1$. Then $C'' = (g(a_2), g(a_1), g(l), g(a_3), g(a_4), g(a_5), \ldots, g(a_r))$ is a C_{k-1} and $C'' + g(a_0)g(a_r)$ contains T.

Case 2.2 *T is not a caterpillar.*

Let $p \in V(T)\backslash V(P)$ be be a *penultimate vertex*, that is, a vertex with at least one leaf neighbor (here, exactly one) and exactly one neighbor that is not a leaf. Let l be the single leaf neighbor of p. Since p must be distance at least 3 (in T) from one of $\{a_1, a_{r-1}\}$, assume $dist(p, a_1) \geq 3$. [If $dist(p, a_1) < 3$ and $dist(p, a_{r-1}) \geq 3$, the proof is identical after we reverse the order of the labels of the vertices on P.]

If $g(a_0)$ or $g(a_1)$ hits all of $g(N(a_2)\backslash\{a_0, a_1\})$, then $g_{a_0 \leftrightarrow a_2}$ or $g_{a_1 \leftrightarrow a_2}$, respectively, is an embedding of T into G; so, assume this is not the case. Thus, $g(a_0)$ hits either both of $g(\{a_{r-1}, a_r\})$ or both of $g(\{p, l\})$ and $g(a_1)$ hits at least one from both of those sets. So, there is a matching from $g(\{a_0, a_1\})$ into either $g(\{a_{r-1}, a_r\})$ or $g(\{p, l\})$. By Lemma 6, $T \subseteq G$.

Proof of Corollary 1.

Since $avedeg(G) > k - 2$, there is a $u \in V(G)$ such that $deg_G(u) \geq k - 1$. The rest of the proof follows from Theorem 6. ■

Proof of Corollary 2.

We already showed that Conjecture 1 (and Corollary 2) hold for $k \leq 7$; so assume $k = 8$. Without loss of generality, assume G has Property $A(6, 1)$ so $\delta(G) \geq 4 = k - 4$. Since $avedeg(G) > 6$, there is a $u \in V(G)$ such that $deg_G(u) \geq 7$. Therefore, by Theorem 6, G contains every tree on 8 vertices. ■

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