Linked reduction systems for permutations

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ABSTRACT For any $n \geq 2$ we let S_n be the set of permutations of the set $\{1, 2, ..., n\}$. A **reduction** $\overline{f}$ on S_n is a set of functions $\{f_i : 1 \leq i \leq n\}$ such that f_n is the identity function on $\{1, 2, ..., n-1\}$ and for $i < n$, f_i is a bijection from the set $\{1, 2, ..., n\} - \{i\}$ to the set $\{1, 2, ..., n-1\}$. The ith reduction of a permutation $p = x_1x_2...x_n$ (with respect to $\overline{f}$) is the permutation $p \downarrow i$ obtained by deleting i from p and then applying the function f_i to each of the remaining elements of p in place. The set $R(p) = \{q \in S_{n-1} : q = p \downarrow i$ for some $i \leq n\}$ is called the **set of reductions** of p. The **simple reduction** on S_n is the one for which $f_i(x) = \begin{cases} x & \text{if } x \neq i \text{ and } x \neq n \\ i & \text{if } x = n \end{cases}$ for all $i \leq n-1$. We say that $\overline{f}$ is **faithful** if $p \neq q \rightarrow R(p) \neq R(q)$. A **reduction system** is a set $\{\overline{f}_n : n \geq n_0\}$ where $\overline{f}_n$ is a reduction on S_n for all $n \geq n_0$. The system is said to be faithful if $\overline{f}_n$ is faithful for all $n \geq n_0$. Such a system is said to be **linked** if there is a integer-valued function $\phi(n)$, defined for $n > n_0$, such that $\phi(n) \leq n$ for all $n \geq n_0$, and for which $p \downarrow \phi(n) \downarrow i = p \downarrow i \downarrow n-1$ for all $n > n_0$, for all $i \leq n-1$ and for all $p \in S_n$. And the system is said to be **amenable** if for every $n > n_0$ there is an integer $k < n$ such that, for all $p \in S_n$, $p \downarrow n \downarrow n-1 = p \downarrow k \downarrow n-1$. The purpose of this paper is to study faithful reductions and linked reduction systems. We characterize amenable, linked reduction systems by means of two types of liftings by which a reduction on S_{n+1} can be formed from one on S_n. And we obtain conditions for a reduction system to be faithful. One interesting consequence is that any amenable, linked reduction system which begins with a simple reduction is faithful.

Key words and phrases: permutation, reduction, set of reductions, reduction system, faithful, linked, amenable

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1. Introduction

For any positive integer $n \geq 2$, we let S_n denote the set of all permutations of the set $\{1, 2, ..., n\}$. We think of a permutation just as an ordered list, and a permutation p is displayed simply by listing its entries in order, sometimes with commas between them for clarity. Thus, for $p \in S_n$, we write $p = p_1, p_2, ..., p_n$. For any integer $i \leq n$, the expression $p - \{i\}$ denotes the $n-1$ permutation obtained by deleting i from p, leaving the other elements in their given order. In other words, if k is the integer for which $p_k = i$, then $p - \{i\} = p_1, p_2, ..., p_{k-1}, p_{k+1}, ..., p_n$. Deleting one entry i, in all possible ways, from a permutation on $\{1, 2, ..., n\}$, to create various $n-1$ permutations, is of course a very commonly used idea. Recent papers in coding theory [7] and permutation graphs [4], [5] use these one-element deletions. In some instances, it is either useful or necessary to represent these one-element deletions as $n-1$ permutations on the "standard $n-1$-element set" $\{1, 2, ..., n-1\}$, that is, as elements of S_{n-1}. Once this is done, one then has a direct way to inductively "lift" properties and constructions from S_{n-1} to S_n. Representing the one-element deletions $p - \{i\}$ of an arbitrary permutation p in S_n by elements of S_{n-1} will be referred to as a **reduction** on S_n. How this is done will, of course, depend on the application that one has in mind. For any such reduction, we will use the notation $p \downarrow i$ to denote the permutation on $\{1, 2, ..., n-1\}$ which represents $p - \{i\}$; we will refer to it as the **ith reduction** of p.

As one basic example, let us describe what we will refer to as the **simple reduction** on S_n. For any $i < n$, let $p \downarrow i$ be the permutation on the set $\{1, 2, ..., n-1\}$ obtained from p as follows: delete i from p and then, in the resulting $n-1$ permutation, change n to i. For $i = n$, we take $p \downarrow n$ to be $p - \{n\}$. To illustrate, let $n = 5$ and let $p = 25413$. We then have
$p \downarrow 1 = 2143$, $p \downarrow 2 = 2413$, $p \downarrow 3 = 2341$, $p \downarrow 4 = 2413$, $p \downarrow 5 = 2413$.

