The Sharp Lower Bound of the Least Eigenvalue of a Bicyclic Graph *

Shu-Guang Guo
School of Mathematical Sciences, Yancheng Teachers University,
Yancheng 224002, Jiangsu, P. R. China
E-mail: ychgsg@yahoo.com.cn

Abstract

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. In this paper we determine the first three graphs among all bicyclic graphs with n vertices, ordered according to their least eigenvalues in increasing order.

AMS Classifications(2000): 05C50
Keywords: Bicyclic graphs; Adjacency matrix; Eigenvalue; Bound

1. Introduction

The graphs in this paper are simple. Let $G = (V, E)$ be a graph on vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, and let $A(G)$ be a $(0,1)$-adjacency matrix of G. The characteristic polynomial of G is just $\det(\lambda I - A(G))$, denoted by $P(G;\lambda)$. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. In particular, we denote the largest eigenvalue $\lambda_1(G)$ by $\rho(G)$, called the spectral radius of G, and the least eigenvalue $\lambda_n(G)$ by $\lambda(G)$.

As G ranges over the collection of all simple graphs with n vertices, it is often required to know how small $\lambda(G)$ gets. Brigham and Dutton [2] proved

$$\lambda(G) \geq \max\left\{ -\sqrt{\frac{2mn_-}{1+n_-}}, -\sqrt{\frac{2mn_+}{1+n_+}} \right\},$$

*Supported by National Natural Science Foundation of China (10771103), Natural Science Foundation of Jiangsu Province (BK2008198) and Qing Lan Project of Jiangsu Province.

where n_+, n_- are the numbers of positive eigenvalues and negative eigenvalues of G, respectively. Constantine [4] showed that $\lambda(G) \geq -n/2$ if n is even or $\lambda(G) \geq -\sqrt{n^2-1}/2$ if n is odd. The same result was obtained by different means in [9] and [12]. For some special families of graphs, the above problem has also been studied by several authors. Let G be a simple graph with $n \geq 3$ vertices, orientable genus g and non-orientable genus h. Hong and Shu [10] defined the Euler characteristic $\mathcal{X}(G)$ of a graph G by $\mathcal{X}(G) = \max\{2-2g, 2-h\}$, and proved that $\lambda(G) \geq -\sqrt{2(n-\mathcal{X}(G))}$. In particular, if G is a planar graph, they obtained $\lambda(G) \geq -\sqrt{2n-4}$ with equality if and only if $G = K_{2,n-2}$. For series-parallel graphs, they obtained a similar result. Xu, Xu and Wang [15] gave a sharp lower bound on the least eigenvalue of a unicyclic graph with n vertices and characterized the graph attained the lower bound.

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. The eigenvalues of bicyclic graphs have been studied by many authors, for example, one may see [3, 7, 8, 16] and the references therein. Denote by $\mathfrak{B}_n$ the set of all bicyclic graphs with n vertices. In this paper we determine the first three graphs in $\mathfrak{B}_n$, ordered according to their least eigenvalues in increasing order.

2. Preliminaries

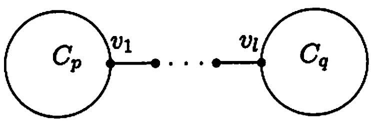

Fig. 1 $B(p,l,q)$

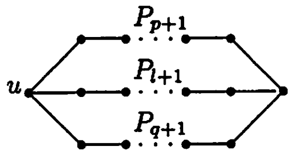

Fig. 2 $\theta(p,l,q)$

Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. Let C_p and C_q be two vertex-disjoint cycles. Suppose that v_1 is a vertex of C_p and v_l is a vertex of C_q. Joining v_1 and v_l by a path $v_1v_2\cdots v_l$ of length $l-1$, where $l \geq 1$ and $l=1$ means identifying v_1 with v_l, the resulting graph (Fig. 1), denoted by $B(p,l,q)$, is called an ∞-graph. Without loss of generality, we may assume that $p \leq q$. Let P_{l+1}, P_{p+1} and P_{q+1} be three vertex-disjoint paths, where $1 \leq l \leq p \leq q$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph (Fig. 2), denoted by $\theta(p,l,q)$, is called a θ-graph. Let $\mathcal{B}_n$ be the set of those graphs on n vertices each of which is an ∞-graph with trees attached, and Θ_n be the set of those graphs on n vertices each of which is a θ-graph with trees attached. Then $\mathfrak{B}_n = \mathcal{B}_n \cup \Theta_n$.

