Minimal forbidden graphs of reducible additive hereditary graph properties

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Abstract

It is known that any reducible additive hereditary graph property has infinitely many minimal forbidden graphs, however the proof of this fact is not constructive. The purpose of this paper is to construct infinite families of minimal forbidden graphs for some classes of reducible properties. The well-known Hajós construction is generalized and some of its applications are presented.
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1 Introduction and preliminaries

A hereditary graph property $\mathcal{P}$ is any proper non-empty isomorphism closed subclass of the class of all finite simple graphs $\mathcal{I}$, which is closed under subgraphs. Such a property is called *additive* if it is additionally closed with respect to disjoint union of graphs. Additive and hereditary graph properties have been investigated mainly in the context of generalized colorings, more details may be found in the surveys [3, 4]. In general we follow the notation and definitions given in [7]. Some well-known and intensively studied additive hereditary graph properties are listed below.

$\mathcal{O} = \{G \in \mathcal{I}\colon E(G) = \emptyset\}$,

$\mathcal{O}_k = \{G \in \mathcal{I}\colon$ each component of G has at most $k+1$ vertices$\}$,

$\mathcal{D}_k = \{G \in \mathcal{I}\colon G$ is k-degenerate, i.e. every subgraph of G has a vertex of degree at most $k\}$,

$\mathcal{I}_k = \{G \in \mathcal{I}\colon G$ does not contain $K_{k+2}\}$,

We denote the set $\{1,2,\ldots,n\}$ by $[n]$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be additive hereditary graph properties. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition (coloring) of G is a partition $(V_1, V_2, \ldots, V_n)$ of the vertex set $V(G)$ such that the subgraph $G[V_i]$ of G induced by V_i has the property $\mathcal{P}_i$, for each $i \in [n]$. A graph G has the property $\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$ (G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable (colorable)) if it has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. For convenience, if $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_n = \mathcal{P}$ we call a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of G briefly a $(\mathcal{P}, n)$-partition (coloring) of G and we write $G \in \mathcal{P}^n$. Thus a proper n-coloring of a graph G is just an $(\mathcal{O}, n)$-coloring of G. A property $\mathcal{R}$ which can be written in the form $\mathcal{R} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$, $n \geq 2$ is said to be *reducible* and it is *irreducible*, otherwise. It is easy to see that if the graph properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$ is an additive hereditary graph property, too.

Each hereditary graph property $\mathcal{P}$ is uniquely determined by the set of its *minimal forbidden graphs* defined as follows:

$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P}$ but for each proper subgraph H of G; $H \in \mathcal{P}\}$.

Obviously, a hereditary graph property $\mathcal{P}$ is additive if and only if $\mathbf{F}(\mathcal{P})$ consists of connected graphs. We will call a property $\mathcal{P}$ 2-connected if each minimal forbidden graph of $\mathcal{P}$ is vertex 2-connected. So that $\mathcal{O}$, $\mathcal{D}_1$, $\mathcal{I}_k$ are 2-connected, while $\mathcal{D}_k, k \geq 2$ and $\mathcal{O}_k, k \geq 1$ are not. If $\mathbf{F}(\mathcal{P}) = \{H\}$, we write $\mathcal{P}$ as $-H$, so that e.g. $\mathcal{O} = -K_2$ and $\mathcal{I}_k = -K_{k+2}$. For a hereditary graph property $\mathcal{P}$ there is always a nonnegative integer k such that $K_{k+1} \in \mathcal{P}$ but $K_{k+2} \notin \mathcal{P}$, which is called the *completeness* of $\mathcal{P}$ and denoted by $c(\mathcal{P})$. The property $\mathcal{O}$, to be edgeless, is the only additive hereditary graph property of completeness 0, $c(\mathcal{D}_k) = c(\mathcal{O}_k) = c(\mathcal{I}_k) = k$. Using the completeness of a hereditary graph property $\mathcal{P}$, it is easy to determine the smallest order of a forbidden graph for $\mathcal{P}$, since it equals