A second natural example will be referred to as the **regular reduction** on S_n. Let $n \geq 2$, and let $p \in S_n$. For any $i \leq n$, let $p \downarrow i$ be the permutation on the set $\{1, 2, ..., n-1\}$ obtained from p as follows: delete i from p and then subtract 1 in place from each of the remaining entries of p which are larger than i. To illustrate, again let $n = 5$ and $p = 25413$. We then have
$p \downarrow 1 = 1432$, $p \downarrow 2 = 4312$, $p \downarrow 3 = 2431$, $p \downarrow 4 = 2413$, $p \downarrow 5 = 2413$.
This form of reduction is employed in [11], pages 85-86, in an inductive description of the Schensted correspondence.

If we are given a reduction on S_n, then, for any $p \in S_n$, we can form the **set of reductions** of p: the set $R(p) = \{q \in S_{n-1} : q = p \downarrow i$ for some $i \leq n\}$. Referring to the above examples, where $p = 25413$, we see that, with respect to the simple reduction we have $R(p) = \{2143, 2341, 2413\}$, and with respect to the regular reduction we have $R(p) = \{1432, 2413, 2431, 4312\}$.

Being able to determine any permutation from its set of reductions would clearly be a desireable property of a reduction. A reduction with this property will be called **faithful**. This notion is related to various kinds of *reconstruction problems*, in which one attempts to reconstruct an object from its one-element deleted sub-objects. For information on reconstruction problems for graphs and ordered sets we refer the reader to [1] and [10]. Interesting recent work on reconstructing sequences from subsequences can be found in [2], [6] and [8]. We also direct the reader to the work in [9] on the reconstruction of subsets of the plane; this paper contains many references to recent work involving reconstructing codes, sets of real numbers, sequences and geometries.

The problem of determining a permutation from its set of reductions has been considered in [3], where it is shown that the regular reduction on S_n is faithful for $n \geq 5$. The inductive argument in [3] makes use of the following three properties of the regular reduction which are proven there:
- for any positive integer $i < n$, we have $p \downarrow n \downarrow i = p \downarrow i \downarrow n-1$.
- $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$
- the position of n in p can be determined from the set $R(p)$.

The simple reduction on S_n is also faithful, for $n \geq 5$. One can establish a similar set of properties for the simple reduction:

- for any positive integer $i < n$, we have $p \downarrow n-1 \downarrow i = p \downarrow i \downarrow n-1$.
- $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$
- the position of $n-1$ in p can be determined from the set $R(p)$.

Having established these properties, it is a simple matter to modify the inductive proof in [3] to show that the simple reduction is faithful.

In this paper we will consider permutation reductions in general. What kinds of reductions satisfy conditions similar to those satisfied by the simple and regular reductions? Can one describe a broad variety of faithful reductions? Is there a useful way to characterize various kinds of unfaithful reductions? It is to these matters that we will turn our attention. Before doing so, we must first give a precise definition of the concept of reduction.

Definition 1.1 Let n be a positive integer such that $n \geq 2$, and let S_n be the set of permutations of the set $\{1, 2, ..., n\}$. A **reduction** $\overline{f}$ on S_n is a set of functions $\{f_i : 1 \leq i \leq n\}$ such that

(i) For all $i \leq n$, f_i is a bijection from the set $\{1, 2, ..., n\} - \{i\}$ to the set $\{1, 2, ..., n-1\}$, and
(ii) f_n is the identity function on $\{1, 2, ..., n-1\}$.

Definition 1.2 Let n be a positive integer such that $n \geq 2$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n. For any permutation $p = p_1 p_2 \cdots p_n$ in S_n, and for any $i \leq n$, we define **the ith reduction of p**, denoted by $p \downarrow i$ as follows: Let k be the integer for which $p_k = i$. Then

$$p \downarrow i \;=\; f_i(p_1) f_i(p_2) \cdots f_i(p_{k-1}) f_i(p_{k+1}) \cdots f_i(p_n)$$

The **set of reductions** of p is the set

$$R(p) = \{q \in S_{n-1} \;:\; q = p \downarrow i \text{ for some } i \leq n\}.$$

Another way to express Definition 1.2 is that, to reduce a permutation p by an entry i, we delete i from p and then rename the remaining elements in place, using the function f_i. It is clear that the result, $p \downarrow i$, is a permutation of the set $\{1, 2, ..., n-1\}$, that is, an element of S_{n-1}.

We also note that, because of condition (ii) in Definition 1, $p \downarrow n$ is just equal to $p - \{n\}$, the result of simply deleting the entry n from p.

