BOOK EMBEDDINGS AND ZERO DIVISORS

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1. Introduction

Throughout this paper all rings are commutative rings with identity. Also, R is a finite local ring with maximal ideal M. We use $\mathbf{Z}$ to denote the set of integers, $\mathbf{N}$ to denote the set of natural numbers and $\mathbf{Z}^+$ to denote $\mathbf{N} - \{0\}$. Following Anderson and Livingston [1] we define the zero-divisor graph, $\Gamma(R)$, to be the graph whose vertices are the nonzero zero-divisors of R, with two different vertices x and y joined by an edge in case $xy = 0$. Since we are only considering finite local rings, the set of vertices of $\Gamma(R)$ is $M - \{0\}$.

Given $i \in \mathbf{Z}^+$,

$$M^i = \{\sum_{j=1}^{t} m_{1j} \cdot \ldots \cdot m_{ij} | t \in \mathbf{Z}^+ \text{ and } m_{1j}, \ldots, m_{ij} \in M\}.$$

Let $M_i = M^i - M^{i+1}$ and let $n_i = |M_i|$. Let κ be the minimal element in $\{i \in \mathbf{Z}^+ | M^i = \{0\}\}$. We observe that if $j \in \{1, 2, \ldots, \kappa\}$, then $M^j - \{0\}$ is the disjoint union of

$$M_j, M_{j+1}, \ldots, M_{\kappa-1},$$

where $M^\kappa = \{0\}$.

A simple graph is a graph with no edges connecting a vertex to itself and at most one edge connecting two distinct vertices. The simple graph with n vertices and all possible $\binom{n}{2}$ edges is called the complete graph K_n. The order of a finite graph is the number of vertices of the graph, so K_n is a graph of order n.

A null graph is a graph in which no two vertices are joined by an edge. In particular, the empty graph is a null graph. A graph G is bipartite if G is null or if there is a partition $V(G) = V_1 \cup V_2$ of the vertices of G so that every edge of G has one endpoint in V_1 and the other endpoint in V_2. The complete bipartite graph $K_{m,n}$ is the bipartite graph of order $m + n$ with vertices $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ and all possible mn edges $\{a_i, b_j\}$.

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Definition 1.1. *Let $B(M_i, M_j)$ be the bipartite graph with vertex set $M_i \cup M_j$ with two vertices x and y joined by an edge if $x \in M_i$, $y \in M_j$, and $xy = 0$.*

If S is a subset of R, we define $\Gamma(S)$ to be the subgraph of $\Gamma(R)$ whose vertices are the nonzero zero-divisors of S with two vertices x and y joined by an edge in case $xy = 0$. The functorial properties of Γ are discussed in DeMeyer, Schneider, McKenzie [3].

Theorem 1.2. *Let $i \in \mathbf{Z}^+$. The graph $\Gamma(M_i) \simeq K_{n_i}$ if and only if $2i \geq \kappa$.*

Proof. Assume $2i \geq \kappa$. Let x and y be elements of M_i. Then xy is in M^{2i}. Since $2i \geq \kappa$, $M^{2i} = \{0\}$, so $xy = 0$. Thus, $\Gamma(M_i) \simeq K_{n_i}$.

To prove the converse, let $x_1, x_2, \ldots, x_t \in M_i$ generate M^i. Then the set

$$\{x_s x_r | 1 \leq s \leq t, 1 \leq r \leq t\}$$

generates M^{2i}. Since $\Gamma(M_i)$ is complete,

$$x_s x_r = 0, \ \forall r, s \in \{1, 2, \ldots, t\}.$$

Hence, $M^{2i} = \{0\}$ and $2i \geq \kappa$.

□

Example 1.3. *If $R = \mathbf{Z}_{p^2}$ then $M^2 = \{0\}$, so $\Gamma(R)$ is a complete graph of order $p - 1$.*

Theorem 1.4. *Let $i, j \in \mathbf{Z}^+$ with $i \neq j$. The graph $B(M_i, M_j) \simeq K_{n_i, n_j}$ if and only if $i + j \geq \kappa$.*

Proof. Suppose $i + j \geq \kappa$. Let $x \in M_i$ and $y \in M_j$. Then xy is in M^{i+j}. Since $i + j \geq \kappa$, $M^{i+j} = \{0\}$. Hence $xy = 0$ and $B(M_i, M_j) \simeq K_{n_i, n_j}$.

