The competition number of a graph with exactly two holes

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Abstract

Let D be an acyclic digraph. The competition graph of D is a graph which has the same vertex set as D and has an edge between x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of G is the smallest number of such isolated vertices.

A hole of a graph is a cycle of length at least 4 as an induced subgraph. In 2005, Kim [5] conjectured that the competition number of a graph with h holes is at most $h + 1$. Though Li and Chang [8] and Kim *et al.* [7] showed that her conjecture is true when the holes do not overlap much, it still remains open for the case where the holes share edges in an arbitrary way. In order to share an edge, a graph must have at least two holes and so it is natural to start with a graph with exactly two holes. In this paper, the conjecture is proved true for such a graph.

Keywords: competition graph; competition number; hole

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1 Introduction

Suppose D is an acyclic digraph (for all undefined graph-theoretical terms, see [1] and [13]). The *competition graph* of D, denoted by $C(D)$, has the same vertex set as D and has an edge between vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. Roberts [12] observed that, for any graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph.

The notion of competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see [4, 9] for surveys). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [11].

Roberts [12] observed that characterization of competition graphs is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for a graph G, as Opsut [10] showed that the computation of the competition number of a graph is an NP-hard problem (see [4, 6] for graphs whose competition numbers are known). It has been one of the important research problems in the study of competition graphs to determine the competition numbers that are possible for various graph classes. A cycle of length at least 4 of a graph as an induced subgraph is called a *hole* of the graph and a graph without holes is called a *chordal graph*. As Roberts [12] showed that the competition number of a chordal graph is at most 1, the competition number of a graph with 0 holes is at most 1. Cho and Kim [2] and Kim [5] studied the competition number of a graph with exactly one hole. Cho and Kim [2] showed that the competition number of a graph with exactly 1 hole is at most 2.

Theorem 1.1 (Cho and Kim [2]). *Let G be a graph with exactly one hole. Then the competition number of G is at most* 2.

Kim [5] conjectured that the competition number of a graph with h holes is at most $h + 1$ from these results. Recently, Li and Chang [8] showed that her conjecture is true for a huge family of graphs. In a graph G, a hole C is *independent* if the following two conditions hold for any other hole C' of G,

(1) C and C' have at most two common vertices.

(2) If C and C' have two common vertices, then they have one common edge and C is of length at least 5.

Theorem 1.2 (Li and Chang [8]). *Suppose that G is a graph with exactly h holes, all of which are independent. Then $k(G) \leq h + 1$.*

After then, Kim, Lee, and Sano [7] generalized the above theorem to the following theorem.

Theorem 1.3 (Kim *et al.* [7]). *Let $C_1, \ldots, C_h$ be the holes of a graph G. Suppose that*

(1) *each pair among $C_1, \ldots, C_h$ share at most one edge, and*

(2) *if C_i and C_j share an edge, then both C_i and C_j have length at least* 5.

Then $k(G) \leq h + 1$.

Thus, it is natural to ask if the bound holds when the holes share arbitrarily many edges. In this paper, we show that the answer is yes for a graph G with exactly two holes. Our main theorem is as follows.

Theorem 1.4. *Let G be a graph with exactly two holes. Then the competition number of G is at most* 3.

This paper is organized as follows. In Section 2, we investigate some properties of graphs with holes. In Section 3, we give a proof of Theorem 1.4.

2 Preliminaries

A set S of vertices of a graph G is called a *clique* of G if the subgraph of G induced by S is a complete graph. A set S of vertices of a graph G is called a *vertex cut* of G if the number of connected components of $G - S$ is greater than that of G.

Cho and Kim [2] showed that for a chordal graph G, we can construct an acyclic digraph D with as many vertices of indegree 0 as there are vertices in a clique so that the competition graph of D is G with one more isolated vertex:

Lemma 2.1 ([2]). *If X is a clique of a chordal graph G, then there exists an acyclic digraph D such that $C(D) = G \cup \{i\}$ where i is an isolated vertex, and the vertices of X have only outgoing arcs in D.*

Theorem 2.2. *Let G be a graph and k be a non-negative integer. Suppose that G has a subgraph G_1 with $k(G_1) \leq k$ and a chordal subgraph G_2 such that $E(G_1) \cup E(G_2) = E(G)$ and $X := V(G_1) \cap V(G_2)$ is a clique of G_2. Then $k(G) \leq k + 1$.*

