On the existence of well-ordered Steiner triple systems

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Abstract

In [1], well-ordered Steiner triple systems were introduced and used to construct 1-perfect partitions of the n-cube. However, non-trivial well-ordered Steiner triple systems were only known to exist when $v = 15$. In this short note, we present a simple construction to give a non-trivial well-ordered Steiner triple system of order $v = 2^n - 1$ for all $n \geq 5$ and this settles a problem in [1].

1 Introduction

A *Steiner triple system* STS(n) is a pair $(A, \mathcal{B})$, where A is a finite set on n elements and $\mathcal{B}$ is a collection of 3-subsets of A, which we call blocks, such that every two different elements $x, y \in A$ are contained in exactly one block of $\mathcal{B}$.

Let A be a finite set. An algebraic structure of a quasigroup consists of A and a binary operation on A defined by the function

$$* : A \times A \to A$$

such that $x * y = x * z$ and $y * x = z * x$ only if $y = z$ for all $x, y, z \in A$.

A quasigroup $(A, *)$ is called a Sloop if

- there exists $0 \in A$ such that $0 * a = a * 0 = a$ for all $a \in A$;
- the operation is totally symmetric, that is, any relation $a*b = c$ implies any other relation obtained by permuting a, b and c.

- It is easy to see that starting from a Sloop A, we can define an STS on the set $A^* = A \setminus \{0\}$ by taking a set of blocks $\mathcal{B} = \{\{x, y, x * y\} |$ for all $x, y \in A^*, x \neq y\}$.
- Conversely, starting from an STS(n)=($A^*, \mathcal{B}$), we can define a Sloop on the set $A = A^* \cup \{0\} = \{0, 1, 2, \ldots, n\}$ by

$$A \times A \to A$$
$$(a, b) \to a * b,$$

  - if $a \neq b$ then $a * b = c$, where $\{a, b, c\} \in \mathcal{B}$,
  - if $a = b$ then $a * b = 0$,
  - if $a = 0$ then $a * b = b$,
  - if $b = 0$ then $a * b = a$.

- Two STSs $(A, \mathcal{B})$ and $(A', \mathcal{B}')$ are isomorphic if $A = A'$ and there exists a permutation of the elements in A such that the triples in $\mathcal{B}$ are converted to the triples in $\mathcal{B}'$.

Suppose that an STS(n) exists on $A_n = \{1, 2, \ldots, n\}$. We define $[c_1 c_2 \ldots c_r]$ to be

$$((\ldots((c_1 * c_2) * c_3) * c_4) * \ldots) * c_r)$$

where $c_1, c_2, \ldots, c_r \in A_n$.

Given $a, x, y \in A$, the equation $(a * x) * y = (a * y^*) * x$ always has a unique solution for y^* that can be calculated as

$$y^* = [axyxa]$$

Given an STS(v) on $\{1, 2, \ldots, v\}$ with the natural ordering $<$ on $1, 2, \ldots, v$. We call an STS *well-ordered* if it is possible to order the elements in A_n such that for all $a, x, y \in A_n$, we have $x < y$ if and only if $x < y^*$, where $y^* = [axyxa]$.

Clearly, if $y^* = y$ for all $x, y, a \in A_n$, the STS must be well-ordered. We call this type of well-ordered STSs *trivial.* In the next section, we show that any trivial well-ordered STS must be the projective triple system. Furthermore, we construct a non-trivial well-ordered STS(v) where $v = 2^n - 1$ for all $n \geq 5$ and it answers a problem in [1].

2 Main Results

Since the existence of a well-ordered STS(n) implies the existence of a 1-perfect binary code, $n = 2^k - 1$ for some k [1].

A (k,l) – *configuration* in an STS($A, \mathcal{B}$) is a subset of l triples of $\mathcal{B}$ whose union is a k-element subset of A.

The *Pasch configuration* or *quadrilateral*, P, is the $(6,4)$-configuration on elements (say) a, b, c, d, e and f with the triples $\{a,b,c\}$, $\{a,d,e\}$, $\{b,d,f\}$ and $\{c,e,f\}$.

Our first theorem characterizes the number of Pasch configurations in any trivial well-ordered STS.

Lemma 2.1 *In any trivial well-ordered STS(n), there are precisely* $\frac{n(n-1)(n-3)}{24}$ *Pasch configurations.*

