The Mean Integrity of Paths and Cycles

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Abstract

Let G be a graph with n vertices. The mean integrity of G is defined as follows: $J(G) = min_{P \subseteq V_G}\{|P| + \bar{m}(G-P)\}$, where $\bar{m}(G-P) = \frac{1}{n-|P|}\sum_{v \in V_{G-P}} n_v$ and n_v is the size of the component containing v. The main result of this article is a formula for the mean integrity of a path P_n of n vertices. A corollary of this formula establishes the mean integrity of a cycle C_n of n vertices.

1 Introduction

Graphs are often used to model real-world problems [2], such as problems in a computer network. In such a network, for example, it is not desirable to have all communication disrupted when only one station is not working. Instead, a single station's failure should have as little effect as possible on communication among the rest of the network. Ideally, in order to minimize the disruption of communication, every station should have a direct connection to every other station; however, the cost of these connections would be high, making that solution impractical in the real world. Some balance must be achieved between connectedness and cost, but fewer connections increase the network's vulnerability. This problem inspired the study of the integrity of graphs, first introduced by Barefoot, Entringer and Swart [3].

Let G be a graph with n vertices. The *integrity* of G is defined as follows: $I(G) = min_{P \subseteq V_G}\{|P| + m(G-P)\}$, where $|P|$ is the size of P and $m(G-P)$ is the size of the largest component of the graph $G-P$.

In graph G, any set P whose removal achieves this minimum is called an *I-set*. In terms of computer networking, the I-set is the set of failed stations, and $m(G-P)$ is the size of the largest remaining group that retains mutual communication. All possible combinations of failed stations are examined, as well as the largest group whose communication is left unhindered by the given set of failures. Finding the integrity means finding the "smallest" set that is the most destructive to the entire network's communication.

The integrities of some special families of graphs are listed below and can be found in [3, 4]:

$$I(G) = \begin{cases} n, & \text{if } G = K_n, \text{ the complete graph on n vertices} \\ \lceil 2\sqrt{n+1} \rceil - 2, & \text{if } G = P_n, \text{ a path of n vertices} \\ \lceil 2\sqrt{n} \rceil - 1, & \text{if } G = C_n, \text{ a cycle of n vertices} \\ 1 + min\{m, n\}, & \text{if } G = K_{m,n}, \text{ the complete bipartite graph} \end{cases}$$

Two significant variations of integrity are edge-integrity, denoted $I'(G)$, and mean integrity, denoted $J(G)$. *Edge-integrity* was also introduced by Barefoot, Entringer and Swart [3], and is defined as follows:

$I'(G) = min_{Q \subseteq E_G}\{|Q| + m(G-Q)\}$, where $m(G-Q)$ remains the size of the largest component of graph $G-Q$.

The edge-integrities of some special families of graphs are listed below and can be found in [1]:

$$I'(G) = \begin{cases} n, & \text{if } G = K_n, \text{ the complete graph on n vertices} \\ \lceil 2\sqrt{n} \rceil - 1, & \text{if } G = P_n, \text{ a path of n vertices} \\ \lceil 2\sqrt{n} \rceil, & \text{if } G = C_n, \text{ a cycle of n vertices, } n \geq 4 \\ m + n, & \text{if } G = K_{m,n}, \text{ the complete bipartite graph} \end{cases}$$

Mean integrity is very similar to integrity in that vertices are deleted. However instead of looking only at the size of the largest remaining component, mean integrity takes into account the sizes of *all* remaining components, replacing the size of the largest component with the weighted average of all components. Mean integrity, then, is a finer measure of vulnerability. Introduced by Chartrand, Kapoor, McKee and Oellermann [5], *mean integrity* for a graph G with n vertices is defined as follows:

$J(G) = min_{P \subseteq V_G}\{|P| + \tilde{m}(G-P)\}$,

where $\tilde{m}(G-P) = \frac{1}{n-|P|}\sum_{v \in V_{G-P}} n_v$ and n_v is the size of the component containing v. Note that $J(G) \in \mathbb{Q}$, whereas $I(G) \in \mathbb{N}$.