$c(\mathcal{P}) + 2$. Sometimes it is very difficult to find any minimal forbidden graph of the smallest order even for a property $\mathcal{P}^2$. It is easy to prove (see [3, 4]), that if $\mathcal{R}$ is the reducible property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, then $c(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n) = c(\mathcal{P}_1) + c(\mathcal{P}_2) + \ldots + c(\mathcal{P}_n) + n - 1$. This also implies that the complete graph $K_{c(\mathcal{R})+2}$ is the (only) minimal forbidden graph for $\mathcal{R}$ of the smallest order if and only if $K_{c(\mathcal{P}_i)+2}$ is the minimal forbidden graph of the smallest order for some $\mathcal{P}_i, i \in [n]$.

For the reducible graph property $\mathcal{O}^k$, to be k-colorable, the minimal forbidden graphs are also called $(k+1)$-critical graphs and have been studied by many authors, see [7]. A constructive characterization of k-critical graphs was obtained by G. Hajós in [6]. The famous *Hajós construction* can be described in the following way:

Let G_1 and G_2 be disjoint graphs with edges u_1v_1 and u_2v_2, respectively. Remove u_1v_1 and u_2v_2, identify vertices v_1 and u_2, and join u_1 and v_2 by a new edge.

Many interesting results and problems related to the Hajós construction may be found in [7]. Among others, if the graphs G_1 and G_2 are $(k+1)$-critical, i.e. $G_1, G_2 \in \mathbf{F}(\mathcal{O}^k)$ and u_1v_1, u_2v_2 are their selected edges then the graph $H = HC[(G_1, u_1v_1), (G_2, u_2v_2)]$ obtained by the Hajós construction is $(k+1)$-critical, too.

This is based on the following well-known facts:

1. for each proper k-coloring of $H - u_1v_2$ the vertices u_1 and v_2 belong to the same color class, which implies $H \notin \mathcal{O}^k$.

2. for each edge $e \in E(H - u_1v_2)$ there is a proper k-coloring (V_1, V_2) of $H - \{e, u_1v_2\}$ with $u_1 \in V_1$ and $v_2 \in V_2$, which implies $HC[(G_1, u_1v_1), (G_2, u_2v_2)] - e \in \mathcal{O}^k$.

Some basic properties and results on minimal forbidden graphs with respect to reducible hereditary properties can be found e.g. in [3, 4, 10]. It is known that $\mathbf{F}(\mathcal{O}^2)$ consists of all odd cycles. However, for $k \geq 2$ there is no property $\mathcal{P}^k$ other than $\mathcal{O}^2$ for which we know all the elements of $\mathbf{F}(\mathcal{P}^k)$. This is to be expected, since A. Farrugia [5] proved that $(\mathcal{P} \circ \mathcal{Q})$-recognition is NP-hard if and only if $\mathcal{P} \circ \mathcal{Q} \neq \mathcal{O}^2$. A. Berger [1] proved that if there are only finitely many non-isomorphic blocks contained in the minimal forbidden graphs of an additive hereditary graph property $\mathcal{P}$, then $\mathcal{P}$ is irreducible. This implies that each reducible additive hereditary graph property $\mathcal{R}$ has infinitely many minimal forbidden graphs. On the other hand, no general construction of an infinite family of minimal forbidden graphs for a given reducible property $\mathcal{R}$ is known. References to papers dealing with constructions of infinite families of minimal forbidden graphs for given reducible graph properties can be found in [2, 3, 4, 7, 9, 11]. In this paper we present new constructions of such families, based on generalizations of the Hajós construction.

First, let us apply the Hajós construction to r disjoint graphs. More precisely let $r \geq 2$ be a positive integer, H_j be disjoint graphs and $e_j = u_jv_j \in E(H_j)$ for $j \in [r]$.

By *a K_2-conjunction* of the r pairs (*a K_2^r-conjunction* of the pairs) (H_j, u_jv_j), $u_jv_j \in E(H_j)$, $j \in [r]$ we mean the graph $K_2^r[H_j, u_jv_j]$ constructed in the following way: for each $j \in [r]$ remove the edge u_jv_j from H_j; for $j \in [r-1]$ identify vertices v_j, u_{j+1} and join u_1 and v_r by a new edge. This construction is illustrated in Figure 1.

