# Domination by Union of Complete Graphs

Vladimir Samodivkin
Department of Mathematics
University of Architecture Civil Engineering and Geodesy
Hristo Smirnenski 1 Blv., 1046 Sofia, Bulgaria,
vlsam_fte@uacg.bg

**Abstract**

Let $G$ be a graph with domination number $\gamma(G)$. A dominating set $S \subseteq V(G)$ has property $\mathcal{UK}$ if all components of the subgraph it induces in $G$ are complete. The *union of complete graphs domination number* of a graph $G$, denoted $\gamma_{uk}(G)$, is the minimum possible size of a dominating set of $G$, which has property $\mathcal{UK}$. Results on changing and unchanging of $\gamma_{uk}$ after vertex removal are presented. Also forbidden subgraph conditions sufficient to imply $\gamma(G) = \gamma_{uk}(G)$ are given.

**Keywords:** conditional domination, acyclic domination, independent domination, induced-paired domination, forbidden graph.

**MSC 2000: 05C69**

# 1 Introduction

We discuss only finite undirected graphs without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [9]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. The number of components of a graph $G$ is denoted by $c(G)$. For a vertex $x$ of $G$, $N(x, G)$ denote the set of all neighbors of $x$ in $G$ and $N[v, G] = N(v, G) \cup \{v\}$. The maximum degree of the graph $G$ is denoted by $\Delta(G)$. $C_n$ will denote the cycle on $n$ vertices, $K_m$ the complete graph on $m$ vertices, $sK_m$ the union of $s$ disjoint copies of $K_m$ and $\overline{K}_m$ the complement of $K_m$. Let $\{G_1, G_2, ..\}$ be a given (finite or infinite) set of non isomorphic undirected graphs. We denote by $Forb(G_1, G_2, ..)$ the class of all graphs containing no induced subgraph isomorphic to any $G_i$, $i \geq 1$. A *dominating set* for a graph $G$ is a set of vertices $D \subseteq V(G)$ such that every vertex of $G$ is either in $D$ or is adjacent to an element of $D$. The *domination number* $\gamma(G)$ of a graph $G$ is the minimum cardinality taken over all dominating sets of $G$. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [9, 10].

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A *graph property* is any non-empty subset of $\mathcal{G}$. We say that a *graph $G$ has property* $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to $G$. We list some properties in order to introduce the notion which will be used in the paper:

$\mathcal{F} = \{H \in \mathcal{G} : H \text{ is a forest }\}$;
$\mathcal{I} = \{\overline{K}_1, \overline{K}_2, \overline{K}_3, ...\}$;
$\mathcal{M} = \{K_2, 2K_2, 3K_2, ...\}$;
$\mathcal{C} = \{K_1, K_2, K_3, ...\}$.

Any dominating set $S \subseteq V(G)$ such that the subgraph $\langle S, G \rangle$ satisfies property $\mathcal{P}$ is called a $\mathcal{P}$-dominating set. Harary and Haynes [8] defined the *conditional domination number* $\gamma(G : \mathcal{P})$ as the smallest cardinality of a $\mathcal{P}$-dominating set of $G$. It follows by this definition that if $\mathcal{P}_1 \subseteq \mathcal{P} \subseteq \mathcal{G}$ and $\gamma(G : \mathcal{P}_1)$ exists then $\gamma(G : \mathcal{P}_1) \geq \gamma(G : \mathcal{P}) \geq \gamma(G : \mathcal{G}) = \gamma(G)$. Any $\mathcal{P}$-dominating set with minimum cardinality is called a *$\gamma(G : \mathcal{P})$-set*. A vertex $v$ of a graph $G$ is *$\gamma(G : \mathcal{P})$-critical* if $\gamma(G - v : \mathcal{P}) \neq \gamma(G : \mathcal{P})$. The graph $G$ is $\gamma(G : \mathcal{P})$ - *critical* if all its vertices are $\gamma(G : \mathcal{P})$ - critical.

Note that the conditional domination numbers $\gamma(G : \mathcal{F})$, $\gamma(G : \mathcal{I})$, $\gamma(G : \mathcal{M})$ and $\gamma(G : \mathcal{C})$ are the well known *acyclic domination number* $\gamma_a(G)$ [11] , *independent domination number* $i(G)$, *induced-paired domination number* $\gamma_{ip}$ [13] and *clique domination number* $\gamma_{cl}(G)$ [6] , respectively. Since $\mathcal{M} \subset \mathcal{F}$ and $\mathcal{I} \subset \mathcal{F}$ then $\gamma_{ip}(G) \geq \gamma_a(G)$ and [11] $\gamma_a(G) \leq i(G)$.

