A note on the neighbor condition for up-embeddability of graphs

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Abstract: Different neighbor conditions are considered in [3,4,9] for a graph up-embeddable. In this paper, we consider the neighbor conditions of all the pairs of vertices with diameter 2 and obtain the following new result: if $|N_G(u) \cap N_G(v)| \geq 2$ for any two vertices $u, v \in D$ where $D = \{(u,v)|d_G(u,v) = 2,\ u,v \in V(G)\}$, then G is up-embeddable.

Key words: Maximum genus, up-embeddable, neighbor condition.

1 Introduction

A graph is often denoted by $G = (V, E)$. A graph is simple if it has neither multiple edges nor loops. In this paper, the graphs are allowed to have multiple edges and loops. The *edge-connectivity* $\kappa_1(G)$ of a connected graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. By a surface, we shall mean a compact connected 2-mainfold without boundary. The maximum genus of graph G, denoted by $\gamma_M(G)$, is the maximum genus among the genus of all orientable surface on which G has a 2-cell embedding. By the Euler formula, we have

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor,$$

Partially supported by NNSFC under Grants No. 60373030 & No. 10571013.
Partially supported by found of Beijing JiaoTong Univeristy 2004SM054.

where $\beta(G) = \varepsilon - \nu + 1$ is called the *Betty number* of G, For a graph G, if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$, then G is said to be *up-embeddable* on the orientable surfaces.

The study of the maximum genus of a graph was inaugurated by Nordhaus, Stewart and White [8]. There are two equivalent characterizations on the maximum genus of a graph, due to Xuong [11] (Liu [6] independently as well) and Nebesky [7], respectively. The characterization of Nebesky is in terms of an edge cut set of the graph. The two characterizations in [11] and [7] are dual to each other.

Let T be a *spanning tree* of a connected graph G. The edge complement $G - T$ of the spanning tree T is called a *co-tree*. A component H of $G - T$ is called an *odd component* if H has an odd number of edges; otherwise, an *even component. The deficiency* $\xi(G,T)$ *of a spanning tree* T of a connected graph G is defined to be the number of odd components of $G - T$. *The deficiency* $\xi(G)$ *of a graph* G is the minimum of $\xi(G,T)$ over all spanning trees T. Now we restated the *characterization theorem.*

Theorem 1 [6,11] *Let* G *be a connected graph. Then*

(1) G *is up-embeddable if and only if* $\xi(G) \leq 1$, and

(2) $\gamma_M(G) = \frac{\beta(G) - \xi(G)}{2}$.

Let A be a subset of E. let $c = c(G - A)$ be the number of components of $G - A$, and let $b = b(G - A)$ be the number of components of $G - A$ with odd Betty number.

Theorem 2 [7] *Let* G *be a connected graph. Then*

(1) G *is up-embeddable if and only if* $c(G - A) + b(G - A) - 2 \leq |A|$, *for any subset* A of E, and

(2) $\xi(G) = \max_{A \subset E}\{c(G - A) + b(G - A) - |A| - 1\}$.

Combining this characterization and a result of Kundu [5], every 4-edge connected graph is up-embeddable. However there are examples of k-edge connected (k-vertex connected) graphs that are not up-embeddable, for $k = 1, 2, 3$. Based on these observations, it leads us to consider what conditions are constrained such that a k-edge-connected is up-embeddable. Let $N_G(u)$ denote the set of neighbors of vertex $u \in G$. In [9], Skoviera showed that every multigraph (without loops) of diameter 2 is up-embeddable. Since a graph without loops has diameter 2 if, and only if, $\forall\ uv \notin E(G)$, $|N_G(u) \cap N_G(v)| \geq 1$. In [3], Huang and Liu considered the neighbor condition of adjacent vertices and proved the following result: Let G be a graph without loops. If $\forall\ uv \in E(G)$ satisfy one of the following condition: (1) $|N_G(u) \cap N_G(v)| \geq 2$; (2) G is 2-connected and $|N_G(u) \cap N_G(v)| \geq 1$, then G is up-embeddable. Let $L \in \{K_{1,3}, K_{1,3} + e\}$. In [4], He and Liu further proved the following result: Let G be a simple graph, for any two vertices u and v of distance 2 in L, i.e., $d_L(u,v) = 2$, satisfy the condition $|N_G(u) \cap N_G(v)| \geq 2$, then G is up-embeddable. Especially, the $L-$ free graphs are up-embeddable. In this paper, we consider the neighbor conditions of all the pairs of vertices with diameter 2 and obtain the following conclusion.

Theorem 3 *Let G be a graph without loops and $D = \{(u,v)|d_G(u,v) = 2,\ u,v \in V(G)\}$. If $|N_G(u) \cap N_G(v)| \geq 2$ for all $u,v \in D$, then G is up-embeddable.*

For any concepts not defined here, we may refer to [1], [6] or [10].