Proof. Since $k(G_1) \leq k$, there exists an acyclic digraph D_1 such that $C(D_1) = G_1 \cup I_k$ where I_k is a set of k isolated vertices with $I_k \cap V(G) = \emptyset$. Since X is a clique of a chordal graph G_2, there exists an acyclic digraph D_2 such that $C(D_2) = G_2 \cup \{a\}$ where a is an isolated vertex not in $V(G) \cup I_k$ and that the vertices in X have only outgoing arcs in D_2 by Lemma 2.1. Now we define a digraph D as follows: $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2)$.

Suppose that there is an edge in $E(C(D))$ but not in $E(C(D_1)) \cup E(C(D_2))$. Then there exist an arc (u, x) in D_1 and an arc (v, x) in D_2 for some $x \in X$. However, this is impossible since every vertex in X has indegree 0 in D_2. Thus $E(C(D)) \subseteq E(C(D_1)) \cup E(C(D_2))$. It is obvious that $E(C(D)) \supseteq E(C(D_1)) \cup E(C(D_2))$ since $E(C(D)) \supseteq E(C(D_i))$ for $i = 1, 2$. Thus

$$E(C(D)) = E(C(D_1)) \cup E(C(D_2)) = E(G_1) \cup E(G_2) = E(G).$$

Hence $C(D) = G \cup I_k \cup \{a\}$. Moreover, since D_1 and D_2 are acyclic, $V(G_1) \cap V(G_2) = X$, and each vertex in X has only outgoing arcs in D_2, it follows that D is also acyclic. Hence $k(G) \leq k + 1$. □

Lemma 2.3 ([7]). *Let G be a graph and C be a hole of G. Suppose that v is a vertex not on C that is adjacent to two non-adjacent vertices x and y of C. Then exactly one of the following is true:*

(1) *v is adjacent to all the vertices of C;*

(2) *v is on a hole C^* different from C such that there are at least two common edges of C and C^* and all the common edges are contained in exactly one of the (x, y)-sections of C.*

For a graph G and a hole C of G, we denote by X_C the set of vertices which are adjacent to all vertices of C. Note that $V(C) \cap X_C = \emptyset$. Given a walk W of a graph G, we denote by W^{-1} the walk represented by the reverse of vertex sequence of W. For a graph G and a hole C of G, we call a walk (resp. path) W a *C-avoiding walk* (resp. *C-avoiding path*) if one of the following holds:

- $|E(W)| \geq 2$ and none of the internal vertices of W are in $V(C) \cup X_C$;
- $|E(W)| = 1$ and one of the two vertices of W is not in $V(C) \cup X_C$.

The following lemma immediately follows from Lemma 2.3.

Lemma 2.4. *Let G be a graph and C be a hole of G. Suppose that there exists a vertex v such that v is adjacent to consecutive vertices v_i and v_{i+1} of C, and that v is not on X_C and not on any hole of G. Then, if there is a C-avoiding path P from v to a vertex in $V(C) \setminus \{v_i, v_{i+1}\}$, then P has length at least 2.*

Proof. Let P be a C-avoiding path from v to a vertex w in $V(C) \setminus \{v_i, v_{i+1}\}$. If $|E(P)| = 1$, then v is adjacent to two non-adjacent vertices of C since $\{v_i, v_{i+1}, w\}$ does not induce a triangle. Then v satisfies the hypothesis of Lemma 2.3 while it does not satisfy none of (1) and (2) in Lemma 2.3, which is a contradiction. Thus, $|E(P)| \geq 2$. □

3 Proof of Theorem 1.4

In this section, we shall show that the competition number of a graph with exactly two holes cannot exceed 3.

Let G be a graph with exactly two holes C_1 and C_2. We denote the holes of G by

$$C_1 : v_0 v_1 \cdots v_{m-1} v_0, \quad C_2 : w_0 w_1 \cdots w_{m'-1} w_0,$$

where m and m' are the lengths of the holes C_1 and C_2, respectively. In the following, we assume that all subscripts of vertices on a cycle are considered modulo the length of the cycle. Without loss of generality, we may assume that $m \geq m' \geq 4$. For $t \in \{1, 2\}$, let

$$X_t := X_{C_t} = \{x \in V(G) \mid xv \in E(G) \text{ for all } v \in V(C_t)\}.$$

In the following, we deal with the case that the two holes have a common edge since Theorem 1.3 covers the case that the two holes are edge disjoint.

