THE MODIFIED ZAGREB INDICES ABOUT JOIN AND COMPOSITION OF GRAPHS

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Abstract: The modified Zagreb indices are topological indices which reflect certain structural features of organic molecules. In this paper we study the modified Zagreb indices of joins and compositions.

INTRODUCTION

Zagreb indices have been introduced in 1972 by Gutman and Trinajstic [1] as terms in their derivation of the pi-electronic energy of conjugated molecules. Three years later Gutman et al. [2] formulated Zagreb indices as the branching indices. After this paper, the Zagreb indices start appearing in the structure-property-activity modeling procedures[3-7]. They are very useful topological indices in QSPR and QSAR [6, 7]. The *first Zagreb index* $M_1(G)$ and the *second Zagreb index* $M_2(G)$ are defined as follows [6, 7]: for a simple connected graph G, let $M_1(G) = \sum_{v\in V(G)} d(v)^2$, $M_2(G) = \sum_{uv\in E(G)} d(u)d(v)$, where d(u) and d(v) are the degrees of vertices u and v respectively.

However, recently some authors found their shortcoming [8]. It has been noted that the contributing elements to the Zagreb indices give greater weights to the inner (interior) vertices and edges and smaller weights to outer (terminal) vertices and edges of a graph. This opposes intuitive reasoning that the outer atoms and bonds should have greater weights than inner vertices and bonds, because the outer vertices and bonds are associated with the larger part of the molecular surface and consequently are expected to make a greater contribution to physical, chemical and biological properties. One way to amend Zagreb indices is to input in the definitions of $M_1(G)$ and $M_2(G)$ inverse values of the vertex-degrees. We call these indices the *modified Zagreb indices* and denoted them by symbols ${}^mM_1$ and ${}^mM_2$ [4]. They are defined as follows [8]: for a simple connected graph G, let ${}^mM_1(G) = \sum_{v\in V(G)} \frac{1}{d(v)^2}$, ${}^mM_2(G) = \sum_{uv\in E(G)} \frac{1}{d(u)d(v)}$, where d(u) and d(v) are the degrees of vertices u and v respectively.

In this paper we study the modified Zagreb indices about join and composition of graphs. In [14], Sagan et al studied Wiener polynomial problem about graph operations containing join and composition problems.

PRELIMINARIES

For further details, see [12, 13].

Definition 2.1[9, 10, 11]. The zeroth-order general Randic index ${}^0R_t(G) = \sum_{v\in V(G)} d(v)^t$ for general real number t, where d(v) is the degree of v. When t = −0.5, ${}^0R_{-0.5}(G)$ is the famous zeroth-order Randic index $R^0(G)$ [6]. Randic index of graph G, denotes $\chi(G)$, is defined as follows: $\chi(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$.

Definition 2.2[12]. The join G + H of graphs G and H with disjoint vertex sets V(G) and V(H) and edge sets E(G) and E(H) is the graph union G ∪ H together with all the edges joining V(G) and V(H).

Definition 2.3[15]. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever u_1 is adjacent with u_2 or $u_1 = u_2$ and v_1 is adjacent with v_2.

Lemma 2.4[12]. Every nontrivial tree has at least two vertices of degree one.

Lemma 2.5[16]. $d_{G[H]}((a, b)) = |V(H)|d_G(a) + d_H(b)$.

MAIN RESULTS ABOUT JOINS

Theorem 3.1. ${}^mM_1(P_m+P_n) = \frac{2}{(n+1)^2} + \frac{m-2}{(n+2)^2} + \frac{2}{(m+1)^2} + \frac{n-2}{(m+2)^2}$, m, n ≥ 2.

Theorem 3.2. ${}^mM_1(K_m+K_n) = \frac{m+n}{(m+n-1)^2}$, where m, n ≥ 2.

Theorem 3.3. ${}^mM_1(C_m+C_n) = \frac{m}{(n+2)^2} + \frac{n}{(m+2)^2}$, where m, n ≥ 3.