The mean integrities of some special families of graphs are listed below and can be found in [5]:

$$J(G) = \begin{cases} n, & \text{if } G = K_n, \text{ the complete graph on n vertices} \\ 1 + min\{m, n\}, & \text{if } G = K_{m,n}, \text{ the complete bipartite graph} \end{cases}$$

For almost all graphs the three integrities take on different values. This difference already occurs for very simple graphs, for example if $G = C_7$ a cycle of seven vertices, then $I(C_7) = 5$, $I'(C_7) = 6$ and $J(C_7) = 9/2$. In general we have the relationship $J(G) \leq I(G) \leq I'(G)$, see [1].

None of the three integrities is easy to calculate, and they are only known for very special graphs [1]. In fact the only algorithms to calculate these integrities that can be used on any graph work by an exhaustive procedure that searches through all subsets of the set of vertices (or edges for I'). If such a program is run on a desktop computer, the running time becomes prohibitive for graphs with more than 40 vertices.

Out of the three integrities the mean integrity is the most difficult to compute since one has to keep track of all different sizes of components created by deleting vertices. This can also be seen in that the list of graph families for which the mean integrity is known is the shortest. Missing from this list is the mean integrity of a path P_n of n vertices. Calculation of the mean integrity of a path is the main result of this paper.

2 Mean Integrity of a Path

The following is the main theorem of this article.

Theorem 1 *For $k \in \mathbb{N}, k \geq 2$,*

$$J(P_n) = \begin{cases} 2k-1, & \text{if } n = k^2+k-1 \\ k + \frac{tk^2+(k+1-t)(k-1)^2}{k^2-1+t}, & \text{if } n = k^2+k-1+t, 1 \leq t \leq k \\ 2k, & \text{if } n = k(k+2) \\ k + \frac{t(k+1)^2+(k+1-t)k^2}{k^2+k+t}, & \text{if } n = k(k+2)+t, 1 \leq t \leq k. \end{cases}$$

Remarks

1. Theorem 1 allows for the computation of the mean integrity of any path. For example, if $n = 82$, then $k = 8, t = 2$, and $k(k+2)+t = 8(10)+2$. From Theorem 1, $J(P_{82}) = k + \frac{t(k+1)^2+(k+1-t)k^2}{k^2+k+t}$. Here, $J(P_{82}) = \frac{601}{37} \approx 16.24$. Note that $I(P_{82}) = \lceil 2\sqrt{82+1} \rceil - 2 = 17$.

2. The result in Theorem 1 is not surprising in the following sense. The integrity or mean integrity of the path P_n is achieved when approximately $\sqrt{n}-1$ vertices are deleted and the remaining approximately $\sqrt{n}$ components all have approximately equal size of approximately $\sqrt{n}-1$. This gives an integrity (or mean integrity) of about $\sqrt{n}-1+\sqrt{n}-1 = 2\sqrt{n}-2$. The exact formula for the integrity of P_n now only needs the addition of the greatest integer function. For the mean integrity one has to exactly count how many components of a given size there are in $G - P$. In more detail the following pattern emerges:

If $n = k(k+2)$, then remove k evenly spaced vertices from P_n to cut the path into $k+1$ pieces of size k. This results into $I(P_n) = J(P_n) = 2k$.

When n increases by one, $n = k(k+2)+1$, there will be one component of size $k+1$ and k components of size k. As n grows larger there will be more and more components of size $k+1$ until all components in $P_n - P$ are of size $k+1$. Now $n = k(k+2)+k+1 = (k+1)^2+(k+1)-1$. The integrity now is $I(P_n) = J(P_n) = 2k+1 = 2(k+1)-1$. However the same integrity can be obtained by removing $k+1$ evenly spaced vertices from P_n to cut the path into $k+2$ pieces of size k. Using this way of obtaining the integrity when n increases by one to $n = (k+1)^2+(k+1)$ there will be one component of size $k+1$ and $k+1$ components of size k. As n grows larger there will be more and more components of size $k+1$ until all $k+2$ components in $P_n - P$ are of size $k+1$. Now $n = (k+1)(k+3) = k'(k'+2)$ for $k' = k+1$ and the pattern starts all over again. Figure 1 shows one such cycle.