Denote by $B_n^*(p,1,q)$ the graph on n vertices obtained from $B(p,1,q)$ by attaching $n-p-q+1$ pendant edges at the vertex of degree 4, by $\theta_n^*(p,l,q)$ the graph on n vertices obtained from $\theta(p,l,q)$ by attaching $n-p-q-l+1$ pendant edges at one vertex of degree 3, and for $p=q=2$ or $p=q=3, l=1$, by $\theta_n^{**}(p,l,q)$ the graph on n vertices obtained from $\theta(p,l,q)$ by attaching $n-p-q-l+1$ pendant edges at one vertex of degree 2.

By the Perron-Frobenius theory of non-negative matrices and Theorem 8.8.2 in [6], we have $\lambda(G) \geq -\rho(G)$ with equality if and only if G is bipartite. If G is connected, then $A(G)$ is irreducible and by the Perron-Frobenius theory of non-negative matrices, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. We shall refer to such an eigenvector as the Perron vector of G. For $v \in V(G)$, $d(v)$ denotes the degree of vertex v and $N(v)$ denotes the set of all neighbors of vertex v in G.

The terminology not defined here can be found in [1, 5, 6]. In order to complete the proofs of our main results, we need the following lemmas.

Lemma 1 ([11, 14]). *Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G and $d(v)$ be the degree of vertex v. Suppose $v_1, v_2, \cdots, v_s \in N(v)\backslash N(u) (1 \leq s \leq d(v))$ and $x = (x_1, x_2, \cdots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i (1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i (1 \leq i \leq s)$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

As immediate consequences of Lemma 1, we have the following.

Lemma 2. *Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \cap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v, and adding a pendant edge to $u(=v)$. Then $\rho(G) < \rho(G^*)$.*

Proof. We use x_u and x_v to denote the components of the Perron vector of G corresponding to u and v. Suppose that $N(u) = \{v, v_1, \cdots, v_s\}$ and $N(v) = \{u, u_1, \cdots, u_t\}$. Since $e = uv$ is a non-pendant edge of G, it follows that $s, t \geq 1$. If $x_u \geq x_v$, let

$$G' = G - \{vu_1, \cdots, vu_t\} + \{uu_1, \cdots, uu_t\}.$$

If $x_u < x_v$, let

$$G'' = G - \{uv_1, \cdots, uv_s\} + \{vv_1, \cdots, vu_s\}.$$

Obviously, $G^* = G' = G''$. By Lemma 1, we have $\rho(G) < \rho(G^*)$. This completes the proof.

The following result is often used to calculate the characteristic polynomials of graphs.

Lemma 3[3, 5, 13]. *Let u be a vertex of G, and let C(u) be the set of all cycles containing u. The characteristic polynomial of G satisfies*

$$P(G;\lambda) = \lambda P(G-u;\lambda) - \sum_{v\in N(u)} P(G-u-v;\lambda) - 2\sum_{Z\in C(u)} P(G\backslash V(Z);\lambda).$$

Lemma 4[10]. *If G is a simple connected graph with n vertices, then there exists a connected bipartite subgraph G′ of G such that*

$$\lambda(G) \geq \lambda(G')$$

with equality if and only if $G = G'$.

For $1 \leq k \leq n-1$, denote by C_k^{n-k} the graph formed by attaching $n-k$ pendant edges at a vertex of the cycle C_k. The following two lemmas were given in [15].

Lemma 5[15]. *Let* $n \geq 4$, *F be a forest on n vertices and* $G \neq K_{1,n-1}$. *Then*

$$\lambda(F) > \lambda(C_4^{n-4}).$$

Lemma 6[15]. *Let* $n \geq 4$, *G be a unicyclic graph on n vertices and* $G \neq C_3^{n-3}$. *Then* $\lambda(G) \geq \lambda(C_4^{n-4})$, *and the equality holds if and only if* $G = C_4^{n-4}$.

Lemma 7. *If* $n \geq 6$, *then* $\rho(\theta^*(2, 2, 2)) > \rho(\theta^{**}(2, 2, 2))$, $\rho(\theta^*(2, 2, 2)) > \rho(\theta^*(3, 1, 3)) > \rho(\theta^{**}(3, 1, 3))$, $\rho(\theta^*(2, 2, 2)) > \rho(B_n^*(4, 1, 4))$.

