On factorisations of cyclic permutations into transpositions

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Abstract

We study the factorisations of a cyclic permutation of length n as a product of a minimal number of transpositions, calculating the number $f(n, m)$ of factorisations in which a fixed element is moved m times. In this way, we also give a new proof–in the spirit of Clarke's proof of Cayley's theorem on the number of labelled trees–of the fact that there are n^{n-2} such factorisations.

It is well known that the number of ways of writing an n-cycle as a product of a minimal number, $n-1$, of transpositions is n^{n-2}, equal to the number of labelled trees on n vertices. For instance, the 3^{3-2} minimal factorisations of the 3-cycle $(1\ 2\ 3)$ are $(1\ 2)(1\ 3)$, $(2\ 3)(1\ 2)$, and $(1\ 3)(2\ 3)$. This was first proved by Dénes ([2]), while Moszkowski ([7]) was the first to give an explicit bijection between such products and labelled trees.

Amongst the later bijections we mention the one found by Goulden and Pepper ([4]), which works by showing that both trees and factorisations satisfy a particular recurrence relation, and the "structural" one by Goulden and Yong ([5]), which translates nicely some of the properties of the tree into the factorisation: for instance, to a tree with k leaves corresponds a factorisation with k transpositions of the form $(i\ i+1)$.

In this paper we compute the number of minimal factorisation moving a fixed element a given number of times. In this way, we shall also give a new proof of the total number of minimal factorisations, in the spirit of Clarke's proof for Cayley's theorem on the number of labelled trees. That proof enumerated trees according to the degree of a fixed vertex.

Given a permutation $\sigma \in S_n$, a *factorisation* of σ is a sequence of transpositions $(\alpha_1, \dots, \alpha_r)$ such that the product $\alpha_1 \dots \alpha_r$ is equal to σ. Multiplication is always from left to right. A factorisation is said *minimal* if it involves the minimum number of transpositions, that is, $n-k$, where k is the number of cycles (including cycles of length 1) of σ written as a product of disjoint cycles.

When σ is an n-cycle, a minimal factorisation is also *transitive*, namely, its transpositions generate the symmetric group. A good deal of research (see for instance [3]) has been done on *minimal transitive* factorisations for arbitrary permutations, that is, factorisations such that the transpositions $\alpha_1, \dots, \alpha_r$ generate S_n and the cardinality r is minimal with respect to this property. In this case, the number of traspositions for a minimal transitive factorisation is $n + k - 2$, where k is the number of cycles.

The main result on minimal factorisations of a cycle is the following.

Theorem 1 (Dénes) *The number of minimal factorisations of a fixed n-cycle is n^{n-2}, for $n \geq 2$.*

We shall enumerate factorisations of a cycle according to the number of their transpositions moving a fixed symbol, say n.

Observe that the claimed total number of minimal factorisations of a fixed cycle (without loss of generality, $(1\,2 \dots n)$) can be written as

$$n^{n-2} = ((n-1)+1)^{n-2} = \sum_{i=0}^{n-2} \binom{n-2}{i} (n-1)^i = \sum_{m=1}^{n-1} \binom{n-2}{m-1} (n-1)^{n-m-1}. \qquad (1)$$

Now, we shall show in Theorem 2 that the term $\binom{n-2}{m-1}(n-1)^{n-m-1}$ has an interpretation in our context: it is the number of factorisations of $(1\,2 \dots n)$ in which exactly m transpositions move the point labelled n. In this way, we also obtain a new proof of Dénes theorem.

This nicely mirrors the fact that the number of labelled trees on n vertices in which the vertex labelled n has exactly m neighbours is also $\binom{n-2}{m-1}(n-1)^{n-m-1}$. This was proved by Clarke in ([1]), who then used equation (1) to give a proof of Cayley's theorem. Unfortunately, it seems that this fact cannot be used directly: the known bijections between trees and factorisations do not behave nicely with respect to the degrees of the vertices: in our case this means that they do not map trees with a m-degree vertex n to factorisations moving n exactly m times.

Observe in particular that there clearly is a unique minimal factorisation of $(1\,2 \dots n)$ in which all the transpositions move n, that is $((1\;n), (2\;n), \dots, (n-1\;n))$. In fact, if $(1\,2 \dots n) = (a_1\;n)(a_2\;n), \dots, (a_{n-1}\;n)$, then a_1 is equal to 1, since a_1 is the image of n. And an easy induction does the trick.

We have the following result about our factorisations.

Theorem 2 *Let $F(n,m)$ be the set of minimal factorisations of $(1\,2 \dots n)$ in which exactly m transpositions have n in their support, and let $f(n,m) = |F(n,m)|$. Then the following holds:*

$$f(n,m) = \binom{n-2}{m-1}(n-1)^{n-m-1}. \qquad (2)$$

The proof of the theorem will follow from two lemmas. We are going to define the numbers $a(n,m)$, which enumerate minimal factorisations of certain suitable permutations; we shall show how $f(n,m)$ and $a(n,m)$ are related, and then give an expression for $a(n,m)$.