Conversely, let

$$x_1, x_2, \ldots, x_{t_1} \in M_i$$

generate M^i and let

$$y_1, y_2, \ldots, y_{t_2} \in M_j$$

generate M^j . Then the set

$$\{x_s y_r | 1 \leq s \leq t_1, 1 \leq r \leq t_2\}$$

generates M^{i+j}. Since

$$B(M_i, M_j) \simeq K_{n_i, n_j},$$

$x_s y_r = 0$ for all $s \in \{1, 2, \ldots, t_1\}$ and $r \in \{1, 2, \ldots, t_2\}$. Hence, $M^{i+j} = \{0\}$ and $i + j \geq \kappa$.

□

Using Theorem 1.2, Theorem 1.4, and the decomposition of $M^i - \{0\}$ as the disjoint union of

$$M_i, M_{i+1}, \ldots, M_{\kappa-1},$$

where $M^\kappa = \{0\}$, we arrive at the following result.

Corollary 1.5. *Let i be an integer with $\kappa/2 \leq i \leq \kappa$, then $\Gamma(M^i)$ is a complete graph.*

2. Finite Principal Rings

In this section, we study finite local rings with principal maximal ideals. If M is a principal ideal generated by z, denoted $M = \langle z \rangle$, then $M^i = \langle z^i \rangle$. Recall that we defined κ to be the smallest positive integer for which $M^\kappa = \{0\}$. Hence, κ is the smallest positive integer for which $z^\kappa = 0$. We make use of the fact that if $M = \langle z \rangle$ and $|R/M| = q$, where q is a prime power, then $|M^i| = q^{\kappa-i}$ whenever $i \in \{1, 2, \ldots, \kappa\}$. Further

$$n_i = |M_i| = |M^i| - |M^{i+1}| = q^{\kappa-i} - q^{\kappa-i-1}.$$

Theorem 2.1. *Assume M is a principal ideal generated by z. If $i \in \mathbf{Z}^+$, then $\Gamma(M_i) \simeq K_{n_i}$ or $\Gamma(M_i)$ is a null graph.*

Proof. Suppose $\Gamma(M_i)$ is not null. Since z is a generator for M, z^i generates M^i. Further, there exist $r, s \in R - M$ such that

$$rz^i, sz^i \in M^i - M^{i+1}$$

with $0 = (rz^i)(sz^i) = (rs)z^{2i}$. Since r and s are units, rs is a unit, so $z^{2i} = 0$. Hence, $M^{2i} = \langle z^{2i} \rangle = \{0\}$. By Theorem 1.2 and since $|M^i| = q^{\kappa-i}$ it follows that $\Gamma(M_i)$ is a complete graph of order $n_i = q^{\kappa-i} - q^{\kappa-i-1}$. □

Combining Theorems 1.2 and 2.1 yields the following result.

Corollary 2.2. *If $i \in \mathbf{Z}^+$ and M is a principal ideal generated by z, then $\Gamma(M_i)$ is a null graph if and only if $i < \kappa/2$.*

Theorem 2.3. *Assume M is a principal ideal generated by z. If $i, j \in \mathbf{Z}^+$ and $i \neq j$, then $B(M_i, M_j) \simeq K_{n_i, n_j}$ or $B(M_i, M_j)$ is a null graph.*

Proof. Suppose $B(M_i, M_j)$ is not null. Since z^i generates M^i and z^j generates M^j, there exist $r, s \in R - M$ such that $rz^i \in M^i - M^{i+1}$ and $sz^j \in M^j - M^{j+1}$ with $0 = (rz^i)(sz^j) = (rs)z^{i+j}$. Since r and s are units, rs is a unit, so $z^{i+j} = 0$. Hence, $M^{i+j} = \langle z^{i+j} \rangle = \{0\}$. By Theorem 1.4 it follows that $B(M_i, M_j)$ is a complete bipartite graph of order $(q^{\kappa-i} - q^{\kappa-i-1}) + (q^{\kappa-j} - q^{\kappa-j-1})$. □

The following corollary follows from Theorem 1.4 and Theorem 2.3.