Theorem 3.4. ${}^mM_1(K_{1,m-1}+K_{1,n-1}) = \frac{m-1}{(n+1)^2} + \frac{n-1}{(m+1)^2} + \frac{2}{(m+n-1)^2}$, m, n ≥ 2.

Theorem 3.5. ${}^mM_1(P_m+C_n) = \frac{2}{(n+1)^2} + \frac{m-2}{(n+2)^2} + \frac{n}{(m+2)^2}$, m ≥ 2, n ≥ 3.

Theorem 3.6. ${}^mM_1(P_m+K_{1,n-1}) = \frac{2}{(n+1)^2} + \frac{m-2}{(n+2)^2} + \frac{n-1}{(m+1)^2} + \frac{1}{(m+n-1)^2}$, where m, n ≥ 2.

Theorem 3.7. ${}^{m}M_1(C_m+K_{1,n-1}) = \frac{m}{(n+2)^2} + \frac{n-1}{(m+1)^2} + \frac{1}{(m+n-1)^2}$, where $m \geq 3, n \geq 2$.

Theorem 3.8. ${}^{m}M_1(P_m+K_n) = \frac{2}{(n+1)^2} + \frac{m-2}{(n+2)^2} + \frac{n}{(m+n-1)^2}$, $m, n \geq 2$.

Theorem 3.9. ${}^{m}M_1(C_m+K_n) = \frac{m}{(n+2)^2} + \frac{n}{(m+n-1)^2}$, where $m \geq 3, n \geq 2$.

Theorem 3.10. ${}^{m}M_1(K_m+K_{1,n-1}) = \frac{m+1}{(m+n-1)^2} + \frac{n-1}{(m+1)^2}$, where $m, n \geq 2$.

Theorem 3.11. Let G and H be simple connected graphs, we have $\frac{m+n}{(m+n-1)^2} \leq {}^{m}M_1(G+H) \leq \min\{\frac{1}{4n}\,{}^{0}R_{-1}(G) + \frac{1}{4m}\,{}^{0}R_{-1}(H),\ \frac{n-1}{(m+1)^2} + \frac{2}{(m+n-1)^2} + \frac{m-1}{(n+1)^2}\}$, ${}^{0}R_{-1}(G)$ is defined in Definition 2.1, $m = |V(G)| \geq 2$, $n = |V(H)| \geq 2$.

Proof. Since $d(u) + d(v) \geq 2(d(u)d(v))^{0.5}$, by the definition of ${}^{m}M_1$ we have

$$
{}^{m}M_1(G+H) = \sum_{u\in V(G)} \frac{1}{(d(u)+n)^2} + \sum_{v\in V(H)} \frac{1}{(d(v)+m)^2}
$$

$\leq \sum_{u\in V(G)} \frac{1}{4nd(u)} + \sum_{v\in V(H)} \frac{1}{4md(v)} = \frac{1}{4n}\,{}^{0}R_{-1}(G) + \frac{1}{4m}\,{}^{0}R_{-1}(H)$. Since G+H is a subgraph of $K_m+K_n = K_{m+n}$, by Theorem 3.2 and the definition of ${}^{m}M_1$ we have ${}^{m}M_1(G+H) \geq \frac{m+n}{(m+n-1)^2}$.

Claim: Let a, b be natural numbers, we have $\frac{1}{a^2} + \frac{1}{b^2} < \frac{1}{(a-1)^2} + \frac{1}{(b+1)^2}$, $a \geq 2$, $a \leq b$

In fact, let $f(x) = \frac{1}{x^2} - \frac{1}{(x+1)^2}$, $x \geq 2$. We have $f'(x) = -2(\frac{1}{x^3} - \frac{1}{(x+1)^3}) < 0$. Hence, f(x) is a decreasing function. Thus, we have $\frac{1}{b^2} - \frac{1}{(b+1)^2} < \frac{1}{(a-1)^2} - \frac{1}{a^2}$. The claim follows. By the claim above we have ${}^{m}M_1(G+H) \leq \frac{m-1}{(n+1)^2} + \frac{n-1}{(m+1)^2} + \frac{2}{(m+n-1)^2}$. The theorem follows.