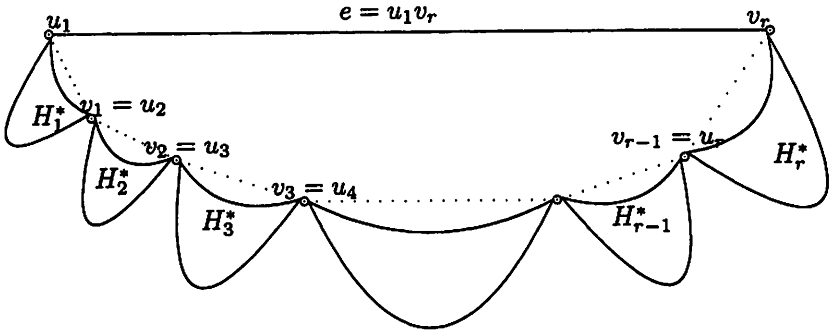

Figure 1: K_2-conjunction of pairs (H_j, u_jv_j), $H_j^* = H_j - u_jv_j$, $j \in [r]$

Obviously a K_2^r-conjunction of $(k+1)$-critical graphs is a $(k+1)$-critical graph, since it is obtained through repeated the Hajós constructions.
A K_2^r-conjunction of arbitrary minimal forbidden graphs of a reducible property $\mathcal{R}$ is not necessarily a minimal forbidden graph of $\mathcal{R}$. It can also happen if the only minimal forbidden graph of the smallest order for $\mathcal{R}$ is complete. In several cases (e.g. $\mathcal{I}_k^2, k \geq 1$), a K_2-conjunction of graphs that do not have a property $\mathcal{R}$ can be a graph with the property $\mathcal{R}$. In some cases, the condition (2), on which criticality of the graphs obtained by the Hajós construction is based, fails. Two examples, when we can use K_2-conjunctions to construct infinitely many forbidden graphs of reducible properties will be given in Section 2. After appropriate definitions a more general construction of minimal forbidden graphs will be presented in Section 3.

2 K_2^r-conjunctions

To produce infinitely many minimal forbidden graphs of a reducible property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ it is useful to consider the structure of minimal forbidden graphs for $\mathcal{R}$ of the smallest order. The Hajós construction of $(k+1)$-critical graphs starts with the $(k+1)$-critical graph K_{k+1}. All 3-critical graphs (i.e. odd cycles) arise from $K_3 - e$ as K_2-conjunctions of its copies.

Let us start with a notion, based on the condition (2) for the Hajós construction of critical graphs.

For given $s \in [n]$ and a graph $H \in \mathbf{F}(\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n)$, an edge $e = uv \in E(H)$ is called *s-eligible* if the following hold:

1. for each $i \in [s]$ there exists a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition $(V_1, V_2, \ldots, V_n)$ of $H - uv$ such that $u, v \in V_i$ and for each $i \in [n] \setminus [s]$ such a partition does not exist;

2. for each edge $e' \in E(H - uv)$ and for each pair distinct indices $i, j \in [s]$, there exists a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition $(V_1, V_2, \ldots, V_n)$ of $(H - uv) - e'$ satisfying $u \in V_i, v \in V_j$.

An n-eligible edge is called *eligible*.

For example, in any $(k+1)$-critical graph F each edge is eligible. Moreover, if $\mathbf{F}(\mathcal{P}_i)$ contains a complete graph for each $i \in [s]$, $2 \leq s \leq n$ and $\mathbf{F}(\mathcal{P}_i)$ does not contain a complete graph for $i \in [n] \setminus [s]$, then each edge of $K = K_{\sum_{i=1}^{n} c(\mathcal{P}_i)+n+1} \in \mathbf{F}(\mathcal{P}_i \circ \ldots \circ \mathcal{P}_n)$ is s-eligible. On the other hand, the graph $H = K_7 - 3K_2$ is the only minimal forbidden graph for the property $(-C_4)^2$ of the smallest order. Its 6 edges incident with the unique vertex of degree 6 are not eligible.