In this paper we introduce the study of a new type conditional domination parameter as follows. Let the *union of complete graphs* property, denoted $\mathcal{UK}$, be:

- $\mathcal{UK} = \{H \in \mathcal{G} : \text{each component of } H \text{ is complete }\}$.

The conditional domination number $\gamma(G : \mathcal{UK})$ will be called the *union of complete graphs domination number* and will be denoted by $\gamma_{uk}(G)$. Since $\mathcal{I} \subseteq \mathcal{UK}$ and $\mathcal{M} \subseteq \mathcal{UK}$, then $\gamma(G) \leq \gamma_{uk}(G) \leq i(G)$ and $\gamma(G) \leq \gamma_{uk}(G) \leq \gamma_{ip}(G)$ (when $\gamma_{ip}(G)$ exists).

We shall consider and the following subsets of $\mathcal{UK}$:

- $\mathcal{UK}_s = \{H \in \mathcal{UK} : \text{each component of } H \text{ has order at most } s\}$, $s \geq 1$.

The conditional domination number $\gamma(G : \mathcal{UK}_s)$ will be denoted by $\gamma_{uk_s}(G)$. By these definitions we immediately have that for any $s \geq 1$, $\gamma_{uk_s}(G) \geq$

$\gamma_{uk_{s+1}}(G) \geq \gamma_{uk}(G)$. Note that since $\mathcal{UK}_1 = \mathcal{I}$ then $\gamma_{uk_1}(G) = i(G)$, and since $\mathcal{UK}_2 \supseteq \mathcal{M}$ and $\mathcal{UK}_2 \subseteq \mathcal{F}$ then $\gamma_{uk_2}(G) \leq \gamma_{ip}(G)$ (when $\gamma_{ip}(G)$ exists) and $\gamma_{uk_2}(G) \geq \gamma_a(G)$.

We proceed as follows. In Section 2, we examine critical vertices in a graph with respect to the union of complete graphs domination number and give a necessary and sufficient condition for a graph to be $\gamma_{uk}$-critical. In Section 3 we present some classes of graphs with equal domination and union of complete graphs domination numbers.

# 2 Vertex Deletion

Much has been written about the effects on a parameter (such connectedness, chromatic number, domination number) when a graph is modified by deleting a vertex. $\gamma(G : \mathcal{P})$ - critical graphs for $\gamma(G : \mathcal{P}) = \gamma, i$ was investigated by Brigham et al. [3] and Ao and MacGillivray (see [10, Chapter 16]) respectively. Further properties on these graphs can be found in [2], [7], [9, Chapter 5], [10, Chapter 16], [12].

Troughout this section, let $\mathcal{K} \in \{\mathcal{UK}; \mathcal{UK}_1, \mathcal{UK}_2, ...\}$ and for any graph $G$, $\gamma(G : \mathcal{K})$ will be denoted by $\gamma_{\cup}(G)$. Here some properties of critical vertices with respect to $\gamma_{\cup}$ will be given.

**Theorem 2.1.** *Let $G$ be a graph of order $n \geq 2$ and $u, v \in V(G)$.*

(i) *Let $\gamma_{\cup}(G - v) < \gamma_{\cup}(G)$.*

 (i.1) *If $uv \in E(G)$ then $u$ belongs to no $\gamma_{\cup}$-set of $G - v$;*

 (i.2) *If $M$ is a $\gamma_{\cup}$-set of $G - v$ then $M \cup \{v\}$ is a $\gamma_{\cup}$-set of $G$ and $v$ is isolated in $\langle M \cup \{v\}, G\rangle$;*

 (i.3) *$\gamma_{\cup}(G - v) = \gamma_{\cup}(G) - 1$;*

(ii) *Let $\gamma_{\cup}(G - v) > \gamma_{\cup}(G)$. Then $v$ belongs to every $\gamma_{\cup}$-set of $G$;*

(iii) *If $\gamma_{\cup}(G - v) < \gamma_{\cup}(G)$ and $u$ belongs to every $\gamma_{\cup}$-set of $G$ then $uv \notin E(G)$;*

(iv) *If $v$ belongs to no $\gamma_{\cup}$-set then $\gamma_{\cup}(G - v) = \gamma_{\cup}(G)$;*

(v) *If $\gamma_{\cup}(G-v) < \gamma_{\cup}(G)$ and $uv \in E(G)$ then $\gamma_{\cup}(G-\{u, v\}) = \gamma_{\cup}(G)-1$;*

(vi) *Let $v$ belong to every $\gamma_{\cup}$-set of $G$ and $\gamma_{\cup}(G - v) = \gamma_{\cup}(G)$. If $uv \in E(G)$ then $u$ belongs to no $\gamma_{\cup}$-set of $G-v$ and $\gamma_{\cup}(G-\{u, v\}) = \gamma_{\cup}(G)$.*

**Proof.** (i.1): Let $uv \in E(G)$ and $M$ be a $\gamma_{\cup}$-set of $G - v$. If $u \in M$ then $M$ will be a $\mathcal{K}$-dominating set of $G$ with $|M| < \gamma_{\cup}(G)$ - a contradiction.