Proof. Since $\theta^*(2, 2, 2)$, $\theta^{**}(2, 2, 2)$, $\theta^*(3, 1, 3)$ and $\theta^{**}(3, 1, 3)$ all have the star $K_{1,3}$ as an induced subgraph, it follows that their spectral radii are greater than $\sqrt{3}$. Applying Lemma 3 to their highest degree vertices respectively, we have

$$\begin{aligned}
P(\theta^*(2, 2, 2);\lambda) &= \lambda^{n-4}[\lambda^4 - (n+1)\lambda^2 + 3(n-5)],\\
P(\theta^{**}(2, 2, 2);\lambda) &= \lambda^{n-4}[\lambda^4 - (n+1)\lambda^2 + 4(n-5)],\\
P(\theta^*(3, 1, 3);\lambda) &= \lambda^{n-6}[\lambda^6 - (n+1)\lambda^4 + (4n-17)\lambda^2 - 3n + 17],\\
P(\theta^{**}(3, 1, 3);\lambda) &= \lambda^{n-6}[\lambda^6 - (n+1)\lambda^4 + (5n-23)\lambda^2 - 2n + 11].
\end{aligned}$$

It follows that when $n > 6$,

$$P(\theta^*(2, 2, 2);\lambda) < P(\theta^{**}(2, 2, 2);\lambda),$$

$$P(\theta^*(3, 1, 3); \lambda) < P(\theta^{**}(3, 1, 3); \lambda)$$

and

$$P(\theta^*(2, 2, 2); \lambda) < P(\theta^*(3, 1, 3); \lambda)$$

hold for $\lambda \geq \sqrt{3}$. These imply that

$$\rho(\theta^*(2, 2, 2)) > \rho(\theta^{**}(2, 2, 2)), \quad \rho(\theta^*(3, 1, 3)) > \rho(\theta^{**}(3, 1, 3)),$$

and

$$\rho(\theta^*(2, 2, 2)) > \rho(\theta^*(3, 1, 3)).$$

Applying Lemma 3 to the highest degree vertex of $B_n^*(4, 1, 4))$, we have

$$P(B_n^*(4, 1, 4); \lambda) = \lambda^{n-6}[\lambda^6 - (n+1)\lambda^4 + 4(n-4)\lambda^2 - 4(n-7)].$$

Since the star $K_{1,4}$ is an induced subgraph of $B_n^*(4, 1, 4)$, it follows that $\rho(B_n^*(4, 1, 4)) > \rho(K_{1,4}) = 2$. Thus, we have

$$P(B_n^*(4, 1, 4); \lambda) - P(\theta^*(2, 2, 2); \lambda) = \lambda^{n-6}[(n-1)\lambda^2 - 4(n-7)] > 0$$

holds for $\lambda \geq \rho(B_n^*(4, 1, 4))$. This implies $\rho(\theta^*(2, 2, 2)) > \rho(B_n^*(4, 1, 4))$. The proof is complete.

Lemma 8. (1) *If* $8 \leq n \leq 27$, *then*

$$\lambda(\theta^*(2, 2, 2)) < \lambda(B_n^*(3, 1, 3)) < \lambda(\theta^*(2, 1, 2)).$$

(2) *If* $28 \leq n \leq 29$, *then*

$$\lambda(\theta^*(2, 1, 2)) < \lambda(\theta^*(2, 2, 2)) < \lambda(B_n^*(3, 1, 3)).$$

(3) *If* $n \geq 30$, *then*

$$\lambda(\theta^*(2, 2, 2)) > \lambda(B_n^*(3, 1, 3)) > \lambda(\theta^*(2, 1, 2)).$$

Proof. Applying Lemma 3 to the highest degree vertices of $\mathcal{B}_n^*(3, 1, 3)$ and $\theta_n^*(2, 1, 2)$, respectively, we have

$$\begin{aligned} P(\mathcal{B}_n^*(3, 1, 3); \lambda) &= \lambda^{n-6}(\lambda^2 - 1)(\lambda^4 - n\lambda^2 - 4\lambda + n - 5), \\ P(\theta_n^*(2, 1, 2); \lambda) &= \lambda^{n-4}[\lambda^4 - (n+1)\lambda^2 - 4\lambda + 2(n-4)]. \end{aligned}$$