n	k	form	J(Pn)
8	2	k(k+2)	4
9	2	k(k+2)+1	31/7
10	2	k(k+2)+2	19/4
11	3	k²+k-1	5
12	3	k²+k	16/3
13	3	k²+k+1	28/5
14	3	k²+k+2	64/11
15	3	k(k+2)	6

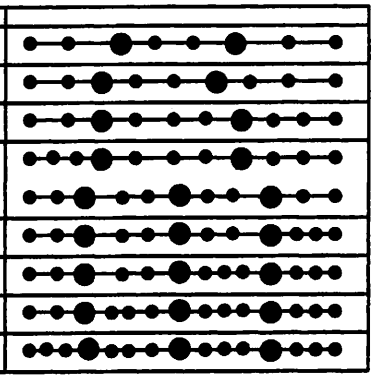

Figure 1: The different mean integrities of paths cycling through the different cases of Theorem 1 from $n = k(k+2)$ to $n = (k+1)(k+3)$ for $k = 2$.

The following are Corollaries of Theorem 1.

Corollary 2 *The following three conditions are equivalent for a path P_n:*
(i) $J(P_n) \in \mathbb{N}$,
(ii) $J(P_n) = I(P_n)$
(iii) $n = k(k+2)$ *or* $n = k^2+k-1$ *for some non negative integer* k.

Proof: (i) ($\Rightarrow$) (iii)
Assume $n = k^2+k+1+t$, for $1 \le t \le k$ and show that $J(P_n) \notin \mathbb{N}$.
By Theorem 1,

$$J(P_n) = k + \frac{tk^2+(k+1-t)(k-1)^2}{k^2-1+t} = k+k-1+\frac{tk}{k^2-1+t}.$$

$1 \le t \le k$ implies that $k \le tk \le k^2$, so $0 < \frac{tk}{k^2-1+t} < 1$.

Therefore, $2k-1 < J(P_n) < 2k$, so $J(P_n) \notin \mathbb{N}$.

Similarly if $n = k(k+2)+t$, for $1 \le t \le k$ then by Theorem 1,

$$J(P_n) = k + \frac{t(k+1)^2 + (k+1-t)k^2}{k^2+k+t} = 2k + \frac{tk+t}{k^2+k+t}.$$

$1 \le t \le k$ implies that $tk+t \le k^2+t < k^2+k+t$, so that $0 < \frac{tk+t}{k^2+k+t} < 1$.

Therefore, $2k < J(P_n) < 2k+1$, so $J(P_n) \notin \mathbb{N}$. Thus $J(P_n) \in \mathbb{N}$ if and only if $n = k(k+2)$ or $n = k^2+k-1$ for some $k \in \mathbb{N}$.

(iii) ($\Rightarrow$) (ii)

Assume $n = k(k+2)$ then $I(P_n) = \lceil 2\sqrt{n+1}\rceil - 2 = \lceil 2\sqrt{k^2+2k+1}\rceil - 2 = 2k = J(P_n)$.

If $n = k^2+k-1$ then $I(P_n) = \lceil 2\sqrt{n+1}\rceil - 2 = \lceil 2\sqrt{k^2+k}\rceil - 2$.

Unfortunately, $\sqrt{k^2+k}$ is not an integer, however $I(P_n) =$

$\lceil 2\sqrt{k^2+k}\rceil - 2 \le \lceil 2\sqrt{k^2+k+1/4}\rceil - 2 = \lceil 2(k+1/2)\rceil - 2 = 2k-1$.

Furthermore $I(P_n) = \lceil 2\sqrt{k^2+k}\rceil - 2 > 2k-2$. So $I(P_n) = 2k-1 = J(P_n)$.

(ii) ($\Rightarrow$) (i) This is obvious.