(i.2) and (i.3): If $M$ is a $\gamma_{\cup}$-set of $G - v$ then (i.1) implies that $M_1 = M \cup \{v\}$ is a $\mathcal{K}$-dominating set of $G$, $v$ is isolated in $\langle M_1, G\rangle$ and $|M_1| = \gamma_{\cup}(G - v) + 1 \leq \gamma_{\cup}(G)$. Hence $M_1$ is a $\gamma_{\cup}$ - set of $G$ and $\gamma_{\cup}(G - v) = \gamma_{\cup}(G) - 1$.

(ii): If $M$ is a $\gamma_{\cup}$-set of $G$ and $v \notin M$ then $M$ is a $\mathcal{K}$-dominating set of $G - v$. But then $\gamma_{\cup}(G) = |M| \geq \gamma_{\cup}(G - v) > \gamma_{\cup}(G)$ and the result follows.

(iii): Let $\gamma_{\cup}(G - v) < \gamma_{\cup}(G)$ and $M$ be a $\gamma_{\cup}$ - set of $G - v$. Then by (i.2), $M \cup \{v\}$ is a $\gamma_{\cup}$-set of $G$. This implies that $u \in M$ and by (i.1) - $uv \notin E(G)$.

(iv): By (ii), $\gamma_{\cup}(G - v) \leq \gamma_{\cup}(G)$ and by (i.2), $\gamma_{\cup}(G - v) \geq \gamma_{\cup}(G)$.

(v): Immediately follows by (i) and (iv).

(vi): Let $M$ be a $\gamma_{\cup}$-set of $G - v$ and $uv \in E(G)$. If $u \in M$ then $M$ will be a $\mathcal{K}$-dominating set of $G$ of cardinality $\gamma_{\cup}(G - v) = \gamma_{\cup}(G)$ with $v \notin M$ - a contradiction. Now by (iv), $\gamma_{\cup}(G - \{u, v\}) = \gamma_{\cup}(G - v) = \gamma_{\cup}(G)$. ■

**Theorem 2.2.** *Let $G$ be a graph of order $n \geq 2$. Then $G$ is a $\gamma_{\cup}$-critical graph if and only if $\gamma_{\cup}(G - v) = \gamma_{\cup}(G) - 1$ for all $v \in V(G)$.*

**Proof.** Necessity is obvious.

Sufficiency: Let $G$ be a $\gamma_{\cup}$-critical graph. Clearly for every isolated vertex $v \in V(G)$, $\gamma_{\cup}(G - v) = \gamma_{\cup}(G) - 1$. Hence if $G$ is isomorphic to $\overline{K}_n$ then $\gamma_{\cup}(G - v) = \gamma_{\cup}(G) - 1$ for all $v \in V(G)$. So, let $G$ have a component of order at least two, say $Q$. Because of Theorem 2.1 (ii), (iii) and (i.3), either for all $v \in V(Q)$, $\gamma_{\cup}(Q - v) > \gamma_{\cup}(Q)$ or for all $v \in V(Q)$, $\gamma_{\cup}(Q - v) = \gamma_{\cup}(Q) - 1$. Suppose, for all $v \in V(Q)$, $\gamma_{\cup}(Q - v) > \gamma_{\cup}(Q)$. But then Theorem 2.1 (ii) implies that $V(Q)$ is a $\gamma_{\cup}$-set of $Q$. This is a contradiction with $\gamma_{\cup}(Q - v) > \gamma_{\cup}(Q)$. ■

**Theorem 2.3.** *Let $G_1$ and $G_2$ be graphs, $V(G_1) \cap V(G_2) = \{x\}$, and $G = G_1 \cup G_2$. Then $\gamma_{\cup}(G) \geq \gamma_{\cup}(G_1) + \gamma_{\cup}(G_2) - 1$.*

**Proof.** Let $M$ be a $\gamma_{\cup}$-set of $G$ and $M_i = M \cap V(G_i)$, $i = 1, 2$. There exist three possibilities:

(a) $x \notin M$ and $M_i$ is a $\mathcal{K}$-dominating set of $G_i$, $i = 1, 2$;

(b) $x \notin M$ and there are $i, j$ such that $\{i, j\} = \{1, 2\}$, $M_i$ is a $\mathcal{K}$-dominating set of $G_i$ and $M_j$ is a $\mathcal{K}$-dominating set of $G_j - x$;

(c) $x \in M$ and $M_i$ is a $\mathcal{K}$-dominating set of $G_i$, $i = 1, 2$.