Similarly, we have

Theorem 3.12. ${}^{m}M_2(P_m+P_n) = \frac{2}{(m+1)(m+2)} + \frac{n-3}{(m+2)^2} + \frac{2}{(n+1)(n+2)} + \frac{m-3}{(n+2)^2} + \frac{4}{(n+1)(m+1)} + \frac{2(n-2)}{(n+1)(m+2)} + \frac{2(m-2)}{(n+2)(m+1)} + \frac{(m-2)(n-2)}{(n+2)(m+2)}$, where $m, n \geq 3$.

Theorem 3.13. ${}^{m}M_2(K_m+K_n) = \frac{m+n}{2(m+n-1)}$, where $m, n \geq 2$.

Theorem 3.14. ${}^{m}M_2(C_m+C_n) = \frac{m}{(n+2)^2} + \frac{n}{(m+2)^2} + \frac{mn}{(n+2)(m+2)}$, $m, n \geq 3$.

Theorem 3.15. ${}^{m}M_2(K_{1,m-1}+K_{1,n-1}) = \frac{2(m-1)}{(m+n-1)(n+1)} + \frac{2(n-1)}{(m+n-1)(m+1)} + \frac{(m-1)(n-1)}{(n+1)(m+1)} + \frac{1}{(m+n-1)^2}$, where $m, n \geq 2$.

Theorem 3.16. ${}^{m}M_2(P_m+C_n) = \frac{2}{(n+2)(n+1)} + \frac{m-3}{(2+n)^2} + \frac{2n}{(m+2)(n+1)} + \frac{n(m-2)}{(m+2)(n+2)} + \frac{n}{(2+m)^2}$, where $m, n \geq 3$.

Theorem 3.17. ${}^{m}M_2(P_m+K_{1,n-1}) = \frac{2}{(n+2)(n+1)} + \frac{m-3}{(2+n)^2} + \frac{n-1}{(m+n-1)(m+1)} + \frac{2(n-1)}{(m+1)(n+1)} + \frac{2}{(m+n-1)(n+1)} + \frac{(m-2)(n-1)}{(m+1)(n+2)} + \frac{m-2}{(m+n-1)(n+2)}$, where $m \geq 3, n \geq 2$.

Theorem 3.18. ${}^{m}M_2(C_m+K_{1,n-1}) = \frac{m}{(n+2)^2} + \frac{n-1}{(m+n-1)(m+1)} + \frac{m(n-1)}{(m+1)(n+2)} + \frac{m}{(m+n-1)(n+2)}$, where $m \geq 3, n \geq 2$.

Theorem 3.19. ${}^{m}M_2(P_m+K_n) = \frac{2}{(n+1)(n+2)} + \frac{m-3}{(n+2)^2} + \frac{n(n-1)}{2(m+n-1)^2} + \frac{2n}{(m+n-1)(n+1)} + \frac{n(m-2)}{(m+n-1)(n+2)}$, where $m \geq 3, n \geq 2$.

Theorem 3.20. ${}^{m}M_2(C_m+K_n) = \frac{m}{(n+2)^2} + \frac{n(n-1)}{2(m+n-1)^2} +$

$\frac{mn}{(m+n-1)(n+2)}$, where $m \geq 3, n \geq 2$.

Theorem 3.21. ${}^{m}M_2(K_m+K_{1,n-1}) = \frac{m(m+1)}{2(m+n-1)^2} + \frac{n-1}{m+n-1}$, where $m, n \geq 2$.