This last case completes the proof of the Corollary 2. □

Corollary 3 *The size of the J-set is unique if and only if $n \ne k^2+k-1$ for some $k \in \mathbb{N}$.*

Proof: The proof of Theorem 1 establishes the J-set size for all values of n, which is k when $n = k(k+2)$, $n = k^2+k-1+t$, and $n = k(k+2)+t$. For $n = k^2+k-1$, there is a J-set of size k and also a J-set of size $k+1$. The preceding corollary directly follows. □

Corollary 4 *For $n \ge 2$, $J(C_n) = J(P_{n-1}) + 1$.*

Proof: A path of length $n-1$ can be obtained from a cycle C_n by removing any vertex. Therefore, at most, the mean integrity of the cycle is one more than the mean integrity of P_{n-1}, so $J(C_n) \le 1 + J(P_{n-1})$.

Similarly, adding one vertex v to a path of length $n-1$, along with two edges, one joining v to each endpoint of P_{n-1}, creates a cycle C_n of length n. Therefore, at most, the mean integrity of the path P_{n-1} is one less than the mean integrity of the cycle $J(C_n)$ described, so $J(P_{n-1}) \le J(C_n) - 1$.

Therefore, $J(C_n) = J(P_{n-1}) + 1$. □

3 Proof of Theorem 1

$J(G) = min_{P \subseteq V_G}\{s + \frac{A}{n-s}\}$, where $A = \sum_{i=1}^{S_P} n_i{}^2$, n_i is the size of the ith component of $G - P$, $s = |P|$, and S_P is the number of components of $G - P$. In graph G, any set P whose removal achieves this minimum is called a J-set. The mean integrity is not difficult to calculate once a J-set has been found; however, finding such a set is not an easy task, as it can be accomplished only through an exhaustive search of all possible subsets of vertices. For ease of notation in these calculations, let $J_s(G) = min_{P \subseteq V_G}\{s + \frac{A}{n-s}\}$, where $A = \sum_{i=1}^{S_P} {n_i}^2$, $s = |P|$, S_P is the number of components of $G - P$, and P is any subset of V_G, not necessarily a J-set. Let $I_s(G)$ be similarly defined as $I_s(G) = min_{P \subseteq V_G}\{s + m(G - P)\}$. The proof of Theorem 1 is split into several lemmas.

Lemma 5 *Let n, s, n_i be whole numbers, for $1 \leq i \leq s+1$, such that $\sum_{i=1}^{s+1} n_i = n - s$. Let $\mu = \frac{n-s}{s+1}$ and $A = \sum_{i=1}^{s+1} {n_i}^2$. Then A is minimized when $|n_i - \mu| < 1$ for all $1 \leq i \leq s+1$.*

Proof: One needs to minimize $A = \sum_{i=1}^{s+1} {n_i}^2$ subject to the constraint $\sum_{i=1}^{s+1} n_i = n - s$. Clearly A is minimized when all n_i are the same size, that is, $n_i = \frac{n-s}{s+1} = \mu$ for $1 \leq i \leq s+1$. However, for the mean integrity, n_i is the size of the ith component, and must be an integer, which means that achieving this absolute minimum A is not always possible.

Assume there exists some n_i such that $n_i - \mu > 1$. Since $\sum_{i=1}^{s+1} n_i = n - s$, there exists some $n_j \neq n_i$ such that $n_j < \mu$. Without loss of generality, assume $n_1 - \mu > 1$ and $n_2 < \mu$. Let $A_1 = \sum_{i=1}^{s+1} {n_i}^2$ and $A_2 = (n_1 - 1)^2 + (n_2 + 1)^2 + \sum_{i=3}^{s+1} {n_i}^2$. Then $A_1 - A_2 = 2(n_1 - n_2 - 1)$. $n_1 - \mu > 1$ and $n_2 < \mu$ imply that $n_1 - n_2 > 1$. Therefore, $A_2 < A_1$. The case when there exists some n_i such that $\mu - n_i > 1$ is dealt with similarly. Therefore, A is minimized when $|n_i - \mu| < 1$, for all $i, 1 \leq i \leq s+1$. □