If (a) holds, then $\gamma_{\cup}(G) = |M| = |M_1| + |M_2| \geq \gamma_{\cup}(G_1) + \gamma_{\cup}(G_2)$. If (b) holds, then $\gamma_{\cup}(G) = |M| = |M_1| + |M_2| \geq \gamma_{\cup}(G_i) + \gamma_{\cup}(G_j - x) \geq \gamma_{\cup}(G_1) + \gamma_{\cup}(G_2) - 1$. If (c) holds then $\gamma_{\cup}(G) = |M| = |M_1| + |M_2| - 1 \geq \gamma_{\cup}(G_1) + \gamma_{\cup}(G_2) - 1$.

Thus, in all casses, $\gamma_{\cup}(G) \geq \gamma_{\cup}(G_1) + \gamma_{\cup}(G_2) - 1$. ■

**Theorem 2.4.** *Let $G_1$ and $G_2$ be graphs, $V(G_1) \cap V(G_2) = \{x\}$, $\gamma_{\mathcal{U}}(G_1 - x) < \gamma_{\mathcal{U}}(G_1)$ and $G = G_1 \cup G_2$. Then:*

(i) $\gamma_{\mathcal{U}}(G) = \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2) - 1$;

(ii) *If $\gamma_{\mathcal{U}}(G_2 - x) < \gamma_{\mathcal{U}}(G_2)$ then $\gamma_{\mathcal{U}}(G - x) = \gamma_{\mathcal{U}}(G) - 1$;*

(iii) *If $\gamma_{\mathcal{U}}(G_2 - x) > \gamma_{\mathcal{U}}(G_2)$ then $x$ belongs to every $\gamma_{\mathcal{U}}$-set of $G$;*

(iv) *If $x$ belongs to no $\gamma_{\mathcal{U}}$-set of $G_2$ then $x$ belongs to no $\gamma_{\mathcal{U}}$-set of $G$.*

**Proof.** (i): Let $U_1$ be a $\gamma_{\mathcal{U}}$-set of $G_1 - x$ and $U_2$ be a $\gamma_{\mathcal{U}}$ - set of $G_2$. It follows by Theorem 2.1 (i.2) that $U_1 \cup U_2$ is a $\mathcal{K}$-dominating set of $G$. Hence $\gamma_{\mathcal{U}}(G) \leq |U_1 \cup U_2| = \gamma_{\mathcal{U}}(G_1 - x) + \gamma_{\mathcal{U}}(G_2) = \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2) - 1$. Now the result follows by Theorem 2.3.

(ii): By Theorem 2.1 (i.3), $\gamma_{\mathcal{U}}(G - x) = \gamma_{\mathcal{U}}(G_1 - x) + \gamma_{\mathcal{U}}(G_2 - x) = \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2) - 2$. Hence by (i), $\gamma_{\mathcal{U}}(G - x) = \gamma_{\mathcal{U}}(G) - 1$.

(iii): $\gamma_{\mathcal{U}}(G-x) = \gamma_{\mathcal{U}}(G_1-x)+\gamma_{\mathcal{U}}(G_2-x) = \gamma_{\mathcal{U}}(G_1)-1+\gamma_{\mathcal{U}}(G_2-x) = \gamma_{\mathcal{U}}(G) + \gamma_(G_2 - x) - \gamma_{\mathcal{U}}(G_2) > \gamma_{\mathcal{U}}(G)$. The result now follows by Theorem 2.1 (ii).

(iv): Let $M$ be a $\gamma_{\mathcal{U}}$-set of $G$ and $M_i = M \cap V(G_i)$, $i = 1, 2$. Suppose $x \in M$. Hence $M_i$ is a $\mathcal{K}$-dominating set of $G_i$, $i = 1, 2$ and then $\gamma_{\mathcal{U}}(G_i) \leq |M_i|$. Since $x$ belongs to no $\gamma_{\mathcal{U}}$-set of $G_2$ then $|M_2| > \gamma_{\mathcal{U}}(G_2)$. Hence $\gamma_{\mathcal{U}}(G) = |M| = |M_1| + |M_2| - 1 \geq \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2)$ - a contradiction with (i). ■

**Theorem 2.5.** *Let $G_1$ and $G_2$ be two connected $\gamma_{\mathcal{U}}$-critical graphs having exactly one common vertex. Then $G = G_1 \cup G_2$ is $\gamma_{\mathcal{U}}$-critical and $\gamma_{\mathcal{U}}(G) = \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2) - 1$.*

**Proof.** Let $\{x\} = V(G_1) \cap V(G_2)$. By Theorem 2.4 (ii) it follows that $\gamma_{\mathcal{U}}(G) - 1 = \gamma_{\mathcal{U}}(G - x)$. Let without loss of generality $y \in V(G_2 - x)$. By Theorem 2.4 (i), applied to the graphs $G_1$ and $G_2 - y$ we have $\gamma_{\mathcal{U}}(G - y) = \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2 - y) - 1 = \gamma_{\mathcal{U}}(G_1) + \gamma_{\mathcal{U}}(G_2) - 2 = \gamma_{\mathcal{U}}(G) - 1$. ■