Corollary 2.4. *Let $i, j \in \mathbf{Z}^+$ with $i \neq j$. If M is a principal ideal generated by z, then $B(M_i, M_j)$ is a null graph if and only if $i + j < \kappa$.*

Theorem 2.5. *Assume M is a principal ideal generated by z. If i is an integer with $1 \leq i < \kappa/2$, then $\Gamma(M - M^{i+1})$ is a null graph.*

Proof. To prove the contrapositive, assume there exist $x, y \in M - M^{i+1}$ with $xy = 0$. Then there exist $r, s \in R - M$ and $j, k \in \{1, 2, \ldots, i\}$ with $x = rz^j$ and $y = sz^k$. Now $0 = (rz^j)(sz^k) = (rs)z^{j+k}$ and $r, s \in R - M$ implies that $z^{j+k} = 0$. Thus $j + k \geq \kappa$ and $j \geq \kappa/2$ or $k \geq \kappa/2$. Hence, $i \geq \kappa/2$. □

Given a positive real number w, the ceiling of w, $\lceil w \rceil$ is the smallest integer that is greater than or equal to w. Similarly, the floor of w, $\lfloor w \rfloor$ iss the greatest integer that is less than or equal to w. Recall that the vertex set of $\Gamma(R)$ is $M - \{0\}$. We can partition this vertex set into the disjoint subsets, $M - M^{\lceil \kappa/2 \rceil}$ and $M^{\lceil \kappa/2 \rceil} - \{0\}$. In the case where M is principal, we can draw two conclusions from this observation. From Theorem 2.5 we have $\Gamma(M - M^{\lceil \kappa/2 \rceil})$ is a null graph. Thus, any maximal complete subgraph of $\Gamma(R)$ contains at most one element from $M - M^{\lceil \kappa/2 \rceil}$.

The second conclusion gives bounds for the clique number of $\Gamma(R)$. The clique number of a graph is the order its largest complete subgraph. It follows from Corollary 1.5 that $\Gamma(M^{\lceil \kappa/2 \rceil} - \{0\})$ is a complete graph. Hence, the clique number of $\Gamma(R)$ is bounded below by $|M^{\lceil \kappa/2 \rceil}| - 1$. Since at most one vertex of $M - M^{\lceil \kappa/2 \rceil}$ can be included in any complete subgraph of $\Gamma(R)$, it follows that the clique number of $\Gamma(R)$ is bounded above by $|M^{\lceil \kappa/2 \rceil}|$.

Theorem 2.6. *Assume M is a principal ideal generated by z and let $|R/M| = q$, where q is a prime power. Then the clique number of $\Gamma(R)$ is $q^{\kappa/2} - 1$ if κ is even and $q^{(\kappa-1)/2}$ if κ is odd.*

Proof. Since $|M^{\lceil \kappa/2 \rceil}| = q^{\kappa - \lceil \kappa/2 \rceil} = q^{\lfloor \kappa/2 \rfloor}$, then the order of this complete graph is $q^{\lfloor \kappa/2 \rfloor} - 1$.

In the case where $\kappa/2$ is even, let x be an element of $M_{\kappa/2}$ and let y be an element of $M - M^{\kappa/2}$. Then $x = rz^{\kappa/2}$ and $y = sz^j$ where $r, s \in R - M$ and $j < \kappa/2$. Now $xy = (rz^{\kappa/2})(sz^j) = (rs)z^{(\kappa/2)+j}$. Since $rs \in R - M$ and $(\kappa/2) + j < \kappa$, then $xy \neq 0$. Thus, the clique number of $\Gamma(R)$ is $q^{\kappa/2} - 1$.

In the case where $\kappa/2$ is odd, let x be any element of $M_{(\kappa-1)/2}$. Then $x = rz^{(\kappa-1)/2}$ where $r \in R - M$. Now let $y \in M^{(\kappa+1)/2} - \{0\}$. Then $y = sz^i$ where $s \in R - M$ and $i \geq (\kappa+1)/2$. Now $xy = (rz^{(\kappa-1)/2})(sz^i) = (rs)z^{[(\kappa-1)/2]+i}$. Since $[(\kappa-1)/2] + i \geq \kappa$, then $xy = 0$. Thus, the clique number of $\Gamma(R)$ is $q^{(\kappa-1)/2}$. □

3. Book Embeddings

In an article published in 1979, Bernhart and Kainen [2] laid the groundwork for further study of book embeddings of graphs. They defined an

n-book as a line L in 3-space, called the spine, and n half-planes, called pages, with L as their common boundary. A book embedding of a graph G is an embedding of G in a book with the vertices of G on the spine and each edge of G within a single page so that no two edges cross. The book thickness $bt(G)$ or page number $pg(G)$ of a graph G is the smallest n so that G has an n-book embedding. An optimal book embedding is one that has $bt(G)$ pages.