A sufficient condition for a K_2^2-conjunction of graphs from $\mathbf{F}(\mathcal{O}_1^k)$ with selected edges to be in $\mathbf{F}(\mathcal{O}_1^k)$ is presented in [2]. In the next example we use K_2-conjunction to construct infinitely many minimal forbidden graphs for the property $\mathcal{O}_k^2, k \geq 1$.

Theorem 1 *Let $r \geq 2$, $k \geq 1$. For $i \in [r]$, let W^i be a copy of the wheel $C_{k(k+2)+1} + K_1$ and let e^i be any edge on the rim of W^i. Then the graph H obtained by the K_2-conjunction of the r pairs (W^i, e^i) is a minimal forbidden graph of $\mathcal{O}_k^2$.*

Proof. For $i \in [r]$, let v^i be the central vertex and $u_0^i u_1^i \ldots u_{k(k+2)}^i$ the vertices of the rim of the wheel W^i. Assume $e^i = u_0^i u_1^i$.

Let us show that $H \notin \mathcal{O}_k^2$. Suppose, to the contrary, that (V_1, V_2) is an $(\mathcal{O}_k, 2)$-partition of H, and let $V_j^i = V_j \cap V(W^i)$, $j \in [2]$, $i \in [r]$. Assume, without loss of generality, that $v^1 \in V_2^1$. In that case at most k vertices out of $u_0^1, u_1^1, \ldots, u_{k(k+2)}^1$ could be in V_2^1 and others have to be in V_1^1. The only possibility is that $u_0^1, u_1^1 \in V_1^1$. By the construction of H, $\{v^i, i \in [r]\} \subseteq V_2$ and vertices u_0^i, u_1^i are contained in V_1^i, $i \in [r]$,. Denote by a^i (b^i) the number of vertices in the component of the subgraph

$H[V_1^i]$ including the vertex u_0^i (u_1^i). Because $H[V_1] \in \mathcal{O}_k$ the inequalities $b^i + a^{i+1} - 1 \le k+1$ for $i \in [r-1]$, $a^1 + b^r \le k+1$ hold. Moreover one can observe that $a^i + b^i \ge k+2$ for each $i \in [r]$. Hence $r(k+2) \le \sum_{i=1}^r a^i + \sum_{i=1}^r b^i \le r(k+2) - 1$, which is a contradiction.

Now, it is enough to show that $H - f \in \mathcal{O}_k^2$ for an arbitrary edge f of H. Let us construct the appropriate $(\mathcal{O}_k, 2)$-partition (V_1, V_2) of $H - f$. We consider two cases:

1. $f = u_0^1 u_1^r$.
   Then we take $V_1 = \bigcup_{i=1}^r \left\{\{v^i\} \cup \bigcup_{j=1}^k \left\{u^i_{j(k+2)}\right\}\right\}$,
   $V_2 = V(H) - V_1$.

2. $f \in E(W^q - u_0^q u_1^q), q \in [r]$.
   Because in this case $u_0^q u_1^q$ is an eligible edge in $W^q \in \mathbf{F}(\mathcal{O}_k^2)$ there is an $(\mathcal{O}_k, 2)$-partition (V_1^q, V_2^q) of $W^q - u_0^q u_1^q - f$ such that $u_0^q \in V_1^q$ and $u_1^q \in V_2^q$.
   For $i < q$, $V_2^i = \{v^i\} \cup \bigcup_{j=1}^k \{u_{j(k+2)}\}$, $V_1^i = V(W^i) - V_2^i$.
   For $i > q$, $V_1^i = \{v^i\} \cup \bigcup_{j=1}^k \{u_{j(k+2)-k}\}$, $V_2^i = V(W^i) - V_1^i$.
   Finally, $V_1 = \bigcup_{i=1}^r V_1^i$, $V_2 = \bigcup_{i=1}^r V_2^i$.