Theorem 3.22. Let G and H be simple connected graphs, we have ${}^{m}M_2(G+H) < \frac{\chi(G)}{4n} + \frac{\chi(H)}{4m} + \frac{R^0(G)R^0(H)}{4\sqrt{mn}}$, where $R^0(G)$ and $\chi(G)$ are defined in Definition 2.1, $m = |V(G)| \geq 2$, $n = |V(H)| \geq 2$.

Proof. $${}^{m}M_2(G+H) = \sum_{uv\in E(G)} \frac{1}{(d(u)+n)(d(v)+n)} + \sum_{xy\in E(H)} \frac{1}{(d(x)+m)(d(y)+m)} + \sum_{u\in V(G)}\sum_{x\in V(H)} \frac{1}{(n+d(u))(d(x)+m)}.$$ Since $a + b \geq 2\sqrt{ab}$, where a and b are positive numbers, we have $${}^{m}M_2(G+H) \leq \sum_{uv\in E(G)} \frac{1}{4\sqrt{d(u)nd(v)n}} + \sum_{xy\in E(H)} \frac{1}{4\sqrt{d(x)md(y)m}} + \sum_{u\in V(G)}\sum_{x\in V(H)} \frac{1}{4\sqrt{d(u)nd(x)m}} = \frac{\chi(G)}{4n} + \frac{\chi(H)}{4m} + \frac{R^0(G)R^0(H)}{4\sqrt{mn}}.$$ Considering $a + b = 2\sqrt{ab}$ if and only if $a = b$, if $${}^{m}M_2(G+H) = \sum_{uv\in E(G)} \frac{1}{4\sqrt{d(u)nd(v)n}} + \sum_{xy\in E(H)} \frac{1}{4\sqrt{d(x)md(y)m}} + \sum_{u\in V(G)}\sum_{x\in V(H)} \frac{1}{4\sqrt{d(u)nd(x)m}},$$ we have $d(u) = n$, $d(v) = n$, $d(x) = m$, $d(y) = m$. Because $d(u) \leq m-1$, $d(x) \leq n-1$, we have $n \leq n-2$, which is a contradiction. Hence, we have ${}^{m}M_2(G+H) < \frac{\chi(G)}{4n} + \frac{\chi(H)}{4m} + \frac{R^0(G)R^0(H)}{4\sqrt{mn}}$. The theorem follows.

MAIN RESULTS ABOUT COMPOSITIONS

Theorem 4.1. ${}^{m}M_1(P_m[P_n]) = \frac{4}{(n+1)^2} + \frac{2(n-2)}{(n+2)^2} + \frac{2(m-2)}{(2n+1)^2} + \frac{(m-2)(n-2)}{(2n+2)^2}$, where $m, n \geq 2$.

Theorem 4.2. ${}^{m}M_1(K_m[K_n]) = \frac{mn}{(mn-1)^2}$, where $m, n \geq 2$.

Theorem 4.3. ${}^{m}M_1(C_m[C_n]) = \frac{mn}{4(n+1)^2}$, where $m, n \geq 3$.

Theorem 4.4. ${}^{m}M_1(K_{1,m-1}[K_{1,n-1}]) = \frac{(m-1)(n-1)}{(n+1)^2} + \frac{m-1}{(2n-1)^2} + \frac{n-1}{(mn-n+1)^2} + \frac{1}{(mn-1)^2}$, where $m, n \geq 2$.

Theorem 4.5. ${}^{m}M_1(P_m[C_n]) = \frac{2n}{(n+2)^2} + \frac{n(m-2)}{(2n+2)^2}$, where $m \geq 2, n \geq 3$.

Theorem 4.6. ${}^{m}M_1(P_m[K_{1,n-1}]) = \frac{2(n-1)}{(n+1)^2} + \frac{2}{(2n-1)^2} + \frac{(m-2)(n-1)}{(2n+1)^2} + \frac{m-2}{(3n-1)^2}$, where $m, n \geq 2$.