Corollary 6 *For a given $n, s \in \mathbb{N}$, where $s = |P|$ and $\frac{n-s}{s+1}$ is not an integer, the number of components in $P_n - P$ of size $\lceil \frac{n-s}{s+1} \rceil$, or $R_{\lceil \frac{n-s}{s+1} \rceil}$, is $n + 1 - (\lceil \frac{n-s}{s+1} \rceil)(s+1)$, and the number of components of size $\lfloor \frac{n-s}{s+1} \rfloor$, or $R_{\lfloor \frac{n-s}{s+1} \rfloor}$, is $(s+1)(\lceil \frac{n-s}{s+1} \rceil + 1) - n - 1$.*

Proof: Removing s vertices from P_n leaves $s+1$ components. By Lemma 5, each of these components must be of size $\lceil \frac{n-s}{s+1} \rceil$ or of size $\lfloor \frac{n-s}{s+1} \rfloor$.

Let $R_{\lceil \frac{n-s}{s+1} \rceil} = y$.

Then $R_{\lfloor \frac{n-s}{s+1} \rfloor} = s + 1 - y$.

$\sum_{i=1}^{s+1} n_i = n - s$ implies that $y(\lceil \frac{n-s}{s+1} \rceil) + (s + 1 - y)(\lfloor \frac{n-s}{s+1} \rfloor) = n - s$.

Let $x = \lceil \frac{n-s}{s+1} \rceil$. Then $x - 1 = \lfloor \frac{n-s}{s+1} \rfloor$. The above equation becomes

$$xy + (s + 1 - y)(x - 1) = n - s.$$

Solving this equation for y yields

$$y = n + 1 - x(s+1)$$

or equivalently

$$R_{\lceil \frac{n-s}{s+1} \rceil} = n + 1 - (\lceil \frac{n-s}{s+1} \rceil)(s+1).$$

Using this formula one obtains:

$$R_{\lfloor \frac{n-s}{s+1} \rfloor} = (s+1)(\lceil \frac{n-s}{s+1} \rceil + 1) - n - 1.$$

□

The next lemma addresses how large a J-set can be.

Lemma 7 *The size of any J-set of path P_n must be within one of $\mu = \frac{n-s^*}{s^*+1}$, where $s^* = -1 + \sqrt{n+1}$.*

Proof: By Lemma 5, A is minimized when all components are of equal size $\mu = \frac{n-s}{s+1}$, and

$J(P_n) \geq s + \frac{\sum_{i=1}^{s+1} (\frac{n-s}{s+1})^2}{n-s} = s + \frac{(s+1)\frac{(n-s)^2}{(s+1)^2}}{n-s} = s + \frac{n-s}{s+1}$.

Let $s^* \in \mathbb{R}$ such that s^* minimizes $g(s) = s + \frac{n-s}{s+1}$ for a given $n \in \mathbb{N}$.

$$g'(s) = 1 + \frac{-(s+1) - (n-s)}{(s+1)^2} = 1 + \frac{-(n+1)}{(s+1)^2}$$

$$g''(s) = \frac{2(n+1)}{(s+1)^3} > 0$$

$g'' > 0$ implies a minimum exists at s^* such that $g'(s^*) = 0$. Then $n + 1 = (s^* + 1)^2$ which implies that $s^* = -1 \pm \sqrt{n+1}$. Since s must be nonnegative, let $s^* = -1 + \sqrt{n+1}$. The equation $n + 1 = (s^* + 1)^2$ can be changed into the equation $s^* = \frac{n-s^*}{s^*+1}$. This means the minimum of $g(s)$ occurs at $g(s^*) = 2s^*$.