# 3 Forbidden Subgraphs

Although the characterization of graphs $G$ for which $\gamma(G) = i(G)$ is still an open problem, several results give sufficient conditions for a graph to have $\gamma(G) = i(G)$. See Allan and Laskar [1] and Topp and Volkmann [14]. Here we give some forbidden subgraph conditions sufficient to imply $\gamma = \gamma_{uk}$. Note that in general, a forbiden subgraph characterization cannot be obtained since for any graph $H$, the join $G = H + K_1$ has $\gamma_{uk}(G) = \gamma(G) = i(G) = 1$. Troughout this section, let the graphs $F_1, F_2, .., F_{15}$ be as is shown in Fig.1. Let $U$ be the graph obtained by $F_2$ by adding an edge connecting the two end-vertices of $F_2$ which are at distance four. Our results are:

**Theorem 3.1.** *If $G \in Forb(C_4, F_1, F_2, U)$ then $\gamma(G) = \gamma_{uk}(G)$.*

**Corollary 3.2.** *If $G$ is bipartite and $G \in Forb(C_4, F_1)$ then $\gamma(G) = \gamma_{uk}(G) = \gamma_{uk_2}(G)$.*

**Corollary 3.3.** *If $T$ is a tree and $T \in Forb(F_1)$ then $\gamma(G) = \gamma_{uk}(G) = \gamma_{uk_2}(G)$.*

**Theorem 3.4.** *Let $G$ be a connected graph of order at most nine. Then $G \in Forb(F_1, .., F_{15})$ if and only if $\gamma(G) = \gamma_{uk}(G) = \gamma_{uk_3}(G)$.*

**Theorem 3.5.** *Let $G$ be a graph, $G \in Forb(F_1, .., F_{15})$ and $\Delta(G) = 4$. If the set of all vertices of maximum degree is independent then $\gamma(G) = \gamma_{uk}(G) = \gamma_{uk_2}(G)$.*

**Corollary 3.6.** *If $G$ is a graph with $\Delta(G) = 3$ then $\gamma(G) = \gamma_{uk}(G) = \gamma_{uk_2}(G)$.*

## 3.1 Proofs

We need the following lemma:

**Lemma 3.1.1.** *Let $G$ be a graph and $s \geq 2$ be an integer.*

(a) *If each complete subgraph of $G$ of order $s$ has a vertex of degree at most $s$ in $G$ then $\gamma_{uk}(G) = \gamma_{uk_{s-1}}(G)$;*

(b) *If $\Delta(G) \geq 2$ then $\gamma_{uk}(G) = \gamma_{uk_{\Delta(G)-1}}(G)$;*

(c) *If $G$ is connected and $|V(G)| \geq 3$ then $\gamma_{uk}(G) = \gamma_{uk_{\lfloor |V(G)|/3 \rfloor}}(G)$.*

**Proof.** (a) Choose a $\gamma_{uk}$-set $D$ of $G$ such that the graph $\langle D, G \rangle$ has the fewest components of order at least $s$. Let $C_D$ be any component of order at least $s$ in $\langle D, G \rangle$. Then there is $x \in V(C_D)$ which has degree at most $s$. Since $s \geq 2$ and $D$ is a $\gamma_{uk}$-set of $G$, then there is $v \in V(G) - D$ with $N[v, G] \cap D = \{x\}$. Hence $|V(C_D)| = s$ and $N[x, G] = V(C_D) \cup \{v\}$. But then $D_1 = (D - \{x\}) \cup \{v\}$ will be a $\gamma_{uk}$-set of $G$ such that $\langle D_1, G \rangle$ has fever components of order at least $s$ than $\langle D, G \rangle$ - a contradiction. Hence any component of $< D, G >$ has order at most $s - 1$ and then $\gamma_{uk}(G) = \gamma_{uk_{s-1}}(G)$.

(b) Immediately follows by (a).