In both cases (V_1, V_2) is an $(\mathcal{O}_k, 2)$-partition of $H - f$. ■

It is easy to see that the K_2-conjunction of r disjoint copies of the minimal forbidden graph K_{2n+1} for the property $\mathcal{D}_1^n$ gives a minimal forbidden graph for $\mathcal{D}_1^n$ (see [3]). This can be generalized as follows:

Theorem 2 *Let $r, n \ge 2$, $p_1, \ldots, p_n \ge 0$ be integers and let $m = n + 1 + \sum_{i=1}^n p_i$. For $i = 1, 2, \ldots, r$ let K_m^i be a copy of the complete graph K_m and let e^i be any edge in K_m^i. Then the K_2-conjunction of the r pairs (K_m^i, e^i) gives a minimal forbidden graph of $\mathcal{D}_{p_1} \circ \mathcal{D}_{p_2} \circ \ldots \circ \mathcal{D}_{p_n}$.*

Proof. Let $e^i = u_i v_i$, $i \in [r]$, and let $H = K_2^r[K_m^i, u_i v_i]$.

First we shall prove that $H \notin \mathcal{D}_{p_1} \circ \mathcal{D}_{p_2} \circ \ldots \circ \mathcal{D}_{p_n}$. Suppose, to the contrary, that $(V_1, V_2, \ldots, V_n)$ is a $(\mathcal{D}_{p_1}, \ldots, \mathcal{D}_{p_n})$-partition of H and let $V_j^i = V(K_m^i) \cap V_j$, $i \in [r]$, $j \in [n]$. Then $(V_1^i, \ldots V_n^i)$ is a $(\mathcal{D}_{p_1}, \ldots, \mathcal{D}_{p_n})$-partition of $K_m^i - u_i v_i$. Since a complete graph $K_j \in D_p$ if and only if $j \le p+1$ and because $m = n + 1 + \sum_{i=1}^n p_i$, the vertices u_i, v_i have to be in the same partite set of $K_m^i - u_i v_i$, say V_t^i, $t \in [n]$ and $|V_t^i| = p_t + 2$. Thus by the construction of H we obtain that $H[V_t]$ is a K_2-conjunction of r pairs (K_{p_t+2}, e^*), with e^* being an arbitrary edge of K_{p_t+2}. It follows that $\delta(H[V_t]) = p_t + 1$. The last equality means that $H[V_t] \notin \mathcal{D}_{p_t}$, a contradiction.

To show that for any $f \in E(H), H - f \in \mathcal{D}_{p_1} \circ \ldots \circ \mathcal{D}_{p_n}$ we consider two cases:

1. $f = u_1 v_r$. Let $(V_1, \ldots, V_n)$ be a partition of $H - f$ satisfying the following conditions:

   - precisely $p_q + 1$ vertices of $K_m^i - u_i v_i$ other than u_i, v_i are contained in V_q, $q \geq 2$, $i \in [r]$,
   - precisely $p_1 + 2$ vertices of $K_m^i - u_i v_i$, including u_i, v_i are contained in V_1, $i \in [r]$.

   Note that $H[V_q]$, $q \geq 2$ is a disjoint union of r copies of K_{p_q+1}, which implies that $H[V_q] \in \mathcal{D}_{p_q}$. Moreover $H[V_1]$ is a graph obtained from r labeled copies $G_i - c_i d_i$ of a complete graph of order $p_1 + 2$ without one edge, by identifying vertices d_i, c_{i+1} for $i \in [r-1]$. Because each subgraph of $H[V_1]$ has a vertex of degree at most p_1 we conclude that $(V_1, \ldots, V_n)$ is a $(\mathcal{D}_{p_1}, \ldots, \mathcal{D}_{p_n})$-partition of $H - f$.