Call the minimum of $g(s)$ $J_{s^*} = s^* + \frac{n-s^*}{s^*+1} = 2s^*$. As previously stated, this s^* may not be an integer. Now suppose some real number b is added to this "ideal set size" s^* to make it into an integer. The minimum possible mean integrity with $s = s^* + b$ is $J_{s^*+b} = s^* + b + \frac{n-s^*-b}{s^*+b+1}$. Compare J_{s^*} to J_{s^*+b} by defining $f(b)$ as follows.

$$f(b) = J_{s^*+b} - J_{s^*}.$$

Substituting $n = (s^*)^2 + 2s^*$ yields

$$f(b) = J_{s^*+b} - J_{s^*} = \frac{b^2}{b + s^* + 1}.$$

$f(b)$ measures how far this J_{s^*+b} is from the ideal J_{s^*}, and the goal is to minimize this difference.

$$f'(b) = \frac{b[b+2(s^*+1)]}{(b+s^*+1)^2}.$$

So $f'(b) = 0$ when $b = 0$ or $b = -2(s^*+1)$. However, since $b + s^* \geq 0$, the latter never occurs. Additionally, $f'(b) < 0$ when $b < 0$, and $f'(b) > 0$ when $0 < b$. Therefore, minimizing the difference between J_{s^*+b} and the absolute minimum requires staying as close as possible to $b = 0$. In other words, the J-set must be as close as possible in size to $s^* = \frac{n-s^*}{s^*+1}$. Exactly s^* vertices can be removed only when s^* is a whole number; otherwise, $s = \lfloor \frac{n-s^*}{s^*+1} \rfloor$ or $s = \lceil \frac{n-s^*}{s^*+1} \rceil$, whichever yields a smaller value for J_{s^*+b}. □

Now all of the necessary tools are in place to prove the main theorem. The proof is split into four cases. Cases (*i*) and (*ii*) explore what happens when $n = k^2 + k - 1$ and $n = k(k+2)$, respectively. Case (*iii*) examines what happens for n between $k^2 + k - 1$ and $k(k+2)$. Similarly, Case (*iv*) addresses the situation for n between $k(k+2)$ and $(k+1)^2 + (k+1) - 1$.

Proof of Case (*i*). Assume $n = k(k+2)$.

By Lemma 7, $s^* = -1 + \sqrt{n+1} = k$. Since $s^* \in \mathbb{N}$, $s = s^* = k$, that is, k vertices must be removed to achieve the mean integrity. By Lemma 5, all component sizes must be within 1 of $\mu = \frac{n-s}{s+1} = \frac{k^2+2k-k}{k+1} = k$, so all components must be size k. Removing k vertices leaves $k+1$ components of size k, so $J(P_n) = k + \frac{(k+1)k^2}{k^2+k} = 2k$.

Proof of Case (*ii*). Assume $n = k^2 + k - 1$.

By Lemma 7, $s^* = -1 + \sqrt{n+1} = -1 + \sqrt{k^2+k}$. The following inequality $-1 + \sqrt{k^2} \leq -1 + \sqrt{k^2+k} \leq -1 + \sqrt{k^2+2k+1}$ implies $k - 1 \leq s^* \leq k$.

Since $s^* \notin \mathbb{N}, s = \lfloor s^* \rfloor = k - 1$ or $s = \lceil s^* \rceil = k$, whichever yields the lowest value for $J_s(P_n)$. For $s_1 = k - 1$, Lemma 5 implies all component sizes must be less than one away from $\mu = \frac{n-s_1}{s_1+1} = \frac{k^2+k-1-(k-1)}{(k-1)+1} = k$, so all components must be size k. Removing $k-1$ vertices leaves k components of size k, so $J_{s_1}(P_n) = k - 1 + \frac{k(k^2)}{k^2} = 2k - 1$. Now check J_s if $s = k$ vertices are removed. For $s_2 = k$, Lemma 5 implies all component sizes must be less than one away from $\mu = \frac{n-s_2}{s_2+1} = \frac{k^2+k-1-k}{k+1} = k - 1$, so all components must be size $k-1$. Removing k vertices leaves $k+1$ components of size $k-1$, so $J_{s_2}(P_n) = k + \frac{(k+1)(k-1)^2}{k^2-1} = k + \frac{(k+1)(k-1)^2}{(k+1)(k-1)} = 2k - 1$. Therefore for $n = k^2 + k - 1$, $J_s(P_n)$ is minimized by two values for s, that is $s = k$ and $s = k - 1$.