The book-embedding problem is difficult since both the ordering of the vertices along the spine and the assignment of edges to pages must be considered. However, for certain families of graphs, the book thickness is known. The following theorem gives the book thickness of K_n.

Theorem 3.1. *If $n \geq 4$, then $bt(K_n) = \lceil n/2 \rceil$.*

Proof. See Bernhart and Kainen [2]. □

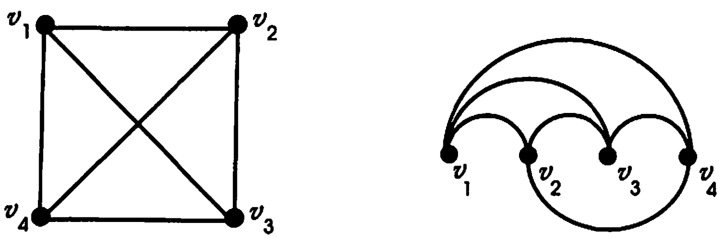

FIGURE 1. Two-page book embedding of K_4.

Figure 1 depicts a two-page book embedding of K_4. The vertices are lined up on the spine. The half-plane above the spine forms one page of the book and the half-plane below the spine forms the second page of the book. By Theorem 3.1, this is the least number of pages possible.

A graph is planar if it can be drawn in the plane so that no two edges cross. It is immediately clear that any two-page embeddable graph must be planar with the two pages forming a plane. However, not every planar graph has a two-page embedding. There are simple examples of planar graphs that require three pages [2]. Yannakakis [4] proves that every planar graph can be embedded in a book with four pages or less.

We now combine the theory of zero divisor graphs with the theory of book embeddings.

Definition 3.2. *A graded ordering of the vertices of $\Gamma(R)$ along the spine of a book is an ordering such that if $x \in M_i$ and $y \in M_j$ and $i < j$, then*

x lies to the left of y on the spine. A graded book embedding of $\Gamma(R)$ is a book embedding of $\Gamma(R)$ using any graded ordering of the vertices.

Note that graded orderings are not unique since there are no restrictions on the ordering of the vertices within a particular M_i.

Example 3.3. *Consider $R = \mathbf{Z}_{16}$. In this case, $M_1 = \{2, 6, 10, 14\}$, $M_2 = \{4, 12\}$, and $M_3 = \{8\}$. In a graded book embedding of $\Gamma(R)$, the vertices can be arranged in $4!2!$ different ways. Two of these ways are*

$$2\ 6\ 10\ 14\ 4\ 12\ 8$$

and

$$10\ 6\ 14\ 2\ 12\ 4\ 8.$$

In the case where M is principal and κ is even, $\Gamma(R)$ contains a complete subgraph of order $\iota = q^{\kappa/2} - 1$, where $q = |R/M|$, by Theorem 2.6. Now by Theorem 3.1, if $\iota \geq 4$, then $bt(\Gamma(R)) \geq bt(K_\iota) = \lceil \iota/2 \rceil$, giving a lower bound for $bt(\Gamma(R))$. We will show that every graded ordering can be used to obtain an optimal book embedding of $\Gamma(R)$ in a book with $\lceil \iota/2 \rceil$ pages. First, we need the following definition.

Definition 3.4. *Let G be a graph embedded in a book. And let x be a vertex of G. We say that x is obstructed on page i if there exist vertices $x_L, x_R \in V(G)$ such that x_L lies to the left of x, x_R lies to the right of x and there exists an edge on page i connecting x_L and x_R. Otherwise, we say that x is unobstructed on page i.*

Now we revisit the proof that when $n \geq 4, bt(K_n) = \lceil n/2 \rceil$. In Bernhart and Kainen's proof, they provide a method to embed K_n in $\lceil n/2 \rceil$ pages. We offer an alternate assignment of edges to pages that also achieves the optimal bound, but has a different set of unobstructed vertices.