Consider the set of the permutations of S_{n-1} consisting of exactly m disjoint cycles—including the 1-cycles—and admitting an expression as a product of disjoint cycles in which $1, 2, \ldots, n-1$ appear consecutively. An example of such a permutation is

$$(1\,2)(3)(4\,5\,6\,7)\ldots(n-3\;n-2\;n-1).$$

Now, let $A(n,m)$ be the set of the minimal factorisations of these permutations, which we are going to call *contiguous factorisations*, and let $a(n,m) = |A(n,m)|$.

Remark. Let us remark some facts we shall use later about manipulating factorisations and, in particular, about making the transpositions moving n appear where we need them. Observe that, if a, b, c are in $[n-1]$ and $a \neq b$, then the following holds: $(a\,b)(c\,n) = (c\,n)^{(a\,b)}(a\,b) = (c^{(a\,b)}\,n)(a\,b)$, where π^σ is equal to $\sigma^{-1}\pi\sigma$, the permutation π conjugated by σ, and c^π is the image of c under the permutation π.

So, if

$$(\alpha_1, \ldots, \alpha_{i-1}, (c\,n), \alpha_{i+1}, \ldots, \alpha_j, \alpha_{j+1}, \ldots, \alpha_{n-1})$$

is a minimal factorisation of the n-cycle, and n is not moved by any of α_{i+1}, $\ldots$, α_j, then

$$(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_j, (c^{\alpha_{i+1}\ldots\alpha_j}\,n), \alpha_{j+1}, \ldots, \alpha_{n-1})$$

is again a minimal factorisation with the same number of transpositions moving n. In general, from a minimal factorisation we can obtain another one with the same transpositions fixing n in the same relative order, and with the transpositions moving n located in arbitrarily chosen positions.

The numbers $f(n,m)$ and $a(n,m)$ are linked in a straightforward way, which relates explicitly the two types of factorisations.

Lemma 3 *With $f(n,m)$ and $a(n,m)$ as above,*

$$f(n,m) = a(n,m)\binom{n-1}{m}. \tag{3}$$

Proof. We shall prove the claimed equality by giving a bijection between $F(n,m)$ and the set $A(n,m) \times \binom{[n-1]}{m}$, that is the set of pairs whose first element is a contiguous factorisation and the second one is an m-subset of $\{1, 2, \ldots, n-1\}$.

Map a minimal factorisation $\mathcal{F} = (\alpha_1, \ldots, \alpha_{n-1})$ to the pair

$$((\alpha_{i_1}, \ldots, \alpha_{i_{n-m-1}}), S),$$

where the α_{i_j}s (with $i_1 > i_2 > \cdots > i_{n-m-1}$) are the transpositions of $\mathcal{F}$ not moving n; and the elements of S are the positions in $\mathcal{F}$ of the transpositions moving n. Call this map φ.

We shall show:

1. that the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_{n-m-1}})$ given by the transpositions of $\mathcal{F}$ not moving n is indeed a contiguous factorisation; and so φ takes values in $A(n, m) \times \binom{[n-1]}{m}$;
2. that φ is a bijection; we shall obtain this by showing that, given any contiguous factorisation $\mathcal{G}$ and any m-subset S of $[n-1]$, one can uniquely determine m transpositions moving n, such that the sequence $\mathcal{F}$ obtained by putting them in the positions given by S and the transpositions of $\mathcal{G}$ in those given by the complement of S is a minimal factorisation of the n-cycle, and all factorisations are obtained this way.

Let us prove the claims:

1. Note that we get a permutation of the kind factorised by contiguous factorisations by multiplying an n-cycle (on the right) by transpositions of the form $(a_m\ n)$, $(a_{m-1}\ n)$, $\ldots$, $(a_2\ n)$, $(1\ n)$ with $a_m > a_{m-1} > \cdots > a_1 = 1$:

   $$\begin{aligned}(1\,2 \ldots n)(a_m\,n)(a_{m-1}\,n) \ldots (a_2\,n)(1\,n) &= \\ = (1 \ldots a_2 - 1)(a_2 \ldots a_3 - 1) \ldots (a_m \ldots n-1)(n).\end{aligned}$$

   So, if in a minimal factorisation $\mathcal{F}$ of the n-cycle the transpositions moving n are in the last positions, we get a contiguous factorisation simply by deleting them. Otherwise, by the procedure described in the Remark, we can get a minimal factorisation of the n-cycle with the same transpositions fixing n as $\mathcal{F}$, and with the transpositions moving n at the end.

2. Let us consider any factorisation $\mathcal{G} = (\alpha_1, \ldots, \alpha_{n-1-m})$ of the permutation $(1 \ldots a_2 - 1)(a_2 \ldots a_3 - 1) \ldots (a_m \ldots n-1)$, in the set $A(n, m)$. We can append to it m transpositions of the form $(a\ n)$ to obtain a factorisation $\mathcal{F}$ in $F(n, m)$, using the transpositions $(a_i\ n)$, with

$1 = a_1 < a_2 < \cdots < a_m$:

$$\underbrace{(\alpha_1, \ldots, \alpha_{n-1-m}}_{\mathcal{G}}, (a_1\ n), \ldots, (a_m\ n)) =: \mathcal{F}.$$

The numbers $a_1, \ldots, a_m$ are uniquely determined: suppose that

$$(\alpha_1, \ldots, \alpha_{n-1-m}, (b_1\ n), \ldots, (b_m\ n))$$

is another minimal transitive factorisation of the cycle $(1\ 2 \ldots n)$ (so that the b_is are different from each other), then

$$(a_1\ n) \ldots (a_m\ n) = (b_1\ n) \ldots (b_m\ n).$$

This implies that $(a_1\ a_2\ \ldots\ a_m\ n) = (b_1\ b_2\ \ldots\ b_m\ n)$. Then $a_1 = b_1$, $\ldots$, $a_m = b_m$.

