AN INDUCTIVE PROOF OF A RESULT ABOUT BULGARIAN SOLITAIRE

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Abstract

Let N be a positive integer and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of N of length l, i.e., $\sum_{i=1}^{l} \lambda_i = N$ with parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$. Define $T(\lambda)$ as the partition of n with parts $l, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1$, ignoring any zeros that might occur. Starting with a partition λ of N, we describe Bulgarian Solitaire by repeatedly applying the shift operation T to obtain the sequence of partitions

$$\lambda, T(\lambda), T^2(\lambda), \ldots.$$

We say a partition μ of N is T-*cyclic* if $T^i(\mu) = \mu$ for some $i \geq 1$. Brandt [2] characterized all T-*cyclic* partitions for Bulgarian Solitaire. In this paper we give an inductive proof of Brandt's result.

1. Introduction and statement of results

The following game, popularized by Gardner in 1983 [4], is called *Bulgarian Solitaire.*

Initially, we are given N cards disposed in several piles. A move consists of removing exactly one card from each pile and forming a new pile. The operation is repeated over and over.

If the number of cards N is a triangular number, i.e., $N = 1 + 2 + \cdots + k$ for some k, a remarkable fact is that, starting from any initial configuration, after a finite number of moves the Bulgarian Solitaire will reach the stable configuration formed by piles of sizes $1, 2, \ldots, k$. This result was proved by J. Brandt ([2], the assertion after the proof of Theorem 4, p. 484). It was also considered in [2] the case when the number of cards is not triangular.

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Since a deck has only finitely many layouts, the game of Bulgarian Solitaire must cycle. Brandt characterizes and counts all cycles for any given deck size ([2], Theorem 5).

Let us now define the game formally. Let N be a positive integer and let λ be a partition of N having l parts written $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ in non-increasing order; that is, $N = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ with positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$. Define $T(\lambda)$ as the partition of n with parts $l, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1$, ignoring any zeros that might occur. So $T^i(\lambda)$ $(i = 1, 2, \ldots)$ denotes the partition obtained by successively applying the shift operation T to λ a total of i times. Starting with a partition λ, we describe Bulgarian Solitaire by repeatedly applying the shift operation to obtain the sequence of partitions

$$\lambda, T(\lambda), T^2(\lambda), \ldots.$$

We say a partition μ of N is *T-cyclic* if $T^i(\mu) = \mu$ for some $i \geq 1$.

If N is arbitrary, Brandt noted that repeated application of T leads into a cycle of partitions, since there are only a finite number of these. Furthermore, a cycle of partitions is completely determined by the sequence of the consecutive lengths of the partitions in the cycle. Motivated by this fact, Brandt ([2], p. 483) defined the set M_n by

$$\begin{aligned} M_n = \{\sigma = (\sigma_i)_{i \in \mathbf{Z}} &: \max \sigma_i = n, \\ \text{where for all } i,\ \sigma_i &= |\{\sigma_j | j < i, \sigma_j \geq i - j\}|\}, \end{aligned} \tag{1}$$

where $|S|$ denotes the cardinality of a set S. If $\sigma \in M_n$, then by Proposition 2 in [2], $\sigma_i \in \{n, n-1\}$ for all $i \in \mathbf{Z}$. As an easy consequence of this fact, Brandt (cf. proof of Theorem 5 in [2]; also see Akin and Davis [1], Theorems 4 and 5, Griggs and Ho [5], Theorem 2.1 and Etienne [3]), characterized all T-cyclic partitions for arbitrary N. This result is given as follows.

Theorem. *Let $N = 1 + 2 + \cdots + k + r$, $0 \leq r \leq k$. Then a partition λ of N is T-cyclic if and only if λ has the form*

$$(k + \delta_k, k - 1 + \delta_{k-1}, \ldots, 1 + \delta_1, \delta_0),$$

where each δ_i is 0 or 1 and $\sum_{i=0}^{k} \delta_i = r$.

In particular (see the assertion after the proof of Theorem 4 in [2]), for a triangular number N we obtain the following result quoted by Gardner in [4]–*Brandt's Equilibrium Theorem.*

Corollary. *If $N = 1 + 2 + \cdots + k$, then $(k, k-1, \ldots, 1)$ is the unique T-cyclic partition of N.*

Recall that the above theorem follows from Theorem 4 of Akin and Davis [1] whose proof is based on Brandt's result. Theorem 5 in [1] which is proved directly, also gives a description of all T-cyclic partitions for arbitrary N as in above theorem. The above corollary is proved by Etienne [3] by introducing a natural array representation of a partition λ. The idea in his proof is applied in the proof of Theorem 2.1 in [5] (the above theorem) to general N.