Theorem 3.5. *Let K_n, $n \in \mathbf{Z}^+$, be the complete graph on the n vertices*

$$x_1, x_2, \ldots, x_n.$$

Then there exists a book embedding of K_n such that

(1) *if $i < j$, then x_i lies to the right of x_j on the spine;*
(2) *the book contains $\lceil n/2 \rceil$ pages;*
(3) *for $1 \leq i \leq \lceil n/2 \rceil$, vertex x_i is unobstructed on page i.*

Proof. We may assume that n is even since K_{n-1} is a subgraph of K_n. Let $n = 2t$, where $t \in \mathbf{Z}^+$. Place the vertices on the spine from left to right in the order $x_{2t}, x_{2t-1}, \ldots, x_t, x_{t-1}, \ldots, x_2, x_1$.

For $1 \leq i \leq t$, we assign edges to page i as follows. Page i will contain the set of non-intersecting edges

$$(x_i, x_{t+i}), \ldots (x_i, x_{2t}),$$

$$(x_i, x_1), \ldots, (x_i, x_{i-1}),$$
$$(x_{t+i}, x_{i+1}), \ldots, (x_{t+i}, x_{t+i-1}).$$

Note that each of t pages contains exactly t edges of the form (x_i, x_j) and $t-1$ additional edges of the form (x_{t+i}, x_j). Since each of these edges are distinct, the t-page embedding will contain all $t[t+(t-1)] = (n/2)(n-1) = \binom{n}{2}$ edges of K_n.

We also note that since the edges of the form (x_{t+i}, x_j) can be placed below the edges of the form (x_i, x_j) on page i, the vertex x_i is unobstructed on page i.

□

Theorem 3.6. *Let R be a finite local ring with a principal maximal ideal M and κ even. Then $bt(\Gamma(R)) = \lceil \iota/2 \rceil$, where $\iota = q^{\kappa/2} - 1$ and $q = |R/M|$.*

Proof. Clearly $bt(\Gamma(R)) \geq bt(K_\iota) = \lceil \iota/2 \rceil$, since $\Gamma(R)$ contains a complete subgraph of order ι with vertex set $M^{\kappa/2} - \{0\}$.

To show $bt(\Gamma(R)) \leq \lceil \iota/2 \rceil$, we make the observation that since $q \geq 2$, $|M_{\kappa/2}| \geq \iota/2$. That is, at least half of the vertices of this complete subgraph come from $M_{\kappa/2}$. So, by Theorem 3.5, this complete graph can be embedded with a graded ordering in an $\lceil \iota/2 \rceil$ page book so that remaining vertices of $M^{(\kappa/2)+1} - \{0\}$ each appear unobstructed on a page of the embedding.

Using a graded ordering of the vertices along the spine, there are no connections between the vertices of the set $M - M^{\kappa/2}$ since by Theorem 2.5, $\Gamma(M - M^{\kappa/2})$ is a null graph. Also, there are no connections between vertices of $M_{\kappa/2}$ and $M - M^{\kappa/2}$ since, by Corollary 2.4, $B(M_i, M_j)$ is a null graph when $i + j < \kappa$.

Now we embed the complete graph with vertex set $M^{\kappa/2} - \{0\}$ in an $\lceil \iota/2 \rceil$ page book using a graded ordering so that the vertices of $M^{(\kappa/2)+1} - \{0\}$ each appear unobstructed on a page of the embedding. The only edges left to embed are those connecting vertices of $M - M^{\kappa/2}$ and $M^{(\kappa/2)+1} - \{0\}$. These vertices can be placed in this book without requiring any additional pages in the following way. Let x be a vertex of $M^{(\kappa/2)+1} - \{0\}$. It is unobstructed on at least one page of the book. Now all edges from x to vertices of $M - M^{\kappa/2}$ can be placed on this page without creating any crossings. We repeat this process for each vertex of $M^{(\kappa/2)+1} - \{0\}$, obtaining an embedding of $\Gamma(R)$ in a book with $\lceil \iota/2 \rceil$ pages.

□

Next we consider an example. Let $R = \mathbf{Z}_{64}$. Then

$$M = \{0, 2, 4, \ldots, 60, 62\},$$
$$M_1 = \{2, 6, 10, \ldots, 58, 62\},$$
$$M_2 = \{4, 12, 20, 28, 36, 44, 52, 60\},$$

$$M_3 = \{8, 24, 40, 56\},$$
$$M_4 = \{16, 48\},$$
and
$$M_5 = \{32\}.$$

Since $M^6 = \{0\}$ we have $\kappa = 6$ and $\iota = 7$. Furthermore, $\Gamma(M - M^3)$ is a null graph and $\Gamma(M^3 - \{0\})$ is a complete graph of order 7. Hence, a 4-page book embedding of $\Gamma(R)$ is optimal.