From these and the proof of Lemma 7, we can see $\lambda(\theta_n^*(2, 2, 2))$, $\lambda(\mathcal{B}_n^*(3, 1, 3))$, $\lambda(\theta_n^*(2, 1, 2))$ are the least roots of following equations respectively:

$$\begin{aligned} &\lambda^4 - (n+1)\lambda^2 + 3(n-5) = 0, \\ &\lambda^4 - n\lambda^2 - 4\lambda + n - 5 = 0, \\ &\lambda^4 - (n+1)\lambda^2 - 4\lambda + 2(n-4) = 0. \end{aligned}$$

By the straightforward calculation via the MATLAB Programming, we can verify that(1) and (2) hold.

Now we show (3) holds. Denote $f(\lambda) = \lambda^4 - n\lambda^2 - 4\lambda + n - 5$ and $g(\lambda) = \lambda^4 - (n+1)\lambda^2 - 4\lambda + 2(n-4)$. Then

$$f(\lambda) - g(\lambda) = \lambda^2 - (n-3),$$

$\lambda(\mathcal{B}_n^*(3,\ 1,\ 3))$ and $\lambda(\theta_n^*(2,\ 1,\ 2))$ are the least roots of the equations $f(\lambda) = 0$ and $g(\lambda) = 0$, respectively. Since

$$f(-\sqrt{n-3}) = -2n + 4 + 4\sqrt{n-3} < 0$$

holds for $n \geq 6$, it follows that $\lambda(\mathcal{B}_n^*(3,\ 1,\ 3)) < -\sqrt{n-3}$. Thus

$$f(\lambda(\mathcal{B}_n^*(3,\ 1,\ 3))) - g(\lambda(\mathcal{B}_n^*(3,\ 1,\ 3))) = \lambda(\mathcal{B}_n^*(3,\ 1,\ 3))^2 - (n-3) > 0,$$

and so $g(\lambda(\mathcal{B}_n^*(3,\ 1,\ 3))) < 0$. Thus $\lambda(\mathcal{B}_n^*(3,\ 1,\ 3)) > \lambda(\theta_n^*(2,\ 1,\ 2))$.

Since

$$f\left(\lambda(\theta_n^*(2,\ 2,\ 2))\right) = f\left(-\sqrt{\frac{n+1+\sqrt{(n-5)^2+36}}{2}}\right)$$

$$= -\frac{3}{2}n + \frac{21}{2} + \frac{1}{2}\sqrt{(n-5)^2+36}$$

$$+ 2(2n + 2 + 2\sqrt{(n-5)^2+36}\ )^{1/2} < 0$$

holds for $n \geq 30$, it follows that when $n \geq 30$,

$$\lambda(\theta_n^*(2,\ 2,\ 2)) > \lambda(\mathcal{B}_n^*(3,\ 1,\ 3)).$$

The proof is complete.

Lemma 9. *Let $n \geq 8$ and G be a bicyclic graph on n vertices with girth $g \geq 4$. Then*

$$\rho(G) \leq \sqrt{\frac{n+1+\sqrt{(n-5)^2+36}}{2}},$$

and the equality holds if and only if $G = \theta^*(2,\ 2,\ 2)$.

Proof. Let G be a bicyclic graph on n vertices with girth $g \geq 4$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \cdots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \cdots, x_n)$, where x_i corresponds to the vertex $v_i (1 \leq i \leq n)$. Let $\mathcal{B}_n(g \geq 4)$ and $\Theta_n(g \geq 4)$ denote the sets of all bicyclic graphs with girth $g \geq 4$ in $\mathcal{B}_n$ and Θ_n, respectively. We consider the following two cases.

Case 1. $G \in \mathcal{B}_n(g \geq 4)$. We first prove that G is an ∞-graph $B(p,1,q)$ with one tree T attached to the vertex of degree 4, denoted by v_1. Let $B(p,l,q)$ be the ∞-graph in G, and $v_1 v_2 \cdots v_l$ be the path joining the cycles C_p and C_q in $B(p,l,q)$.