Proof of Case (*iii*). Assume $n = k^2 + k - 1 + t$, where $1 \leq t \leq k$.

$s^* = -1 + \sqrt{n+1} = -1 + \sqrt{k^2+k+t}$. The following inequality $-1 + \sqrt{k^2} < -1 + \sqrt{k^2+k+t} < -1 + \sqrt{k^2+2k+1}$ implies that $k-1 < s^* < k$.

Here $s^* \notin \mathbb{N}$, so $s = k-1$ or $s = k$, that is, the mean integrity is achieved by removing either a set of $k-1$ vertices or a set of k vertices, whichever removal yields the smaller value for J_s. Let $s_1 = k$ and $s_2 = k-1$.

For $s_1 = k$, $\frac{n-s_1}{s_1+1} = \frac{k^2-1+t}{k+1} = k - 1 + \frac{t}{k+1}$

$0 < t \leq k$ implies that $0 < \frac{t}{k+1} < 1$, so $\lfloor \frac{n-s_1}{s_1+1} \rfloor = k-1$ and $\lceil \frac{n-s_1}{s_1+1} \rceil = k$.

By Corollary 6,

$$R_{k-1} = (k+1)(k+1) - (k^2+k-1+t) - 1 = k+1-t,$$

and

$$R_k = k^2 + k - 1 + t + 1 - k(k+1) = t.$$

When k vertices are removed, $(k+1-t)$ components of size $k-1$ and t components of size k remain, so

$J_{s_1} = k + \frac{tk^2+(k+1-t)(k-1)^2}{k^2-1+t}$.

For $s_2 = k-1$, $\frac{n-s_2}{s_2+1} = \frac{k^2+k-1+t-k+1}{k} = k + \frac{t}{k}$

$0 < t \leq k$ implies that $0 < \frac{t}{k} \leq 1$. If $\frac{t}{k} = 1$, then $t = k$, and all components are size $k+1$. If $t \neq k$, then $0 < \frac{t}{k} < 1$ and $\lfloor \frac{n-s_2}{s_2+1} \rfloor = k$ and $\lceil \frac{n-s_2}{s_2+1} \rceil = k+1$. By Corollary 6,

$$R_k = k(k+2) - (k^2+k-1+t) - 1 = k-t,$$

and

$$R_{k+1} = k^2 + k - 1 + t + 1 - (k+1)k = t.$$

In either case, when $k-1$ vertices are removed, $(k-t)$ components of size k and t components of size $k+1$ remain, so

$J_{s_2} = k - 1 + \frac{(k-t)k^2+t(k+1)^2}{k^2+t}$.

Here,

$$\begin{aligned} J_{s_2} &= k - 1 + \frac{(k-t)k^2 + t(k^2+2k+1)}{k^2+t} \\ &= k + \frac{tk^2 + k^3 - 2k^2 + k + k^2 - 2k + 1 - tk^2 + 2tk - t}{k^2-1+t}, \end{aligned}$$

and

$$J_{s_1} = k + \frac{(k^3 - k + 2tk) - (k^2 - 1 + t)}{k^2-1+t} = 2k - 1 + \frac{tk}{k^2-1+t}.$$

Now assume $J_{s_2} \leq J_{s_1}$. Then $J_{s_2} - J_{s_1} \leq 0$. That is,

$$2k - 1 + \frac{kt+t}{k^2+t} - [2k - 1 + \frac{tk}{k^2-1+t}] \leq 0,$$

or equivalently

$$\frac{t^2 + tk^2 - (t + tk)}{(k^2 + t)(k^2 - 1 + t)} \le 0.$$

Since $(k^2+t)(k^2-1+t) > 0$, $t^2+tk^2-(t+tk) \le 0$. This is a contradiction because for $k \ge 2$, $k < k^2$. Also, $t \ge 1$, so $k^2 + t > k + 1$, which implies that $tk^2 + t^2 - (tk + t) > 0$, so $J_{s_2} \not\le J_{s_1}$. Therefore, $J_{s_2} > J_{s_1}$, and s_1 is a J-set, so $J(P_n) = J_{s_1} = k + \frac{tk^2+(k+1-t)(k-1)^2}{k^2-1+t}$.