Theorem 4.7. ${}^{m}M_1(C_m[K_{1,n-1}]) = \frac{m(n-1)}{(2n+1)^2} + \frac{m}{(3n-1)^2}$, where $m \geq 3, n \geq 2$.

Theorem 4.8. ${}^{m}M_1(P_m[K_n]) = \frac{2n}{(2n-1)^2} + \frac{n(m-2)}{(3n-1)^2}$, where $m, n \geq 2$.

Theorem 4.9. ${}^{m}M_1(C_m[K_n]) = \frac{mn}{(3n-1)^2}$, where $m \geq 3, n \geq 2$.

Theorem 4.10. ${}^{m}M_1(K_m[K_{1,n-1}]) = \frac{m(n-1)}{(mn-n+1)^2} + \frac{m}{(mn-1)^2}$, where $m, n \geq 2$.

Theorem 4.11. Let G and H be simple connected graphs, we have $\frac{mn}{(mn-1)^2} \leq {}^{m}M_1(G[H]) \leq \min\{\frac{1}{4n}\,{}^{0}R_1(G)\,{}^{0}R_1(H),\ \frac{(m-1)(n-1)}{(n+1)^2} + \frac{m-1}{(2n-1)^2} + \frac{n-1}{(mn-n+1)^2} + \frac{1}{(mn-1)^2}\}$, where ${}^{0}R_1(G)$ is defined in Definition 2.1, $m = |V(G)| \geq 2$, $n = |V(H)| \geq 2$.

Proof. Because $n > d_H(b)$, we have $nd_G(a) + d_H(b) > 2(nd_G(a)d_H(b))^{0.5}$, where $b \in V(H)$, we have ${}^{m}M_1(G[H]) = \sum_{(a,b)\in V(G[H])} \frac{1}{[nd_G(a)+d_H(b)]^2}$ $< \sum_{(a,b)\in V(G[H])} \frac{1}{4nd_G(a)d_H(b)} = \frac{1}{4n}\,{}^{0}R_1(G)\,{}^{0}R_1(H)$. Since G[H] is a subgraph of $K_m[K_n] = K_{mn}$, by Theorem 3.2 we have ${}^{m}M_1(G[H]) \geq \frac{mn}{(mn-1)^2}$.

Claim: Let a, b and k be natural numbers, $a \geq k + 1$, $a \leq b$, $k \geq 1$, we have $\frac{1}{a^2}$

$+\frac{1}{b^2} < \frac{1}{(a-k)^2} + \frac{1}{(b+k)^2}$.

In fact, let $f(x) = \frac{1}{x^2} - \frac{1}{(x+k)^2}$, $x > 0$, we have $f'(x) = -2(\frac{1}{x^3} - \frac{1}{(x+k)^3})$ < 0. Hence, f(x) is a decreasing function. Thus, we have $\frac{1}{b^2} - \frac{1}{(b+k)^2} < \frac{1}{(a-k)^2} - \frac{1}{a^2}$. The claim follows.

By Lemma 2.5 and the definition of ${}^mM_1$, if edge e does not belong to E(G), we have ${}^mM_1((G+e)[H]) < {}^mM_1(G[H])$. Similarly, if edge e does not belong to E(H), we have ${}^mM_1(G[H+e]) < {}^mM_1(G[H])$. Hence, when ${}^mM_1(G[H])$ attains its maximum, both G and H are trees. If $G \neq K_{1,m-1}$, by Lemma 2.4, without loss of generality, let $d_G(a_1) = 1$, $a_1a_2 \in E(G)$, $d_G(a_i) = \Delta(G)$. Then, we let $G' = G - a_1a_2 + a_1a_i$. By Lemma 2.5, in G[H] we have $d_{G[H]}(a_1, b) = n + d_H(b)$, $d_{G[H]}(a_2, b) = nd_G(a_2) + d_H(b)$, $d_{G[H]}(a_i, b) = nd_G(a_i) + d_H(b)$, $d_{G[H]}(a, b) = nd_G(a) + d_H(b)$, where $a \neq a_1, a_2, a_i$. Similarly, in $G'[H]$ we have $d_{G'[H]}(a_1, b) = n + d_H(b)$, $d_{G'[H]}(a_2, b) = n(d_G(a_2) - 1) + d_H(b)$, $d_{G'[H]}(a_i, b) = n(d_G(a_i) + 1) + d_H(b)$, $d_{G'[H]}(a, b) = nd_G(a) + d_H(b)$, where $a \neq a_1, a_2, a_i$. By the claim above we have ${}^mM_1(G[H]) < {}^mM_1(G'[H])$. If $G' \neq K_{1,n-1}$, we can do as above. Hence, considering the maximum of ${}^mM_1(G[H])$, we can suppose $G = K_{1,m-1}$.