We claim that $l = 1$. Assume, on the contrary, that $l > 1$. Applying Lemma 2 to the edge $e = v_1 v_2$, we obtain a graph $G^* \in \mathcal{B}_n(g \geq 4)$ such that $\rho(G^*) > \rho(G)$, a contradiction. Hence $l = 1$.

Assume that there exists a vertex v_i of $B(p,1,q)$ such that $v_i \neq v_1$ and there exist a tree T attached to v_i. By symmetry, we may assume that v_i is a vertex of C_p. Denote $N(v_i) = \{v_{i-1}, v_{i+1}, z_1, \cdots, z_s\}$, and $N(v_1) = \{v_{j-1}, v_{j+1}, w_1, \cdots, w_t\}$, where $v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}$ are vertices of C_p. Then $s \geq 1$ and $t \geq 2$. If $x_1 \geq x_i$, let

$$G^* = G - \{v_i z_1, \cdots, v_i z_s\} + \{v_1 z_1, \cdots, v_1 z_s\}.$$

If $x_1 < x_i$, let

$$G^* = G - \{v_1 w_1, \cdots, v_1 w_t\} + \{v_i w_1, \cdots, v_i w_t\}.$$

Then in either case $G^* \in \mathcal{B}_n(g \geq 4)$. By Lemma 1, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence G has a unique attached tree.

We second prove that each vertex of T not in $V(B(p,1,q))$ has degree 1, i.e., G is an ∞-graph $B(p,1,q)$ with some pendant edges attached to v_1. On the contrary, if there exists one vertex v_i of T such that $v_i \notin V(B(p,1,q))$ and $d(v_i) \geq 2$, then there exist a path joining v_1 and v_i. Without loss of generality, we may assume that v_i is adjacent to v_1. Applying Lemma 2 to edge $v_1 v_i$, we get a graph $G^* \in \mathcal{B}_n(g \geq 4)$ such that $\rho(G^*) > \rho(G)$, a contradiction. Hence G is an ∞-graph $B(p,1,q)$ with some pendant edges attached to v_1.

Finally, we show that both C_p and C_q have length 4. Assume that $p \geq 5$. Let $C_p = v_1 v_2 \cdots v_p v_1$. Applying Lemma 2 to edge $v_1 v_2$, we get a graph $G^* \in \mathcal{B}_n(g \geq 4)$ such that $\rho(G^*) > \rho(G)$, a contradiction. Therefore $p = 4$. Similarly, we can verify that $q = 4$.

From the above arguments, we have $G = B_n^*(4,1,4)$. This contradicts Lemma 7.

Case 2. $G \in \Theta_n(g \geq 4)$. Using Lemma 1, Lemma 2 and Lemma 7, by similar arguments to the proof of Case 1, we can show $G = \theta_n^*(2,2,2)$.

Combining Case 1 and Case 2, we have $G = \theta_n^*(2,2,2)$. From the proof of Lemma 7, we have

$$\rho(G) = \sqrt{\frac{n+1+\sqrt{(n-5)^2+36}}{2}}.$$

This completes the proof.

3. Main results

Theorem 1. *Let $n > 8$, G be a bicyclic graph on n vertices and $G \neq B_n^*(3,1,3), \theta_n^*(2,1,2)$. Then*

$$\lambda(G) \geq -\sqrt{\frac{n+1+\sqrt{(n-5)^2+36}}{2}},$$

and the equality holds if and only if $G = \theta_n^*(2,2,2)$.

Proof. **Case 1.** The girth of G is greater than or equal to 4. Since $\theta_n^*(2,2,2)$ is bipartite graph, it follows from Lemma 9 that

$$\lambda(\theta_n^*(2,2,2)) = -\rho(\theta_n^*(2,2,2)) \leq -\rho(G) \leq \lambda(G),$$

and $\lambda(G) = \lambda(\theta_n^*(2,2,2))$ if and only if $G = \theta^*(2, 2, 2)$.

Case 2. The girth of G is equal to 3. By Lemma 4, there exists a spanning bipartite subgraph G' of G such that $\lambda(G) \geq \lambda(G')$. Obviously, G' is either a tree or a bipartite unicyclic graph. Since $G \neq B_n^*(3,1,3), \theta_n^*(2,1,2)$, it follows that $G' \neq K_{1,n-1}$. By Lemma 5 and Lemma 6, we have

$$\lambda(G') \geq \lambda(C_4^{n-4}),$$

and so $\lambda(G) \geq \lambda(C_4^{n-4})$.