The minimal factorisation $\mathcal{F}$ will correspond in this way to the pair $(\mathcal{G}, \{n-m, n-m+1, \ldots, n-1\})$.

To get the factorisation $\mathcal{F}$ corresponding to the pair $(\mathcal{G}, S)$ for an arbitrary set S, we start by completing $\mathcal{G}$ with m transpositions at the end as above, obtaining $(\alpha_1, \ldots, \alpha_{n-1-m}, (a_1\ n), \ldots, (a_m\ n))$. Then, by the Remark, we can move the transpositions $(a_i\ n)$ leftwards: if s_1 is the minimum element in S, we "push" the transposition $(a_1\ n)$ in position s_1, conjugating it along the way:

$$(\alpha_1, \ldots, \alpha_{s_1-1}, (b_1\ n), \alpha_{s_1}, \alpha_{s_1+1}, \ldots, \alpha_{n-1-m}, (a_2\ n), \ldots, (a_m\ n)),$$

with b_1 equal to $a_1^{\alpha_{n-1-m}\alpha_{n-2-m}\cdots\alpha_{s_1+1}\alpha_{s_1}}$. By an analogous procedure, all the transpositions moving n will appear in the positions dictated by S.

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Example. Take the permutation $\sigma = (1\ 2\ 3)(4)(5\ 6\ 7\ 8)$, with $n = 9$ and $m = 3$. Let us choose the factorisation $\mathcal{G} = ((1\ 3), (5\ 8), (2\ 3), (6\ 8), (7\ 8))$. It can be completed in just one way to a factorisation of $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$ in $F(9, 3)$, by adding at the end three transpositions moving 9:

$$(\mathbf{(1\ 3)}, \mathbf{(5\ 8)}, \mathbf{(2\ 3)}, \mathbf{(6\ 8)}, \mathbf{(7\ 8)}, (1\ 9), (4\ 9), (5\ 9)).$$

Each choice of five positions for the transpositions of $\mathcal{G}$, out of eight, gives a single factorisation of the cycle; for instance, the positions 2, 3, 5, 7, and 8 yield

$$((3\ 9), \mathbf{(1\ 3)}, \mathbf{(5\ 8)}, (4\ 9), \mathbf{(2\ 3)}, (5\ 9), \mathbf{(6\ 8)}, \mathbf{(7\ 8)}).$$

The transpositions involving 9 are uniquely determined by conjugation.

In the next lemma we calculate the value of $a(n, m)$.

Lemma 4 *With $a(n,m)$ as above,*

$$a(n,m) = m(n-1)^{n-m-2}. \tag{4}$$

Proof. First, it is possible to show that the following equality holds:

$$a(n,m) = \sum_{\substack{l_1,\dots,l_m \\ \sum l_i = n-1 \\ \forall i\ l_i \geq 1}} \binom{n-1-m}{l_1-1,\dots,l_m-1} \prod_{i=1}^{m} l_i^{l_i-2}.$$

In fact, by Dénes theorem, each of the m cycles (of length, say, l_i), admits $l_i^{l_i-2}$ factorisations consisting of $l_i - 1$ transpositions. The multinomial coefficient counts in how many ways these transpositions can be intertwined–more formally, the ways of choosing the $l_1 - 1$ positions for the transpositions factorising the first cycle, and so on.

(Note that this follows immediately from Theorem 3 in the classical paper by Dénes [2], which gives an expression for the number of minimal factorisations of a permutation of given cycle structure. The equality then follows by summing on all ordered partitions of $n-1$.)

Now, a closed form for the above sum is obtained by noting that it is evaluated by a multinomial identity of Abel type due to Hurwitz (eq. (36), p. 26 in the book by Riordan [8], or eq. (VI) in the original paper by Hurwitz [6]). We have

$$\sum_{\substack{l_1,\dots,l_m \\ \sum l_i = n-1 \\ \forall i\ l_i \geq 1}} \binom{n-1-m}{l_1-1,\dots,l_m-1} \prod_{i=1}^{m} l_i^{l_i-2} = m(n-1)^{n-m-2}.$$

◇

Remark. Note that, although we use Dénes theorem in this proof, we have given an original proof of it, in which we reason by induction on n, the length of the cycle: this lemma invokes the theorem only for permutations on at most $n-1$ points (because $A(n,m)$ consists of permutations in S_{n-1}).

This completes the proof of Theorem 2: we obtain the claimed result about the number of factorisations with exactly m transpositions moving n from equalities (3) and (4).

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