(c) Choose $D$ to be a $\gamma_{uk}$-set of $G$ such that (1) the largest component $C_D$ of $\langle D, G \rangle$ to have minimum order over all $\gamma_{uk}$-sets of $G$; (2) subject to

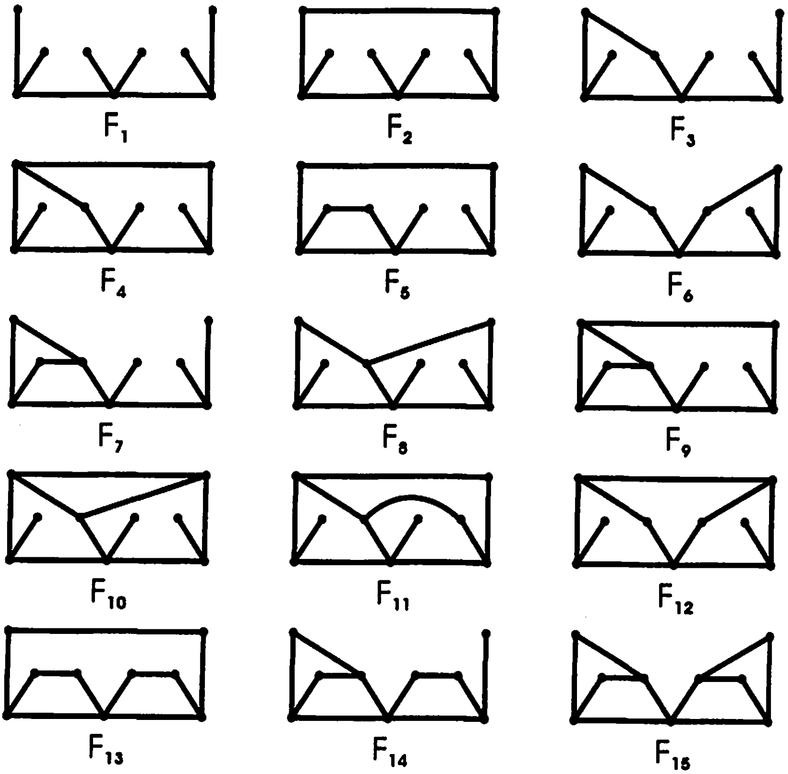


Figure 1:

(1), $\langle D, G\rangle$ to have minimum number of largest components. If $|V(C_D)| = 1$ then $\gamma_{uk}(G) = \gamma_{uk_1}(G)$ and the result is obvious. So, let $|V(C_D)| > 1$. Since $D$ is a $\gamma_{uk}$-set then for any $x \in V(C_D)$, the set $p_x = \{y \in V(G) - D : N(y, G) \cap D = \{x\}\} \neq \emptyset$. Suppose $C_D$ has order at least $\lfloor |V(G)|/3 \rfloor + 1$. Then there is $z \in V(C_D)$ with $p_z = \{u\}$. But then $D_2 = (D - \{z\}) \cup \{u\}$ will be a $\gamma_{uk}$-set of $G$ which contradicts the choice of $D$. ■

**Proof of Theorem 3.1, Theorem 3.4, Theorem 3.5 and Corollary 3.6:** It is easy to see that for all $i = 1, 2, .., 15$, $\gamma(F_i) = 3 < \gamma_{uk}(F_i) = 4$.

Let $G$ be a graph with $\gamma(G) < \gamma_{uk}(G)$. Choose $D_0$ to be a $\gamma$-set of $G$ such that the number of components of $\langle D_0, G\rangle$ is the maximum number taken over all $\gamma$-sets of $G$. Then there is a component $Q$ of $\langle D_0, G\rangle$ which is not complete. Hence there exist three vertices $x_0, x_1, x_2 \in V(Q)$ such that $\langle \{x_0, x_1, x_2\}, G\rangle \cong P_3$. Let $Y_i = \{u \in V(G) - D_0 : N(u, G) \cap D_0 = \{x_i\}\}$, $i = 0, 1, 2$. Since $D_0$ is a $\gamma$-set then $Y_i \neq \emptyset$, $i = 0, 1, 2$. If $y_i \in Y_i$ and $N[y_i, G] \supseteq Y_i$ for some $i \in \{0, 1, 2\}$, then $D_1 = (D_0 - \{x_i\}) \cup \{y_i\}$ will be a

$\gamma$-set of $G$ with $c(\langle D_0, G\rangle) < c(\langle D_1, G\rangle)$ which is a contradiction with the choice of $D_0$. Hence $Y_i \neq \emptyset$ and for any $y_i \in Y_i$ there is $z_i \in Y_i - \{y_i\}$ such that $y_i z_i \notin E(G)$, $i = 0, 1, 2$. Thus if $F$ is a graph with $|V(F)| \leq 8$ or $\Delta(F) = 3$ then $\gamma(F) = \gamma_{uk}(F)$. Further if $\Delta(F) = 3$ then by Lemma 3.1.1 (b), $\gamma_{uk}(F) = \gamma_{uk_2}(F)$. So, Corollary 3.6 is proved.