Applying Lemma 3 to the highest degree vertex of C_4^{n-4}, we have

$$P(C_4^{n-4};\lambda) = \lambda^{n-4}[\lambda^4 - n\lambda^2 + 2(n-4)],$$

and so

$$\lambda(C_4^{n-4}) = -\sqrt{\frac{n+\sqrt{(n-4)^2+16}}{2}}.$$

By the proof of Lemma 7, we have

$$\lambda(\theta^*(2,2,2)) = -\sqrt{\frac{n+1+\sqrt{(n-5)^2+36}}{2}} < \lambda(C_4^{n-4}).$$

Combining the above arguments, we obtain a proof of Theorem 1.

By Theorem 1 and Lemma 8, we have the following Theorem 2 immediately.

Theorem 2. *Let G be a bicyclic graph on n vertices.*

(1) *If* $8 \leq n \leq 27$ *and* $G \neq \theta_n^*(2, 2, 2)$, *then* $\lambda(G) > \lambda(\theta_n^*(2, 2, 2))$.

(2) *If* $28 \leq n \leq 29$ *and* $G \neq \theta_n^*(2, 2, 2)$, $\theta_n^*(2,1,2)$, *then*

$$\lambda(G) > \lambda(\theta_n^*(2, 2, 2)) > \lambda(\theta_n^*(2, 1, 2)).$$

(3) *If* $n \geq 30$ *and* $G \neq \theta_n^*(2, 2, 2)$, $B_n^*(3,1,3)$, $\theta_n^*(2,1,2)$, *then*

$$\lambda(G) > \lambda(\theta_n^*(2, 2, 2)) > \lambda(B_n^*(3, 1, 3)) > \lambda(\theta_n^*(2, 1, 2)).$$

Acknowledgments

The author would like to thank the anonymous referees for valuable suggestions and corrections which result in an improvement of the original manuscript.

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, The Macmillan Press Ltd,1976.

[2] R. C. Brigham, R. D. Dutton, Bounds on graph spectra, J. Combin. Theory Ser. B37(1984)228-234.

[3] A. Chang, F. Tian, A. Yu, On the index of bicyclic graphs with perfect matchings, Discrete. Math.283(2004)51-59.

[4] G. Constantine, Lower bounds on the spectra of symmetric matrices with nonnegative entries, Linear Algebra Appl.65(1985)33-39.

[5] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1980.

[6] C. Godsil, G. Royle, Algebraic Graph Theory, Grad. Texts in Math. 207, Springer-Verlag, 2001.

[7] S. G. Guo, On bicyclic graphs whose second largest eigenvalue does not exceed 1, Linear Algebra Appl.407(2005)201-210.

[8] S. G. Guo, The spectral radius of unicyclic and bicyclic graphs with n vertices and k pendant vertices, Linear Algebra Appl.408(2005)78-85.

[9] Y. Hong, Bounds of eigenvalues of a graph, Acta Math. Appl. Sinica4(1998)165-168.

[10] Y. Hong, J. L. Shu, Sharp lower bounds of the least eigenvalue of planar graphs, Linear Algebra Appl.296(1999)(1-3)227-232.

[11] H. Liu, M. Lu, F. Tian, On the spectral radius of graphs with cut edges, Linear Algebra Appl.389(2004)139-145.

[12] P. L. Powers, Graph partitioning by eigenvectors, Linear Algebra Appl.101(1988)121-133.

[13] A. Schwenk, Computing the characteristic polynomial of a graph. In:R. Bary , F. Harary ed., Graphs and Combinatorics–Lecture Notes in Mathematics, No. 406, Berlin: Springer-Verlag, 1974, 153-172.

[14] B. F. Wu, E. L. Xiao, Y. Hong, The spectral radius of trees on k pendant vertices, Linear Algebra Appl.395(2005)343-349.

[15] G. H. Xu, Q. F. Xu, S. K. Wang, A sharp lower bound on the least eigenvalue of unicyclic graph, J. of Ningbo University(NSEE).16(3)(2003)225-227. (in Chinese)

[16] A. M. Yu, F. Tian, On the spectral radius of bicyclic graphs, MATCH Commun. Math. Comput. Chem.52(2004)91-101.