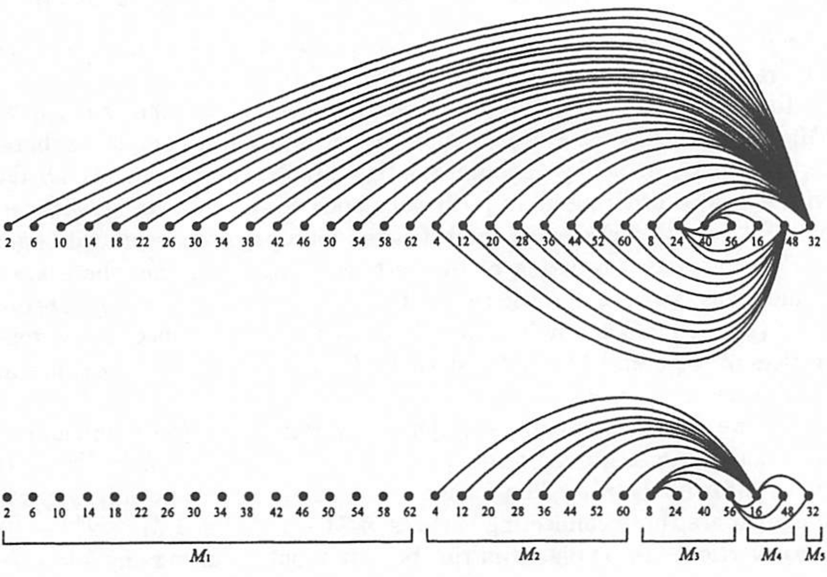

FIGURE 2. Four-page graded book embedding of $\Gamma(\mathbf{Z}_{64})$.

Figure 2 depicts a 4-page graded book embedding of $\Gamma(R)$. The edges above the first copy of the spine lie on page one, the edges below the first copy of the spine lie on page two, the edges above the second copy of the spine lie on page three, and the edges below the second copy of the spine lie on the fourth page.

We conclude this paper by considering which book embeddings correspond to graded book embeddings of zero divisor graphs of finite local principal rings. Let G be a finite simple graph with vertex set $V(G)$. We can define an equivalence relation on $V(G)$ as follows. The degree of a vertex x is the number of distinct vertices which are connected to x by an edge. Two vertices are equivalent if they have the same degree.

This induces a partition of the vertices

$$V(G) = V_1 \cup \ldots \cup V_t.$$

We further assume that if $x \in V_i$ and $y \in V_j$ and $i < j$, then the degree of x is less than the degree of y. We use this notation in the following theorem.

Theorem 3.7. *Let G be a finite simple graph with vertex set $V(G) = V_1 \cup \ldots \cup V_t$, embedded in a book. The embedding of G is a graded book embedding of $\Gamma(R)$ where R is a finite local principal ring if and only if the following hold:*

(1) *For all $x, y \in V(G)$ if x lies to the left of y on the spine, then the degree of x is less than or equal to the degree of y.*
(2) *There is a prime power q such that $|V_i| = q^{t-i+1} - q^{t-i}$ for all i with $1 \leq i \leq t$.*
(3) *For all $x, y \in V(G)$ with $x \in V_i$, $y \in V_j$, and $x \neq y$, there is an edge between x and y in the graph if and only if $i + j \geq t + 1$.*

Proof. Suppose G is a finite graph embedded in a book such that conditions (1), (2), and (3) hold. Let $\mathbf{F}_q$ be the finite field with q elements and let $R = \mathbf{F}_q[x]/(x^{t+1})$. One checks that any graded book embedding of $\Gamma(R)$ is isomorphic to the given book embedding of G.

To prove the converse, we consider any graded book embedding of a finite local principal ring R. Let $V_i = M_i$. Condition (1) follows from Definition 3.2. Condition (2) follows from the fact that

$$|M_i| = |M^i| - |M^{i+1}| = q^{\kappa-i} - q^{\kappa-i-1}$$

where $t = \kappa - 1$. Condition (3) follows from Theorems 2.1 and 2.3. □

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