2. for $f = ab \in E(K_m^j - u_j v_j)$, $j \in [r]$, construct a $(\mathcal{D}_{p_1}, \ldots, \mathcal{D}_{p_n})$-partition $(V_1, \ldots, V_n)$ of $H - f$ in the following way:

   - $\{u_1, v_1 = u_2, v_2 = u_3, \ldots, u_{j-1}, v_{j-1} = u_j, a, b\} \subseteq V_1$,
   - $\{v_j = u_{j+1}, v_{j+1}, \ldots, u_r, v_r\} \subseteq V_2$,
   - precisely $p_q + 1$ vertices of $K_m^i - u_i v_i$ are contained in $V_q, q > 2$, $i \in [r]$,
   - precisely $p_1 + 2$ vertices of $K_m^i - u_i v_i$ are contained in V_1 for $i \in [j]$, and $p_1 + 1$ otherwise, i.e. for $j < i$,
   - precisely $p_2 + 1$ vertices of $K_m^i - u_i v_i$ are contained in V_2 for $i \in [j]$ and $p_2 + 2$ for $r \geq i > j$,

This finishes the proof. ■

3 A generalization of the Hajós construction

Let G be a graph of order m and T be any spanning tree of G, i.e. $V(T) = V(G)$ and $E(T) \subseteq E(G)$, with the edges $E(T) = \{e_1 = x_1 y_1, e_2 = x_2 y_2, \ldots, e_{m-1} = x_{m-1} y_{m-1}\}$. For given positive integers $r_1, \ldots, r_{m-1}$, $r_i \geq 2$, by a *G^T-conjunction* of the pairs $(H_{i,j}, u_{i,j} v_{i,j}), i \in [m-1], j \in [r_i]$ we mean the graph $H = G^T[H_{i,j}, u_{i,j} v_{i,j}]$ obtained as an issue of the following operations. For each $i \in [m-1]$, construct the graph H_i from the graphs $H_{i,1}, \ldots, H_{i,r_i}$ by removing the edge $u_{i,j} v_{i,j}$, for each $j \in [r_i]$, and identifying the vertex $v_{i,j}$ with the vertex $u_{i,j+1}$ for each $j \in [r_i - 1]$. Then combine the graphs $G, H_1, \ldots, H_{m-1}$ to form the graph H by identifying the vertex x_i of G with $u_{i,1}$ and the vertex y_i of G with v_{i,r_i} for each $i \in [m-1]$.

Let us remark that if $G = K_2$, then a G^T-conjunction is just a K_2-conjunction. By an $r-$ *cyclic* property, $r \geq 2$, we mean such a hereditary property $\mathcal{P}$ that for each $F \in \mathbf{F}(\mathcal{P})$, F does not contain an induced cycle of length greater than r. It is worth noting that e.g. the property $\mathcal{D}_p, p \geq 1$, is not r-cyclic for $r \geq 2$. On the other hand, let F be a 2-connected graph of order r, then obviously the additive hereditary graph property $-F$ is 2-connected and r-cyclic simultaneously.

Theorem 3 *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be 2-connected, r-cyclic additive hereditary graph properties, $r \geq 2$. Moreover, let $G \in \mathbf{F}(\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s), 2 \leq s \leq n$, be a graph of order m and let T be a spanning tree of G, such that for any edge $e \in E(T)$ there exist integers $i, j, 1 \leq i < j \leq s$, satisfying $G[V(T_1)] \in \mathcal{P}_i, G[V(T_2)] \in \mathcal{P}_j$ where T_1, T_2 are the components of T-e. Let $r_1, \ldots, r_{m-1}$ be integers all greater than $\frac{r-1}{2}$ and let $H_{i,j} \in \mathbf{F}(\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n)$ be graphs with s-eligible edges $u_{i,j}v_{i,j}$, $i \in [m-1], j \in [r_i]$, respectively.*

Then a G^T-conjunction of the pairs $(H_{i,j}, u_{i,j}v_{i,j})$, $i \in [m-1], j \in [r_i]$, is a minimal forbidden graph of $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$.

Proof. Let $H = G^T[H_{i,j}, u_{i,j}v_{i,j}]$ be the G^T-conjunction described above. Suppose, contrary to our claim, that $H \notin \mathbf{F}(\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n)$. This implies that $H \in \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$ or there exists an edge $f \in E(H)$, satisfying $H - f \notin \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$.