Let $\{y_{i1}, y_{i2}\} \subseteq Y_i$ and $y_{i1}y_{i2} \notin E(G)$, $i = 0, 1, 2$. Denote $H = \langle\{x_0, x_1, x_2, y_{01}, y_{02}, y_{11}, y_{12}, y_{21}, y_{22}\}, G\rangle$. Let $R_1 = \langle\{y_{01}, y_{02}, y_{21}, y_{22}\}, G\rangle$ and $R_k = \langle\{y_{k1}, y_{k2}, y_{11}, y_{12}\}, G\rangle$ for $k = 0, 2$.

First assume $G \in Forb(C_4, F_1, F_2, U)$. Since $H \in Forb(C_4)$ then $E(R_0) = E(R_2) = \emptyset$ and $\Delta(R_1) < 2$; since $H \in Forb(F_2, U)$ then $E(R_1) = \emptyset$. Hence $H = F_1$ - a contradiction. Thus Theorem 3.1 is proved.

Now, let $G \in Forb(F_1, \ldots, F_{15})$ and one of the following holds:

(i) $|V(G)| \leq 9$;

(ii) $\Delta(G) = 4$ and no two vertices of degree four are adjacent.

Note that if (i) holds then $G = H$ and if (ii) holds then $N(x_j, G) = \{x_1, y_{j1}, y_{j2}\}$ for $j = 0, 2$ and $N(x_1, G) = \{x_0, x_2, y_{11}, y_{12}\}$.

**Claim 1.** $|E(R_1)| \leq 1$.

**Proof.** Suppose $|E(R_1)| \geq 2$. If $R_1$ has a matching then $D_2 = (D_0 - \{x_0, x_2\}) \cup \{y_{01}, y_{02}\}$ will be a $\gamma$-set of $G$ with $c(\langle D_2, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction. Hence there is $y_{ks} \in V(R_1)$ with two neighbors in $R_1$ and then $D_3 = (D_0 - \{x_0, x_2\}) \cup \{y_{k1}, y_{k2}\}$ will be a $\gamma$-set of $G$ with $c(\langle D_3, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction. □

**Claim 2.** *Let $k \in \{0, 2\}$. Then $|E(R_k)| \leq 2$ and if equality holds then $E(R_k) = \{y_{k1}y_{1j}, y_{k2}y_{1j}\}$ for some $j \in \{1, 2\}$.*

**Proof.** If $y_{11}, y_{12} \in N(y_{k1}, R_k) \cup N(y_{k2}, R_k)$ for some $k \in \{0, 2\}$ then $D_4 = (D_0 - \{x_k, x_1\}) \cup \{y_{k1}, y_{k2}\}$ will be a $\gamma$-set with $c(\langle D_4, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction. □

**Claim 3.** *Let $k \in \{1, 2\}$. Then $y_{1k}$ has at most two neighbors among $y_{01}, y_{02}, y_{21}, y_{22}$.*

**Proof.** Without loss of generality, let $k = 1$ and $y_{01}, y_{02}, y_{21} \in N(y_{11}, G)$. Then $D_5 = (D_0 - \{x_0, x_2\}) \cup \{y_{11}, y_{22}\}$ will be a $\gamma$-set of $G$ with $c(\langle D_5, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction. □

Since $G \in Forb(F_1)$ then at least one of the sets $E(R_i)$, $i = 0, 1, 2$ is nonempty.

Case $E(R_1) \neq \emptyset$: By Claim 1, $|E(R_1)| = 1$ and without loss of generality, let $E(R_1) = \{y_{01}y_{21}\}$. Since $G \in Forb(F_2)$ it follows that at least one of the sets $E(R_0)$ and $E(R_2)$ is nonempty. Let without loss of generality, $0 \neq |E(R_0)| \geq |E(R_2)|$. By Claim 2 we additionally have $2 \geq |E(R_0)|$.

SUBCASE $E(R_2) = \emptyset$ : If $|E(R_0)| = 1$ then $H$ will be isomorphic to either $F_4$ or $F_5$ - a contradiction. Hence by Claim 2, $E(R_0) = \{y_{01}y_{1j}, y_{02}y_{1j}\}$ for some $j \in \{1, 2\}$. But then $H$ will be isomorphic to $F_9$ - a contradiction.

SUBCASE $|E(R_0)| = |E(R_2)| = 1$: If $y_{01}y_{1i}, y_{21}y_{1i} \in E(G)$ for some $i \in \{1, 2\}$ then $H$ will be isomorphic to $F_{10}$ - a contradiction. If either $y_{01}y_{1i}, y_{1i}y_{22} \in E(G)$ or $y_{21}y_{1i}, y_{1i}y_{02} \in E(G)$ for some $i \in \{1, 2\}$ then the graph $H$ will be isomorphic to $F_{11}$ - a contradiction. If $y_{01}y_{1k}, y_{21}y_{1l} \in E(G)$, where $\{k, l\} = \{1, 2\}$ then the graph $H$ will be isomorphic to $F_{12}$ - a contradiction. If $y_{02}y_{1k}, y_{22}y_{1l} \in E(G)$, where $\{k, l\} = \{1, 2\}$ then the graph $H$ will be isomorphic to $F_{13}$ - a contradiction.