Lemma 3.1. *If a graph G has exactly two holes C_1 and C_2, then both X_1 and X_2 are cliques.*

Proof. Suppose that two distinct vertices x_1 and x_2 in X_1 are not adjacent. Then $x_1 v_0 x_2 v_2 x_1$ and $x_1 v_1 x_2 v_3 x_1$ are two holes other than C_1. That is, G has at least three holes, which is a contradiction. □

Lemma 3.2. *Let G be a graph having exactly two holes C_1 and C_2. If C_1 and C_2 have a common edge, then the subgraph of G induced by $E(C_1) \cap E(C_2)$ is a path.*

Proof. Suppose that $G[E(C_1) \cap E(C_2)]$ is not a path. Without loss of generality, we may assume that $v_0 v_1$ is a common edge but $v_1 v_2$ is not common. Let v_i be the first vertex on C_1 after v_1 common to C_1 and C_2. Then $i \in \{2, \ldots, m-2\}$. Let w be the vertex on C_2 that is adjacent to v_1 and that is not v_0. Let Z be the (w, v_i)-section of C_2 which does not contain v_0. Now, consider the (w, v_{m-1})-walk $W := Z v_{i+1} \cdots v_{m-1}$. Let P be a shortest (w, v_{m-1})-path among (w, v_{m-1})-paths such that $V(P) \subseteq V(W)$. We shall claim that $C := v_0 v_1 P v_0$ is a hole. Since neither v_0 nor v_1 is on W, none of v_0, v_1 is on P. Thus C is a cycle. By the definition of P, there is no chord between any pair of non-consecutive vertices on P. Since C_1 is a hole, v_0 is not adjacent to any of $v_{i+1}, \ldots, v_{m-2}$. Since $\{v_0\} \cup V(Z) \subset V(C_2)$, v_0 is not adjacent to any vertex on Z. Thus v_0 is not adjacent to any vertex on P. By a similar argument, we can show that v_1 is not adjacent to any vertex in $V(P) \setminus \{w\}$. Hence C is a hole of G. Since $v_1 v_2 \notin E(C)$, we have $C \neq C_1$ and so $C = C_2$.

If v_j is adjacent to a vertex v on Z for some $j \in \{i+1, \ldots, m-1\}$, then $v_j v$ is shorter than any (v, v_j)-path containing v_i in $G[W]$ and so P does not contain

v_i. Therefore $v_i \notin V(C)$, and so $C \neq C_2$, which is a contradiction. Thus, v_j is not adjacent to any vertex on Z for any $j \in \{i+1, \ldots, m-1\}$. Hence v_j is not on Z for any $j \in \{i+1, \ldots, m-1\}$. This implies that no vertex on W repeats and that no two non-consecutive vertices in W are adjacent. Thus $W = P$. Then $G[E(C_1) \cap E(C_2)] = v_i v_{i+1} \cdots v_{m-1} v_0 v_1$ is a path and we reach a contradiction. □

Lemma 3.3. *Let G be a graph having exactly two holes C_1 and C_2. If $|E(C_1) \cap E(C_2)| \geq 2$, then $X_1 = X_2$.*

Proof. By Lemma 3.2, we have $G[E(C_1) \cap E(C_2)] = w_i w_{i+1} \cdots w_j$ where $|j - i| \geq 2$. We take any vertex $x \in X_1$. If $x \in V(C_2)$, then C_2 has a chord xw_{i+1}, which is a contradiction. Therefore $x \notin V(C_2)$. Then x must be contained in X_2 by the Lemma 2.3 since x is adjacent to non-adjacent vertices w_i and w_j in $V(C_2)$. Thus, $X_1 \subseteq X_2$. Similarly, it can be shown that $X_2 \subseteq X_1$. □

Lemma 3.4. *Let G be a graph having exactly two holes C_1 and C_2. If there is no C_t-avoiding (u, v)-path for consecutive vertices u, v on C_t for $t \in \{1, 2\}$, then $G - uv$ has at most one hole.*