1. Assume that $(V_1, \ldots, V_n)$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-coloring of H. Since $u_{i,j}v_{i,j}$ are s-eligible edges of $H_{i,j}$, we have that there exists $q \in [s]$ such that for all $i \in [m-1], j \in [r_i], u_{i,j}v_{i,j} \in V_q$. This gives $G \in \mathcal{P}_q$, contrary to $G \notin \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s$.

2. It remains to prove that for any edge $f \in E(H)$ the graph $H - f \in \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$. Consider two cases:

   (a) $f \in E(G)$.
   Since $G \in \mathbf{F}(\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$ there is an index $t \in [s]$ such that $G - f \in \mathcal{P}_t$. Without loss of generality let $t = 1$. Since the edge $u_{i,j}v_{i,j}$ is s-eligible in $H_{i,j}$, the graph $H_{i,j} - u_{i,j}v_{i,j}$ has a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition $(V^1_{i,j}, \ldots, V^n_{i,j})$ satisfying $u_{i,j}, v_{i,j} \in V^1_{i,j}$. Let $V_k = \cup_{i \in [m-1], j \in [r_i]} V^k_{i,j}, k \in [n]$. We shall show that $(H - f)[V_k] \in \mathcal{P}_k$ for each $k \in [n]$, with $V(G) \subseteq V_1$. Suppose, to the contrary, that there exists $k \in [n]$ such that $(H-f)[V_k] \notin \mathcal{P}_k$. Then there is a graph $F \in \mathbf{F}(\mathcal{P}_k)$ satisfying $F \subseteq H[V_k]$. Since F is 2-connected, $\mathcal{P}_k$ is r-cyclic and $r_i \geq \frac{r-1}{2}$ for all permissible i, it implies that $V(F) \subseteq V^k_{p,j}$ for some fixed p, j, which contradicts the assumption that $(V^1_{p,j}, \ldots, V^n_{p,j})$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $H_{p,j} - u_{p,j}v_{p,j}$.

(b) $f \in E(H_{p,q} - u_{p,q}v_{p,q})$ for some indices p, q.
Recall that there exist indices $l_1, l_2 \in [s]$ such that the components T_1, T_2 of the graph $T - u_{p,1}v_{p,r_p}$, with $u_{p,1} \in V(T_1)$, satisfy $G[V(T_1)] \in \mathcal{P}_{l_1}$ and $G[V(T_2)] \in \mathcal{P}_{l_2}$. Let $J_w = \{i \in [m-1] : u_{i,1}, v_{i,r_i} \in V(T_w)\}$, $w \in [2]$. Because $u_{i,j}v_{i,j}$ is an s-eligible edge in $H_{i,j}$, for each $i \in J_1$, $j \in [r_i]$ and for $i = p$, $j \in [q-1]$ there is a $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-partition $(V_{i,j}^1, V_{i,j}^2, \ldots, V_{i,j}^n)$ of $H_{i,j} - u_{i,j}v_{i,j}$ in which $\{u_{i,j}, v_{i,j}\} \subseteq V_{i,j}^{l_1}$. Again, as before, for each $i \in J_2$, $j \in [r_i]$ and for $i = p$, $j \in [r_i] \setminus [q]$ we can find a $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-partition $(V_{i,j}^1, V_{i,j}^2, \ldots, V_{i,j}^n)$ of $H_{i,j} - u_{i,j}v_{i,j}$ in which $\{u_{i,j}, v_{i,j}\} \subseteq V_{i,j}^{l_2}$. Also, following the definition of s-eligibility we can find a $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-partition $(V_{p,q}^1, V_{p,q}^2, \ldots, V_{p,q}^n)$ of $(H_{p,q} - u_{p,q}v_{p,q}) - f$ in which $u_{p,q} \in V_{p,q}^{l_1}$ and $v_{p,q} \in V_{p,q}^{l_2}$. Our proof finishes with the observation that we can put together all such $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-partitions. Precisely, we construct the partition $(V_1, \ldots, V_n)$ of $V(H - f)$ such that $V_k = \bigcup_{i \in [m-1], j \in [r_i]} V_{i,j}^k$, $k \in [n]$. It is easy to verify that $(V_1, \ldots V_n)$ is a $(\mathcal{P}_1, \ldots \mathcal{P}_n)$-partition of $H - f$.