Similarly, If $H \neq K_{1,n-1}$, by Lemma 2.4, without loss of generality, let $d_H(b_1) = 1$, $b_1b_2 \in E(H)$, $d_H(b_i) = \Delta(H)$. Then, we let $H' = H - b_1b_2 + b_1b_i$. By Lemma 2.5, in G[H] we have $d_{G[H]}(a, b_1) = nd_G(a) + d_H(b_1)$, $d_{G[H]}(a, b_2) = nd_G(a) + d_H(b_2)$, $d_{G[H]}(a, b_i) = nd_G(a) + d_H(b_i)$, $d_{G[H]}(a, b) = nd_G(a) + d_H(b)$, where $b \neq b_1, b_2, b_i$. Similarly, in $G[H']$ we have $d_{G[H']}(a, b_1) = nd_G(a) + d_H(b_1)$, $d_{G[H']}(a, b_2) = nd_G(a) + d_H(b_2) - 1$, $d_{G[H']}(a, b_i) = nd_G(a_i) + d_H(b_i) + 1$, $d_{G[H']}(a, b) = nd_G(a) + d_H(b)$, where $b \neq b_1, b_2, b_i$. By the claim above we have ${}^mM_1(G[H]) < {}^mM_1(G[H'])$. If $H' \neq K_{1,n-1}$, we can do as above. Hence, considering the maximum of ${}^mM_1(G[H])$, we can suppose $H = K_{1,n-1}$. Thus, the maximum of ${}^mM_1(G[H])$ is attained by $K_{1,m-1}[K_{1,n-1}]$. By Theorem 3.4 the theorem follows.

Similarly, we have

Theorem 4.12. ${}^mM_2(P_m[P_n]) = \frac{mn^2 - 3n^2 - 3mn + m + 18n - 22}{(2n+2)^2} + \frac{2n-6}{(n+2)^2} +$

$$\frac{n^2-4n+8}{(n+1)(n+2)} + \frac{2mn-3m-6n+18}{(n+1)(2n+1)} + \frac{4m-12}{(2n+1)^2} + \frac{4(n-2)}{(n+2)(2n+1)}, \quad m, n \geq 3.$$

Theorem 4.13. $^{m}M_2(K_m[K_n]) = \frac{mn}{4(n+1)}$, where $m, n \geq 2$.

Theorem 4.14. $^{m}M_2(C_m[C_n]) = \frac{mn}{2(mn-1)}$, where $m, n \geq 3$.

Theorem 4.15. $^{m}M_2(K_{1,m-1}[K_{1,n-1}]) = \frac{1}{2}\{\frac{n-1}{mn-n+1}(\frac{(m-1)(n-1)}{n+1} + \frac{m-1}{2n-1} + \frac{1}{mn-1}) + \frac{1}{mn-1}(\frac{(m-1)(n-1)}{n+1} + \frac{m-1}{2n-1} + \frac{n-1}{mn-n+1}) + \frac{m-1}{2n-1}(\frac{n-1}{n+1} + \frac{n-1}{mn-n+1} + \frac{1}{mn-1}) + \frac{(m-1)(n-1)}{n+1}(\frac{n-1}{mn-n+1} + \frac{1}{2n-1} + \frac{1}{mn-1})\}$, where $m, n \geq 2$.