In this paper we give an inductive proof of the above theorem. For the proof we define a sequence which is analogous to the set M_n given by (1).

2. Proof of the Theorem

Let N be a positive integer and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of N having l parts with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 1$. Bulgarian Solitaire is based on a function T defined on the partition λ as above:

$$T(\lambda) = (l, \lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_l - 1),$$

where all zeros are omitted and the parts may need to be reordered to be non-increasing. For a partition λ, we associate a sequence

$$\mathrm{seq}_T(\lambda) =< \sigma_1, \sigma_2, \dots, \sigma_n, \dots >,$$

where σ_n is the number of parts in $T^{n-1}(\lambda)$ ($T^0(\lambda) = \lambda$, $n = 1, 2, \dots$). Then applying the shift operation T to λ n times, we obtain

$$T^n(\lambda) = (\lambda_1 - n, \lambda_2 - n, \dots, \lambda_l - n, \sigma_n, \sigma_{n-1} - 1, \dots, \sigma_{n-i} - i, \dots \sigma_1 - (n-1)),$$

where all negative integers and zeros are omitted.

Note that, if $n \geq N$ then $\lambda_i - n \leq N - n \leq 0$, and hence

(2) $T^n(\lambda) = (\sigma_n, \sigma_{n-1} - 1, \dots, \sigma_{n-i} - i, \dots, \sigma_1 - (n-1))$ for all $n \geq N$,

where all negative integers and zeros are omitted.

Proposition. *Let* $\mathrm{seq}_T(\lambda) =< \sigma_1, \sigma_2, \dots, \sigma_n, \dots >$ *be a sequence associated to the partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ *of a positive integer* N*. Then for all* $s \in \mathbf{N}$ *there exists sufficiently large* $q \in \mathbf{N}$ *with* $q > s$ *such that*

(3) $$\sigma_{m-j} + 1 \geq \sigma_m \text{ for all } m \geq q \text{ and for all } j \leq s.$$

Proof. Let $N = \lambda_1 + \lambda_2 + \dots + \lambda_l$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 1$. We proceed by induction on s. It follows from the definition of the shift operation T that $\sigma_m \leq \sigma_{m-1} + 1$ for all $m \geq 2$, and hence (3) is satisfied for $s = 1$ assuming $q = 2$.

Now suppose that for a fixed $s \in \mathbf{N}$ there exists $q \in \mathbf{N}$ such that (3) holds. If we put $t = q + N$, then $\lambda_i - t \leq N - t < 0$ for all $i = 1, 2, \dots, l$, and hence (2) yields

(4) $$T^n(\lambda) = (\sigma_n, \sigma_{n-1} - 1, \dots, \sigma_1 - (n-1)) \text{ for all } n \geq t,$$

where all negative integers and zeros are omitted. It follows immediately from (4) that

$$\begin{aligned} \sigma_{m+1} &= |\{i : 1 \leq i \leq m \text{ and } \sigma_i - (m-i) \geq 1\}| \\ (5) \qquad &= |\{i : 1 \leq i \leq m \text{ and } \sigma_i + i \geq m+1\}| \text{ for all } m \geq t. \end{aligned}$$

Furthermore, if for a fixed $m \geq t$, $\sigma_i + i \geq m+1$ holds for some i, then $i \geq m+1-\sigma_i \geq t+1-N = q+1$. Now from this fact and (5) we have

$$(6) \qquad \sigma_{m+1} = |\{i : q+1 \leq i \leq m \text{ and } \sigma_i + i \geq m+1\}| \text{ for all } m \geq t.$$

By the inductive hypothesis, (3) with $j = s$ implies that

$$(7) \qquad \sigma_{i-s} + 1 \geq \sigma_i \text{ for all } i \geq q+1.$$

Therefore, if for a fixed $i \geq q+1$, there holds $\sigma_i + i \geq m+1$, (7) implies

$$\sigma_{i-s} + i + 1 \geq \sigma_i + i \geq m+1,$$

whence we conclude that

$$(8) \quad \sigma_{i-s} + (i-s) \geq m-s \text{ whenever } i \geq q+1 \text{ such that } \sigma_i + i \geq m+1.$$