This establishes the fact $H \in \mathbf{F}(\mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n)$. ■

A method, for finding a graph $G \in \mathbf{F}(\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$, was presented in [3]: let X be a set of graphs. We write $min_{\subseteq}[X]$ for the set of graphs in X that are minimal with respect to the partial order $\subseteq$ (to be a subgraph) on $\mathcal{I}$. Then, using this notation, $\mathbf{F}(\mathcal{P}) = min_{\subseteq}[\mathcal{I} - \mathcal{P}]$ for any hereditary property $\mathcal{P}$. Thus we can express $\mathbf{F}(\mathcal{P}_1 \cup \mathcal{P}_2) = min_{\subseteq}[\{H \in \mathcal{I}$: there exists a pair of graphs $G_1 \in \mathbf{F}(\mathcal{P}_1)$ and $G_2 \in \mathbf{F}(\mathcal{P}_2)$ such that $G_1 \subseteq H$ and $G_2 \subseteq H\}]$. Let us remark that if $\mathcal{P}_1, \mathcal{P}_2$ are additive hereditary graph properties then $\mathcal{P}_1 \cup \mathcal{P}_2$ is a hereditary graph property but not necessarily additive, since $\mathbf{F}(\mathcal{P}_1 \cup \mathcal{P}_2)$ can contain disconnected graphs.

4 Concluding remarks

The condition "T be a spanning tree of G" in the construction of G^T-conjunction and in Theorem 3 can be weakened to "T be a tree satisfying $V(T) = V(G)$". This new assumption does not change the proof and gives the possibility to use in a G^T-conjunction disconnected graphs from $G \in \mathbf{F}(\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_s)$. In order to make the proof easier to follow, we presented only the case where T is a spanning tree.

In the case that $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_k = \mathcal{P}$ we can simplify the construction given in Theorem 3 as follows:

Corollary 4 *For $r \geq 2$ let $\mathcal{P}$ be a 2-connected r-cyclic additive hereditary graph property, $G \in \mathbf{F}(\mathcal{P})$ be a graph with m vertices and T be an arbitrary tree satisfying $V(T) = V(G)$. Let $r_1, \ldots, r_{m-1}$ be integers all greater than $\frac{r-1}{2}$ and $H_{i,j}$ be minimal forbidden graphs of $\mathcal{P}^k, k \geq 2$ with eligible edges $e_{i,j}; i \in [m-1], j \in [r_i]$. Then a G^T-conjunction of the pairs $(H_{i,j}, e_{i,j}); i \in [m-1], j \in [r_i]$ is a minimal forbidden graph of $\mathcal{P}^k$.*

Corollary 4 may be applied for infinitely many reducible properties of the type $\mathcal{P}^k$ taking each $H_{i,j}$ to be the same minimal forbidden graph for $\mathcal{P}^k$ of the smallest order.

For example, let $\mathcal{P}$ be a property with $\mathbf{F}(\mathcal{P}) = \{C_{p_1}, C_{p_2}, \ldots, C_{p_m}\}, p_1 < p_2 < \ldots < p_m$. We can verify, that the graph $H = C_{p_1} + P_{p_1-1}$ is a minimal forbidden graph of $\mathcal{P}^2$ in which any edge e of the cycle C_{p_1} is eligible. Then a $C_{p_1}^{P_{p_1}}$-conjunction based on the pairs (H, e) with $r_i \geq \frac{p_m-1}{2}$ is a minimal forbidden graph of $\mathcal{P}^2$.

We have succeeded in finding eligible edges in the minimal forbidden graphs of several specific 2-connected reducible additive hereditary properties. However, it is an open problem whether the eligible edges required by Corollary 4 exist in general.

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