Theorem 4.16. $^{m}M_2(P_m[C_n]) = \frac{n^2}{(n+2)(n+1)} + \frac{2n}{(2+n)^2} + \frac{n(mn-3n+m-2)}{4(n+1)^2}$, where $m, n \geq 3$.

Theorem 4.17. $^{m}M_2(P_m[K_{1,n-1}]) = \frac{2(n-1)}{(n+1)(2n-1)} + \frac{2(n-1)^2}{(1+n)(2n+1)} + \frac{2(n-1)}{(3n-1)(n+1)} + \frac{2(n-1)}{4n^2-1} + \frac{2}{(2n-1)(3n-1)} + \frac{(3m-8)(n-1)}{(2n+1)(3n-1)} + \frac{(n-1)^2(m-3)}{(2n+1)^2} + \frac{m-3}{(3n-1)^2}$, where $m \geq 3, n \geq 2$.

Theorem 4.18. $^{m}M_2(C_m[K_{1,n-1}]) = \frac{m(n-1)^2}{(2n+1)^2} + \frac{m}{(3n-1)^2} + \frac{3m(n-1)}{(2n+1)(3n-1)}$, where $m \geq 3, n \geq 2$.

Theorem 4.19. $^{m}M_2(P_m[K_n]) = \frac{2n^2}{(2n-1)(3n-1)} + \frac{n(n-1)}{(2n-1)^2} + \frac{n(2n-1)}{(3n-1)^2} + \frac{n(m-4)}{2(3n-1)}$, where $m \geq 3, n \geq 2$.

Theorem 4.20. $^{m}M_2(C_m[K_n]) = \frac{mn}{2(3n-1)}$, where $m \geq 3, n \geq 2$.

Theorem 4.21. $^{m}M_2(K_m[K_{1,n-1}]) = \frac{m^2(n-1)}{(mn-n+1)(mn-1)}$

$$+\frac{m(m-1)(n-1)^2}{2(mn-n+1)^2}+\frac{m(m-1)}{2(mn-1)^2}$$, where m , n ≥ 2.

Theorem 4.22. Let G and H be simple connected graphs, we have ${}^{m}M_2(G[H]) < \frac{{}^{0}R_{-1}(G)\chi(H)+\chi(G)(R^0(H))^2}{4n}$, where $R^0(G)$ and $\chi(G)$ are defined in Definition 2.1, m = |V(G)| ≥ 2, n = |V(H)| ≥ 2.

Proof. Since $nd_G(w) + d_H(v) > 2\sqrt{nd_G(w)d_H(v)}$, we have ${}^{m}M_2(G[H]) =$

$$\sum_{w\in V(G)}\sum_{uv\in E(H)}\frac{1}{nd_G(w)+d_H(u)}\frac{1}{nd_G(w)+d_H(v)}+$$

$$\sum_{ab\in E(G)}\sum_{v\in V(H)}\sum_{u\in V(H)}\frac{1}{nd_G(a)+d_H(u)}\frac{1}{nd_G(b)+d_H(v)}<$$

$$\frac{1}{4n}\sum_{w\in V(G)}\sum_{uv\in E(H)}\frac{1}{d_G(w)\sqrt{d_H(u)d_H(v)}}+$$

$$\frac{1}{4n}\sum_{ab\in E(G)}\sum_{v\in V(H)}\sum_{u\in V(H)}\frac{1}{\sqrt{d_G(a)d_G(b)}}\frac{1}{\sqrt{d_H(u)}}\frac{1}{\sqrt{d_H(v)}}=$$

$$\frac{{}^{0}R_{-1}(G)\chi(H)+\chi(G)(R^0(H))^2}{4n}$$. The theorem follows.

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