2 The proof of Theorem 3

Let $F_{i_1}, \cdots, F_{i_k}$ be some connected components of $G-A$, and $E(F_{i_1}, \cdots, F_{i_k})$ be the set of edges each of which has one end vertex in in $V(F_{i_m})$, the other in $V(F_{i_n})(1 \leq m,n \leq k, m \neq n)$. In [2], Huang and Liu obtained the following theorem.

Theorem 4 [2] *Let G be a connected graph. if G is not up-embeddable, then there exists an edge subset A satisfying the following properties,*

(a) *$c(G-A) \geq 2$, and $\beta(F) = 1$ (mod 2) for any connected component F of $G-A$;*

(b) *F is an vertex-induced graph of G for any connected component F of $G-A$;*

(c) *$|E(F_{i_1}, \cdots, F_{i_k})| \leq 2k-3$ for any k distinct components $F_{i_1}, \cdots, F_{i_k}$ of $G-A$;*

(d) *$\xi(F) = 1$ for any connected component F of $G-A$;*

(e) *$\xi(G) = 2c(G-A) - |A| - 1$.*

The following fact is obvious.

$$|A| = \frac{1}{2}\sum_{i=1}^{l}|E(F_i, A)| \tag{1}$$

Proof (of Theorem 3). Suppose that G is not up-embeddable. By Theorem 4, there exists $A \subset E$ such that the properties (a)-(e) of Theorem 4 are satisfied. Let $F_1, F_2, \cdots, F_l$ $(l \geq 2)$ be the all connected components of $G-A$.

Claim $\kappa_1(G) \geq 2$.

Suppose to the contrary, $\kappa_1(G) = 1$, thus G has a cut-edge $e = uv$. If $d(u) = d(v) = 1$, in this case, G is K_2. Thus G is up-embeddable, it's a contradiction to our hypothesis. Without loss of generality, we may suppose w is a neighbor of u and $w \neq v$. Since $d_G(w,v) = 2$, by our assumption, $|N_G(w) \cap N_G(v)| \geq 2$. Let x be a neighbor of w and v and $x \neq u$. Thus uv lies in a cycle $uvxw$ and uv is not a cut-edge of G, it's impossible.

If we prove that $|E(F_i, A)| \geq 4$, for $i = 1, 2, \cdots, l$, By (1),

$$|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, A)| \geq 2l.$$

By Theorem 4,

$$|A| \leq 2l - 3.$$

It's impossible. Then we complete the proof.

Suppose to the contrary, there exists a F_i such that $|E(F_i, A)| \leq 3$, $i \in \{1, 2, \cdots, l\}$. By the claim, we know that $2 \leq |E(F_i, A)| \leq 3$.

Case 1 $|E(F_i, A)| = 2$. Let u, v be two contacting vertex of F_i. Let $E(F_i, F_j) = \{ux\}$ and $E(F_i, F_k) = \{vy\}$, $(x \in F_j, y \in F_k)$.

Subcase 1 $u = v$. By Theorem 4, F_i is a vertex-induced subgraph of G and $\beta(F_i) = 1$ $(mod\ 2)$, we know $|V(F_i)| \geq 2$. Let w be another neighbor of u in F_i. Since $d_G(w, x) = 2$ and $|N_G(w) \cap N_G(x)| \geq 2$, there exist a vertex z such that z is a neighbor of w and x. If $w \in F_i$, it has $|E(F_i, F_j)| \geq 2$, otherwise, $|E(F_i, A)| \geq 3$. By Theorem 4 and $|E(F_i, A)| = 2$, it's impossible.

Subcase 2 $u \neq v$. By Theorem 4, we know $|V(F_j)| \geq 2$. let h be a neighbor of x in F_j. Since $d_G(u, h) = 2$, we have $|N_G(v) \cap N_G(x)| \geq 2$, in this case, $|E(F_i, A)| \geq 3$, it's contrary to our assumption $|E(F_i, A)| = 2$.

Case 2 $|E(F_i, A)| = 3$. Let u, v, w be three contacting vertex of F_i. Let $E(F_i, F_j) = \{ux\}$, $E(F_i, F_k) = \{vy\}$, and $E(F_i, F_l) = \{wz\}$ $(x \in F_j, y \in F_k, z \in F_l)$.

Subcase 1 $u = v = w$. By Theorem 4, F_i is a vertex-induced subgraph of G and $\beta(F_i) = 1$ $(mod\ 2)$, we know $|V(F_i)| \geq 2$. Let h be a neighbor of u in F_i. Since $d_G(x, h) = 2$, we have $|N_G(h) \cap N_G(x)| \geq 2$, in this case $|E(F_i, A)| \geq 4$, it's a contradiction to our assumption $|E(F_i, A)| = 3$.