Proof: Let $(A_n, \mathcal{B})$ be any trivial well-ordered STS(n). Let 3 distinct elements of A_n be x, y and a. By the property of trivial well-ordered STS, we must have $y = [axyxa]$. Therefore, we have $\{a, x, [ax]\}$, $\{[ax], y, [axy]\}$, $\{[axy], x, [axyx]\}$ and $\{[axyx], a, y\}$ as blocks in the STS. If the six points $x, y, a, [ax], [axy], [axyx]$ are distinct, then these four blocks form a Pasch configuration. If $\{a, x, y\}$ is not a block in the STS, then $[ax] \notin \{x, y, a\}$, $[axy] \notin \{a, x, y, [ax]\}$ and $[axyx] \notin \{a, x, y, [ax], [axy]\}$. This shows that whenever $\{a, x, y\}$ is not on a block of the STS, it will generate a Pasch configuration. Next, we show that given any Pasch configuration, there are precisely 24 triples of (a, x, y) which will generate this Pasch configuration. Suppose a is any point in the Pasch configuration. For the block $\{a, x, [ax]\}$ to be in the Pasch configuration, there are only four choices for x, namely, any of the four points each of which belongs to a block with a. Finally, y must be the unique point that $\{y, a\}$ and $\{y, [ax]\}$ are subset of a block in the Pasch configuration. Hence, there are precisely $6 \times 4 = 24$ ways to generate the same Pasch configuration. Therefore, the number of Pasch configuration in any trivial well-ordered STS is $\frac{n(n-1)(n-3)}{24}$ where $n(n-1)(n-3)$ counts the number of 3 points configurations in n points which do not lie in a block and 24 counts the number of 3 points configuration that generates the same Pasch configuration. □

Theorem 2.2 *Any trivial well-ordered STS(n) is a PG(k, 2) where* $n = 2^k - 1$.

Proof: In [2], the authors have shown that any STS(n) contains at most $\frac{n(n-1)(n-3)}{24}$ Pasch configurations with equality holds if and only if the STS(n) is a PG(k, 2) where $n = 2^k - 1$. □

Theorem 2.3 *There exists a non-trivial well-ordered STS($2^n - 1$) for all $n \geq 4$.*

Proof: When $n = 4$, the result is established by Rifà in [1]. We shall prove the existence inductively. Suppose $n \geq 5$. Take a PG(n, 2) on $V = \{1, 2, \ldots, 2^n - 1\}$ with a sub-PG($n-1$, 2) on $\{1, 2, \ldots, 2^{n-1} - 1\}$. Take out all the blocks of the sub-PG($n-1$, 2) on $\{1, 2, \ldots, 2^{n-1} - 1\}$ and replace them with a non-trivial well-ordered STS($2^{n-1} - 1$) on $\{1, 2, \ldots, 2^{n-1} - 1\}$ with the natural ordering (which exists by induction). We claim that the resulting STS is well-ordered. First, it is clear that the STS is not the projective triple system since we replaced a subsystem of the PG(n, 2). To show that the STS is well-ordered, we do a simple case-by-case analysis. Let $A = \{1, 2, \ldots, 2^{n-1} - 1\}$ and $B = \{2^{n-1}, 2^{n-1} + 1, \ldots, 2^n - 1\}$. There are eight cases to consider.

Case	$a \in$	$x \in$	$y \in$
1	A	A	A
2	A	A	B
3	A	B	A
4	B	A	A
5	A	B	B
6	B	B	A
7	B	A	B
8	B	B	B

Before we proceed, we note that any block intersects B in either 0 or 2 points. Also, if $x \in A$ and $y \in B$, then $x < y$.

If $\{a, x, y\}$ were a block in the STS, then $x < y$ if and only if $x < y^\star$ since $y^\star = y$. W.L.O.G., we assume $\{a, x, y\}$ is not a block in the STS.

1. In case 1, it is clear that $x < y$ if and only if $x < y^\star$ since the subsystem STS($2^{n-1} - 1$) on A is well-ordered.

2. In case 2, $a, x \in A$ and $y \in B$. This implies $[ax] \in A$, $[axy] \in B$, $[axyx] \in B$ and $[axyxa] \in B$. Therefore, $x < y^\star$.

3. In case 3, $a, y \in A$ and $x \in B$. This implies $[ax] \in B$, $[axy] \in B$, $[axyx] \in A$ and $[axyxa] \in A$. Therefore, $x > y^\star$.

4. In case 4, $a \in B$ and $x, y \in A$. This implies $[ax] \in B$, $[axy] \in B$, $[axyx] \in B$ and $[axyxa] \in A$. In this case, we note that none of the four blocks $\{a, x, [ax]\}$, $\{[ax], y, [axy]\}$, $\{[axy], x, [axyx]\}$ and $\{[axyx], a, [axyxa]\}$ is in the subsystem on A. Hence, they must form a Pasch configuration and $y^\star = y$.

5. Case 5 is the same as case 4.

6. In case 6, $[ax] \in A$, $[axy] \in A$, $[axyx] \in B$ and $[axyxa] \in A$. Therefore $x > y^\star$.

7. In case 7, we can simply deduce that $[axyxa] \in B$ using the above argument. Hence, $x < y^\star$.

8. Case 8 is the same as case 4.

□

References

[1] J. Rifà, Well-ordered Steiner triple systems and 1-perfect partitions of the n-cube, *SIAM J. Discrete Math.* Vol. 12 (1999), 35-47.

[2] D.R. Stinson and Y.J. Wei, Some results on quadrilaterals in Steiner triple systems, *Discrete Math.* 105 (1992), 207-219.