Proof. First, we consider the case where $uv \notin E(C_1) \cap E(C_2)$. We may assume that uv is an edge of C_1. Suppose that $G - uv$ has at least two holes. Let C^* be a hole of $G - uv$ different from C_2. Then $C^* + uv$ contains two cycles C_1 and C' sharing exactly one edge uv. Note that $C' \neq C_2$ since uv does not belong to C_2. If $|E(C')| \geq 4$, then C' is a hole, which is a contradiction. Thus it follows that $C' - uv$ is a path of length 2. Let x be the internal vertex of $C' - uv$. Since there is no C_1-avoiding (u, v)-path, it holds that $x \in X_1$. However, this implies that C^* has a chord joining x and every vertex in $V(C_1) \setminus \{u, v\}$, which is a contradiction.

Second, we consider the case where $uv \in E(C_1) \cap E(C_2)$. Then $G - uv$ contains neither C_1 nor C_2. If there exists a vertex $x \in X_1 \setminus X_2$ (resp. $x \in X_2 \setminus X_1$), uxv is a C_2-avoiding (resp. C_1-avoiding) path, which is a contradiction. Thus we can let $X = X_1 = X_2$. Suppose that $G - uv$ contains a hole C^*. Since C^* is not a hole of G, uv is a chord of C^* in G. In fact, uv is the unique chord of C^* in G. Let Z_1^* and Z_2^* be the two (u, v)-sections of C^*. If $|E(Z_1^*)| = |E(Z_2^*)| = 2$, then the internal vertices x_1 and x_2 of the (u, v)-paths Z_1^* and Z_2^*, respectively, are contained in X since there is no hole-avoiding (u, v)-path in G. So x_1 and x_2 are adjacent by Lemma 3.1, which contradicts the assumption that C^* is a hole of $G - uv$. If $|E(Z_i^*)| = 2$ and $|E(Z_j^*)| \geq 3$ where $\{i, j\} = \{1, 2\}$, then the internal vertex x_i of Z_i^* is in X and Z_j^* is one of $C_1 - uv$ and $C_2 - uv$ since $Z_j^* + uv$ is a hole of G. This implies that the vertex x_i is adjacent to all the internal vertices of Z_j^*, which also contradicts the assumption that C^* is a hole of $G - uv$. Hence, $|E(Z_1^*)| \geq 3$ and $|E(Z_2^*)| \geq 3$. This implies that C^* is composed of $C_1 - uv$ and $C_2 - uv$ and so $G - uv$ has at most one hole. □

Lemma 3.5. *Let G be a graph with exactly two holes C_1 and C_2 sharing at least one edge. Suppose that there exists a C_1-avoiding (v_i, v_{i+1})-path for each $i \in \{0, 1, \dots, m-1\}$. Then G has a subgraph G_1 which has exactly one hole and an induced subgraph G_2 which is chordal such that $E(G_1) \cup E(G_2) = E(G)$ and $V(G_1) \cap V(G_2) = X_1 \cup \{v_j, v_{j+1}\}$ for some $j \in \{0, 1, \dots, m-1\}$.*

Proof. By Lemma 3.2, $G[E(C_1) \cap E(C_2)]$ is a path. Without loss of generality, we may assume that $G[E(C_1) \cap E(C_2)] = v_0 v_1 \dots v_k = w_0 w_1 \dots w_k$ for some integer $k \geq 1$. We let

$$j = \begin{cases} 2 & \text{if } k = 1; \\ 0 & \text{if } k \geq 2. \end{cases}$$

Then $\{v_j, v_{j+1}\} \subseteq V(C_1) \setminus V(C_2)$ if $k = 1$, and $\{v_j, v_{j+1}\} \subseteq V(C_1) \cap V(C_2)$ if $k \geq 2$. Let L be a shortest C_1-avoiding (v_j, v_{j+1})-path. If $|E(L)| \geq 3$, then $L + v_{j+1}v_j$ is a hole of G sharing exactly one edge with C_1, which is a contradiction. Thus $|E(L)| = 2$ and so $L = v_j v v_{j+1}$ for some $v \in V(G) \setminus V(C_1)$. Now we show that $v \notin V(C_2)$ by contradiction. Suppose that $v \in V(C_2)$. We first consider the case $k = 1$. If $v = w_{k+1}$, then v is adjacent to two non-adjacent vertices $v_k (= v_1)$ and $v_{j+1} (= v_3)$ in $V(C_1)$. By Lemma 2.3, v is in X_1 or G has two holes which have at least two common edges, and we reach a contradiction. Therefore $v \neq w_{k+1}$. Then v_j is adjacent to two non-adjacent vertices v_k and v in $V(C_2)$, which is also a contradiction. Thus $v \notin V(C_2)$ in either case.