SUBCASE $|E(R_2)| = 1$ and $|E(R_0)| = 2$: By Claim 2, $E(R_0) = \{y_{01}y_{1j}, y_{02}y_{1j}\}$ for some $j \in \{1, 2\}$. Let without loss of generality $j = 1$. By Claim 3, $y_{11}y_{21}, y_{11}y_{22} \notin E(G)$. Hence either $y_{12}y_{21} \in E(G)$ or $y_{12}y_{22} \in E(G)$. If $y_{12}y_{21} \in E(G)$ then $D_6 = (D_0 - \{x_0, x_1\}) \cup \{y_{02}, y_{21}\}$ will be a $\gamma$-set with $c(\langle D_6, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction. If $y_{12}y_{22} \in E(G)$ then $D_7 = (D_0 - \{x_0, x_1, x_2\}) \cup \{y_{01}, y_{11}, y_{22}\}$ will be a $\gamma$-set with $c(\langle D_7, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction.

SUBCASE $|E(R_2)| = |E(R_0)| = 2$: By Claim 2, $E(R_0) = \{y_{01}y_{1j}, y_{02}y_{1j}\}$ and $E(R_2) = \{y_{21}y_{1s}, y_{22}y_{1s}\}$ for some $j, s \in \{1, 2\}$. By Claim 3, $j \neq s$, say $j = 1$, $s = 2$. But then $D_8 = (D_0 - \{x_0, x_1, x_2\}) \cup \{y_{01}, y_{11}, y_{22}\}$ will be a $\gamma$-set of $G$ with $c(\langle D_8, G\rangle) > c(\langle D_0, G\rangle)$ - a contradiction. □

CASE $E(R_1) = \emptyset$: Without loss of generality, let $|E(R_0)| \geq |E(R_2)|$. Hence $E(R_0) \neq \emptyset$ and without loss of generality, let $y_{01}y_{11} \in E(G)$. Since $H \neq F_3$ then $|E(H)| > |E(F_3)|$. By Claim 2, if $|E(R_0)| > 1$ then $E(R_0) = \{y_{01}y_{11}, y_{02}y_{11}\}$.

SUBCASE $y_{02}y_{11} \notin E(G)$: Since $H \neq F_3$ and $|E(R_0)| \geq |E(R_2)|$ we have $|E(R_2)| = 1$. Thus, exactly one of $y_{11}y_{21}, y_{11}y_{22}, y_{12}y_{21}, y_{12}y_{22}$ is an edge of $G$. But then $H$ will be isomorphic to either $F_8$ or $F_6$ - a contradiction.

SUBCASE $y_{02}y_{11} \in E(G)$: Since $H$ is not isomorphic to $F_7$ it follows that $E(R_2) \neq \emptyset$. By Claim 3, $y_{11}$ is an isolate vertex in $R_2$. Hence there are three possibilities, namely $E(R_2) = \{y_{12}y_{21}\}$, $E(R_2) = \{y_{12}y_{22}\}$ and $E(R_2) = \{y_{12}y_{21}, y_{12}y_{22}\}$. But then $H$ will be isomorphic to $F_{14}, F_{14}$ and $F_{15}$ respectively. □

Hence we prove that if $G \in Forb(F_1, .., F_{15})$ and at least one of the (i) and (ii) holds, then $\gamma(G) = \gamma_{uk}(G)$. Further if (i) holds then by Lemma 3.1.1 (c), $\gamma_{uk}(G) = \gamma_{uk_3}(G)$, if (ii) holds then by Lemma 3.1.1 (a), $\gamma_{uk}(G) = \gamma_{uk_2}(G)$. This proves Theorem 3.4 and Theorem 3.5. ■

Finally, note that Corollary 3.2 and Corollary 3.3 immediately follow by Theorem 3.1.

# 4 Remarks

Hedetniemi and al. [11, Theorem 4.3] proved that if $G$ is 3-regular then $\gamma(G) = \gamma_a(G)$. This result follows immediately by Corollary 3.6.

Cockayne and Mynhardt [5] showed that there is a class of 3-connected cubic graphs for which the difference $i - \gamma$ is unbounded. Hence this is true and for $i - \gamma_{uk_2} = i - \gamma_{uk}$ (for such class of graphs).

We conclude with:

**Conjecture 4.1.** *There exists a class of 4-connected 4-regular graphs for which the differences $\gamma_{uk} - \gamma$ and $i - \gamma_{uk}$ are unbounded.*

**Problem 4.2.** *Characterize the class of all bipartite graphs $G$ with $\gamma(G) = \gamma_{uk_2}(G)$.*

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