Proof of Case (iv). Assume $n = k(k+2) + t$, for $1 \le t \le k$.

$s^* = -1 + \sqrt{n+1} = -1 + \sqrt{k^2 + 2k + t + 1}$ The following inequality $-1 + \sqrt{k^2 + 2k + 1} < -1 + \sqrt{k^2 + 2k + t + 1} < -1 + \sqrt{k^2 + 4k + 4}$, implies that $k < s^* < k + 1$.

Here s^* is not a whole number, so $s = k$ or $s = k + 1$, that is, the mean integrity is achieved by removing either a set of k vertices or a set of $k + 1$ vertices, whichever removal yields the smaller value for J_s. Let $s_1 = k$ and $s_2 = k + 1$. A similar argument to case (iii) can now be carried out.

For $s_1 = k$, $\frac{n-s_1}{s_1+1} = k + \frac{t}{k+1}$.

$0 < t \le k$ implies that $0 < \frac{t}{k+1} < 1$, so $\lfloor \frac{n-s_1}{s_1+1} \rfloor = k$ and $\lceil \frac{n-s_1}{s_1+1} \rceil = k + 1$.

By Corollary 6,

$$R_k = (k+1)(k+2) - (k^2 + 2k + t) - 1 = k + 1 - t,$$

and

$$R_{k+1} = k^2 + 2k + t + 1 - (k+1)^2 = t.$$

When k vertices are removed, $(k + 1 - t)$ components of size k and t components of size $k + 1$ remain, so

$J_{s_1} = k + \frac{t(k+1)^2+(k+1-t)k^2}{k^2+k+t} = \frac{k^3+k^2+2tk+t}{k^2+k+t}$.

For $s_2 = k + 1$, $\frac{n-s_2}{s_2+1} = k - 1 + \frac{t+1}{k+2}$.

$0 < t \le k$ implies that $0 < \frac{t+1}{k+2} < 1$, so $\lfloor \frac{n-s_2}{s_2+1} \rfloor = k - 1$ and $\lceil \frac{n-s_2}{s_2+1} \rceil = k$.

By Corollary 6,

$$R_{k-1} = (k+2)(k+1) - (k^2 + 2k + t) - 1 = k + 1 - t,$$

and

$$R_k = k^2 + 2k + t + 1 - k(k+2) = t + 1.$$

When $k + 1$ vertices are removed, $(k + 1 - t)$ components of size $k - 1$ and $t + 1$ components of size k remain, so

$J_{s_2} = k + 1 + \frac{(k+1-t)(k-1)^2+(t+1)k^2}{k^2+k+t-1} = \frac{k^3+k^2+2tk+t}{k^2+k+t}$.

Now assume $J_{s_2} \le J_{s_1}$. Then $J_{s_2} - J_{s_1} \le 0$. That is,

$$\frac{k^3 + k^2 + 2tk}{k^2 + k + t - 1} - \frac{k^3 + k^2 + 2tk + t}{k^2 + k + t} \le 0.$$

This inequality is equivalent to

$$\frac{k^3 + k^2 + tk + t - (k^2t + t^2)}{(k^2 + k + t - 1)(k^2 + k + t)} \leq 0.$$

Since $(k^2+k+t-1)(k^2+k+t) > 0$, $k^3+k^2+tk+t-(k^2t+t^2) \leq 0$. This is a contradiction because $0 < t \leq k$ implies that $k^3+k^2+tk+t > k^2t+t^2$, so $k^3 + k^2 + tk + t - (k^2t + t^2) > 0$ and $J_{s_2} \nleq J_{s_1}$. Therefore, $J_{s_2} > J_{s_1}$, s_1 is a J-set, and $J(P_n) = J_{s_1} = k + \frac{t(k+1)^2+(k+1-t)k^2}{k^2+k+t}$. This completes the proof of case (iv) and also completes the proof of Theorem 1.

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