Now we will show that $X_1 \cup \{v_j, v_{j+1}\}$ is a vertex cut by contradiction. Suppose that v is connected to a vertex in $V(C_1) \setminus \{v_j, v_{j+1}\}$ by a C_1-avoiding path. Let v_ℓ be the first vertex on the (v_{j+1}, v_j)-path $C_1 - v_j v_{j+1}$ such that there is a C_1-avoiding (v, v_ℓ)-path, and let P be a shortest C_1-avoiding (v, v_ℓ)-path. By Lemma 2.4, $|E(P)| \geq 2$. In the following, we will show that v_{j+1} is adjacent to every internal vertex on P. Let Q be the (v_{j+1}, v_ℓ)-section of C_1 which does not contain v_j. Then $v_{j+1}PQ^{-1}$ is a cycle of length at least 4 different from C_1. Note that $v_{j+1} \in V(v_{j+1}PQ^{-1})$ while $v_{j+1} \notin V(C_2)$ if $k = 1$, and that $v_j \in V(C_2)$ while $v_j \notin V(v_{j+1}PQ^{-1})$ if $k \geq 2$. Therefore $v_{j+1}PQ^{-1}$ is also different from C_2. Thus $v_{j+1}PQ^{-1}$ cannot be a hole and so it has a chord. By the choice of v_ℓ, no internal vertex of Q is adjacent to any internal vertex of P. Since P is a shortest path, any two non-consecutive vertices of P are not adjacent. In addition, since Q is a part of a hole, any two non-consecutive vertices are not adjacent. Thus v_{j+1} is adjacent to an internal vertex of P. Let x be the first internal vertex on P adjacent to v_{j+1} and let P' be the (v, x)-section of P. Then $v_{j+1}P'v_{j+1}$ is a hole or a triangle. However, if $k = 1$, then $v_{j+1}P'v_{j+1}$ is different from C_1 and v_{j+1} is not on any hole other than C_1. If $k \geq 2$, then $v_j \in V(C_1) \cap V(C_2)$ but v_j is not contained in $v_{j+1}P'v_{j+1}$. Therefore $v_{j+1}P'v_{j+1}$ cannot be a hole whether $k = 1$ or $k \geq 2$. Thus $v_{j+1}P'v_{j+1}$ is a triangle and so x immediately follows v on P. Now consider the cycle consisting of v_{j+1}, the (x, v_ℓ)-section of P, and Q^{-1}. If this cycle is a triangle, then we are done. Otherwise, we apply the same argument to conclude that v_{j+1} is adjacent to the vertex immediately following x on P. By repeating this argument, we can show that v_{j+1} is adjacent to every internal vertex on P. Then the cycle C' consisting of v_{j+1}, the vertex immediately proceeding v_ℓ on P, Q^{-1} is either a hole or a triangle. If $k = 1$, then v_{j+1} is not on any hole other than C_1. However, $C' \neq C_1$ and so C' cannot be a hole. If $k \geq 2$, then v_j is not on C' while it is on both C_1 and C_2, and so C' cannot be a hole. Thus C' must be triangle and so $\ell = j + 2$.