Finally, if $m \geq t$, then $m - s > t - q = N$, and so by (2) we have

$$\begin{aligned} \sigma_{m-s} &= |\{i-s : 1 \leq i-s \leq m-s-1 \text{ and } \sigma_{i-s} + i - s \geq m-s\}| \\ &= |\{i : 1+s \leq i \leq m-1 \text{ and } \sigma_{i-s} + i - s \geq m-s\}| \\ &\geq |\{i : q+1 \leq i \leq m \text{ and } \sigma_{i-s} + i - s \geq m-s\}| - 1 \\ &\quad \text{(because of } q > s) \\ &\geq |\{i : q+1 \leq i \leq m \text{ and } \sigma_i + i \geq m+1\}| - 1 \text{ (because of (8))} \\ &= \sigma_{m+1} - 1 \text{ (because of (6)).} \end{aligned}$$

Therefore, $\sigma_{m-s} + 1 \geq \sigma_{m+1}$ for all $m \geq t$, or equivalently, $\sigma_{m-(s+1)} + 1 \geq \sigma_m$ for all $m \geq t+1$. The last inequality and the inductive hypothesis given by (3) imply

$$\sigma_{m-j} + 1 \geq \sigma_m \text{ for all } m \geq t+1 \text{ and for all } j \leq s+1.$$

This concludes the proof. ■

Corollary (cf. [2], Lemma 1). *Let $N = 1 + 2 + \cdots + k + r$, $0 \leq r \leq k$, with the same assumptions as in the above Proposition. Then there exists $t \in \mathrm{N}$ such that $\sigma_n \in \{k, k+1\}$ for all $n > t$.*

Proof of the above Corollary and the Theorem. It is easy to see that a sequence $\mathrm{seq}_T(\lambda) =< \sigma_1, \sigma_2, \ldots, \sigma_n, \ldots >$ is periodic, that is, there exist $p, v \in \mathrm{N}$ such that $\sigma_{n+p} = \sigma_n$ for all $n > v$.

By Proposition, there exists sufficiently large $q \in \mathbf{N}$ with $q > p$ such that

(9) $$\sigma_{m-j} + 1 \geq \sigma_m \text{ for all } m \geq q \text{ and for all } j \leq p.$$

Put $t = \max\{v, q\}$. Then $< \sigma_{t+1}, \sigma_{t+2}, \ldots, \sigma_{t+p} >$ is a period of seq_T. Assume i and n such that $t+1 \leq i < n \leq t+p$. Then since $n > q$ and $1 \leq n-i \leq p-1$, by (9) we get $\sigma_n \leq \sigma_{n-(n-i)} + 1 = \sigma_i + 1$ and $\sigma_i = \sigma_{i+p} \leq \sigma_{i+p-(p-(n-i))} + 1 = \sigma_n + 1$. Hence, $|\sigma_i - \sigma_n| \leq 1$ and thus, $\sigma_n \in \{u, u+1\}$ for some fixed $u \in \mathbf{N}$ and all $n \geq t+1$. It remains to show that $u = k$. If we choose $m \in \mathbf{N}$ such that $mp \geq N$, then since $\sigma_n \leq u+1$ for $t+1 \leq n \leq mp+t-1$, for such a n we have

(10) $$\sigma_n - ((mp+t+u) - n) \leq n + 1 - mp - t \leq 0.$$

On the other hand, if $1 \leq n \leq t$, then since $\sigma_i \leq N \leq mp$, we obtain

(11) $$\begin{aligned}(\sigma_n - ((mp+t+u) - n) &\leq N - mp - t - u + n \\ &\leq mp - mp - t - u + t = -u < 0.\end{aligned}$$

In view of (10) and (11), by (4) we get

(12) $$\begin{aligned}T^{mp+t+u}(\lambda) = (&\sigma_{mp+t+u}, \sigma_{mp+t+u-1} - 1, \ldots, \\ &\sigma_{mp+t+1} - (u-1), \sigma_{mp+t} - u).\end{aligned}$$

Since $\sigma_{mp+t+u-i} = u + \delta_{u-i}$ with $\delta_{u-i} \in \{0, 1\}$ for all $i = 0, 1, \ldots, u$, it follows from (12) that the sum of all parts of the partition $T^{mp+t+u}(\lambda)$ is equal to

$$u + (u-1) + \cdots + 1 + \sum_{i=0}^{u} \delta_{u-i}.$$

It is easily see that the above sum is equal to $N = 1 + 2 + \cdots + k + r$ if and only if $u = k$ and $\sum_{i=0}^{k} \delta_{k-i} = r$. Hence $\sigma_{mp+t+k-i} = k + \delta_{k-i}$ for all $i = 0, 1, \ldots, k$, which together with (12) yields

$$T^{mp+t+k}(\lambda) = (k + \delta_k, k - 1 + \delta_{k-1}, \ldots, 1 + \delta_1, \delta_0).$$

This completes both proofs. ■

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