Subcase 2 $u = v \neq w$. (1) $uw \in E(G)$. Since $d_G(w, x) = 2$, we have $|N_G(w) \cap N_G(x)| \geq 2$. Let t be a neighbor of w and x and $t \neq u$. If $t \neq z$, then $|E(F_i, A)| \geq 4$, a contradiction. Thus $t = z$, i.e. $xz \in E(G)$. Similarly we have $yz \in E(G)$. Let s be a neighbor of x in F_j. Since $d_G(u, s) = 2$, we have $|N_G(u) \cap N_G(s)| \geq 2$. Let s' be a neighbor of u and s and $s' \neq x$. If $s' \neq y$, then $|E(F_i, A)| \geq 4$, a contradiction. Thus $s' = y$. In this case, $E(F_i, F_j, F_k, F_l) \supseteq \{sy, xz, yz, ux, uy, wz\}$, i.e.,$|E(F_i, F_j, F_k, F_l)| \geq 6$, by (c) of Theorem 4, it's impossible. (2) $uw \notin E(G)$, Since F_i is connected, we suppose h be a neighbor of u in F_i. Because $d_G(h, x) = 2$, we have $|N_G(h) \cap N_G(x)| \geq 2$. Let t be a neighbor of h and x and $t \neq u$. By (c) of Theorem 4, we have $t \notin F_i \cup F_j \cup F_k \cup F_l$. Thus $|E(F_i, A)| \geq 4$. It's a contradiction to our hypothesis $|E(F_i, A)| = 3$.

Subcase 3 $u \neq v \neq w$. We first show that $u \in N_G(v)$ or $w \in N_G(v)$. If $u, w \notin N_G(v)$, by the connectivity of F_i, let h be a neighbor of v in F_i.

In this case, $d_G(h,y)=2$ and we have $|N_G(h)\cap N_G(y)|\geq 2$, in this case, $|E(F_i,A)|\geq 4$. It's impossible. Without loss of generality, let $uv\in E(G)$. If $vw\notin E(G)$, Similarly, we have $u\in N_G(w)$ or $v\in N_G(w)$. By above discussion, we may suppose $uv, vw\in E(G)$.

Since $d_G(w,y)=2$ and $d_G(u,y)=2$, we have $|N_G(w)\cap N_G(y)|\geq 2$ and $|N_G(u)\cap N_G(y)|\geq 2$. Let r be a neighbor of u and y, s be a neighbor of w and y. By (c) of Theorem 4 and $|E(F_i,A)|=3$, we must have $r=x$ and $s=z$. If $uw\notin E(G)$, then $d_G(u,w)=2$. Thus, let t be a neighbor of u and w ($t\neq v$). Since $d_G(t,x)=2$, we have $|N_G(t)\cap N_G(x)|\geq 2$, In this case $|E(F_i,A)|\geq 4$. It's a contradiction to our assumption $|E(F_i,A)|=3$. Otherwise $uw\in E(G)$, in this case $d_G(x,w)=2$, we have $|N_G(x)\cap N_G(w)|\geq 2$. let p be a neighbor of w and x such that $p\neq u$. Since $|E(F_i,A)|=3$, we must have $p\in F_l$ and $p=z$. But $E(F_i,F_j,F_k,F_l)\supseteq\{xy,xz,yz,ux,vy,wz\}$, i.e., $|E(F_i,F_j,F_k,F_l)|\geq 6$, it's contrary to Theorem 4 (c). ♮

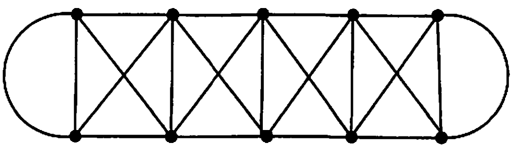

Fig. 1.

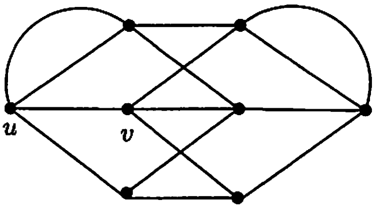

Fig. 2.

Remark 1 Note that the graph G in Fig. 1 has multiple edge and with diameter 4. By Theorem 3, we know that G is up-embeddable. But, we can't decide the up-embeddability of G by using the methods of [4,9].

Remark 2 By Theorem 3, the graph G shown in Fig. 2 is up-embeddable. Since $uv \in E(G)$ and $|N_G(u) \cap N_G(v)| = 0$, we can't use the methods of [3,4].

Thus, Theorem 3 is independent of the methods of [3,4,9].

By Theorem 3, we have the following corollary.

Corollary 5 *Let G be a graph without loops and $D = \{(u,v)|d_G(u,v) = 3,\ u,v \in V(G)\}$. Let $H = G \cup \{uv|uv \in D\}$. Then H is up-embeddable.*

Proof. If for all $u, v \in V(G), d_G(u,v) \leq 2$, then the diameter of G is less than 2. By a result of [9], we know that G is up-embeddable. Otherwise $D = \{(u,v)|d_G(u,v) = 3,\ u,v \in V(G)\}$ is not empty, it's routine task to check that H satisfied the condition of Theorem 3.

Acknowledgements: The authors wish to thank the referees for their careful reading of the paper, and for their valuable comments.

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