Let y be the last vertex on P that is adjacent to v_j. Such y exists since v is adjacent to v_j. Let P'' be the (y, v_{j+2})-section of P and C'' be the cycle resulting from deleting v_{j+1} from C_1 and then adding path P''. Then $|E(C'')| \geq 4$. If $k = 1$, then it holds that $C'' \neq C_1$ since $v_{j+1} \notin V(C'')$ and that $C'' \neq C_2$ since $v_j \in V(C'')$ and $v_j \notin V(C_2)$. If $k \geq 2$, then C'' is different from both C_1 and C_2 since $v_{j+1} \notin V(C'')$. Thus C'' cannot be a hole in either case and so C'' has a chord. Recall that any two non-consecutive vertices on P cannot be adjacent and that any two non-consecutive vertices in $V(C') \cap V(C_1) = V(C_1) \setminus \{v_{j+1}\}$ cannot be adjacent. Thus a vertex u on P'' must be adjacent to a vertex v_r on C'' to form a chord if $k = 1$ while a vertex u on P'' must be adjacent to a vertex $v_r \in V(C_1) \setminus \{v_{j+1}\}$ if $k \geq 2$. Obviously $r \neq j + 2$. Moreover, by the choice of u, $r \neq j$. Then u is adjacent to two nonconsecutive vertices v_{j+1} and v_r on C_1. If $k = 1$, then, by Lemma 2.3, $u \in X_1$ or G contains two holes which have at least two common edges, either of which is a contradiction. Now suppose that $k \geq 2$. Since $u \notin X_1$, by Lemma 2.3, u is on C_2 and all the edges common to C_1 and C_2 are contained in exactly one of the (v_{j+1}, v_r)-section of C_1. However, edges v_jv_{j+1} and $v_{j+1}v_{j+2}$ belong to distinct (v_{j+1}, v_r)-sections of C_1 even though they are shared by C_1 and C_2 by the hypothesis. Thus we have reached a contraction. Consequently, there is no C_1-avoiding path between v and a vertex in $V(C_1) \setminus \{v_j, v_{j+1}\}$. This implies that $X_1 \cup \{v_j, v_{j+1}\}$ is a vertex cut.

Now we define the subgraphs G_1 and G_2 of the graph G as follows. Let Q be the component of $G - (X_1 \cup \{v_j, v_{j+1}\})$ that contains $V(C_1) \setminus \{v_j, v_{j+1}\}$. Let G_2 be the subgraph of G induced by the vertex set $V(G) \setminus V(Q)$. Then, since v_0 (resp. v_2) is a vertex in $V(C_1) \cap V(C_2) \cap V(Q)$ for $k = 1$ (resp. $k \geq 2$), C_2 is not contained in G_2 and so G_2 is chordal. Let G_1' be the subgraph induced by $V(Q) \cup X_1 \cup \{v_j, v_{j+1}\}$. Then G_1' contains no C_1-avoiding (v_j, v_{j+1})-path. Therefore the subgraph $G_1 := G_1' - v_jv_{j+1}$ has exactly one hole by Lemma 3.4. By the definitions of G_1 and G_2, we can check that $E(G_1) \cup E(G_2) = E(G)$ and $V(G_1) \cap V(G_2) = X_1 \cup \{v_j, v_{j+1}\}$. Hence the lemma holds. □

Now, we are ready to complete the proof of the main theorem.

Proof of Theorem 1.4. If C_1 and C_2 do not share an edge, then $k(G) \leq 3$ by Theorem 1.3. Thus we may assume that C_1 and C_2 share at least one edge. By Lemma 3.2, $G[E(C_1) \cap E(C_2)]$ is a path. Suppose that there is no C_1-avoiding

(v_i, v_{i+1})-path for some $i \in \{0, \dots, m-1\}$. Then $G_1 := G - v_iv_{i+1}$ has at most one hole by Lemma 3.4 and so $k(G_1) \leq 2$ by Theorem 1.1. Let $G_2 := v_iv_{i+1}$. Then G_2 is chordal, $E(G_1) \cup E(G_2) = E(G)$, and $V(G_1) \cap V(G_2) = \{v_i, v_{i+1}\}$ is a clique of G_2. By Theorem 2.2, we have $k(G) \leq 3$.

Now we suppose that there is a C_1-avoiding (v_i, v_{i+1})-path for any $i \in \{0, 1, \dots, m-1\}$. By Lemma 3.5, G has a subgraph G_1 which has exactly one hole and an induced subgraph G_2 which is chordal such that $E(G_1) \cup E(G_2) = E(G)$ and $V(G_1) \cap V(G_2) = X_1 \cup \{v_j, v_{j+1}\}$ for some $j \in \{0, 1, \dots, m-1\}$. Note that $X_1 \cup \{v_j, v_{j+1}\}$ is a clique of G_2. By Theorem 1.1, we have $k(G_1) \leq 2$. Hence $k(G) \leq 3$ by Theorem 2.2. □

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