We will let $\overline{\sigma}_n$ and $\overline{\rho}_n$ respectively denote the simple and regular reductions on S_n. Note that the simple reduction $\overline{\sigma}_n$ is given by the bijections $\{\sigma_i^n : 1 \leq i \leq n\}$ defined as follows: for $i \neq n$, $\sigma_i^n(x) = \begin{cases} x & \text{if } x \neq n \text{ and } x \neq \\ i & \text{if } x = n \end{cases}$

The regular reduction $\overline{\rho}_n$ is given by the bijections $\rho_i^n(x) = \begin{cases} x & \text{if } x < i \\ x-1 & \text{if } x > i \end{cases}$

Notation: (i) To indicate that an element has been deleted from an ordered list, we will place a "hat" above that element. Thus, if $x_1, x_2, ..., x_m$ is a given list, and if $i \leq m$, the notation $x_1, x_2, ..., \hat{x}_i, ..., x_m$ denotes the following list of length $m-1$: $x_1, x_2, ..., x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}, ..., x_m$.

(ii) Let $x_1, x_2, ..., x_m$ be an ordered list of elements and let f be a function whose domain includes these elements. We will use the expression $f : [x_1, x_2, ..., x_m]$ to denote the ordered list $f(x_1), f(x_2), ..., f(x_m)$. In this context we may also place square brackets around the latter list.

(iii) The identity permutation in S_n will be denoted by e_n. Thus we have $e_n = 123...n$.

(iv) For any permutation p we will let p^{opp} denote the permutation obtained by writing the entries of p in reverse order. Thus if $p = p_1, p_2, ..., p_n$ then $p^{opp} = p_n, p_{n-1}, ..., p_2, p_1$. For any set of permutations S, we let $S^{opp} = \{p^{opp} : p \in S\}$.

(v) Compositions of functions are applied from right to left: $f \circ g(x) = f(g(x))$.

Applying this notation, the ith reduction in Definition 1.2 above can be written as

$$p \downarrow i = f_i : [p_1, p_2, ..., \hat{p_k}, ..., p_n].$$

The reader can easily verify the following useful facts.

Lemma 1.1 Let $n \geq 2$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n. Then for any $p \in S_n$ we have
(i) $p^{opp} \downarrow i = (p \downarrow i)^{opp}$, and
(ii) $R(p^{opp}) = R(p)^{opp}$.

Definition 1.3 Let n be a positive integer such that $n \geq 2$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n. This reduction is said to be **faithful** if for any $p, q \in S_n$, $p \neq q \rightarrow R(p) \neq R(q)$.

Another way to express Definition 1.3 is this: a reduction on S_n is faithful if, for any p in S_n, the set $R(p)$ determines p. This is the point of view in [3], where the main argument(for the regular reduction) leads directly to an algorithm for reconstructing p from the set $R(p)$.

Next let us describe a few more examples of reductions.

Example 1.1 (i) Let $n \geq 2$. There is an obvious way to generalize the regular reduction on S_n. Let $x_1, x_2, ..., x_n$ be any ordering of the integers $1, 2, ..., n$ such that $x_n = n$. For any $i \leq n$, we define

$$f_{x_i}(x) = \begin{cases} x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$$

The reduction $\{f_{x_i} : 1 \leq i \leq n\}$ will be referred to as **a reduction of regular type** on S_n.
This reduction is faithful for $n \geq 5$, as follows isomorphically from the result in [3].

(ii) Let $n \geq 3$ and let a be an integer with $1 \leq a < n$. Let $x_1, x_2, ..., x_{n-1}$ be any ordering of the set $\{1, 2, ..., n\} - \{a\}$ such that $x_{n-1} = n$. We define

$$f_a(x) = \begin{cases} x & \text{if } x \neq a \text{ and } x \neq n \\ a & \text{if } x = n \end{cases} \quad \text{and for } i \leq n-1,$$

$$f_{x_i}(x) = \begin{cases} a & \text{if } x = a \\ x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$$

The integer a will be referred to as a **simple element** for this reduction, and by analogy with Example (i), we say that this reduction is the sum of **a reduction of regular type and one simple element.** Such a reduction is faithful for $n \geq 5$; this will follow from our work in the Section 3, so we will not verify it here.

(iii) Let $n \geq 4$ and let a and b be distinct integers such that $1 \leq a < n$ and $1 \leq b < n$. Let $x_1, x_2, ..., x_{n-2}$ be any ordering of the set $\{1, 2, ..., n\} - \{a, b\}$ such that $x_{n-2} = n$. We define

$$f_a(x) = \begin{cases} x & \text{if } x \neq a \text{ and } x \neq n \\ a & \text{if } x = n \end{cases} \qquad f_b(x) = \begin{cases} x & \text{if } x \neq b \text{ and } x \neq n \\ b & \text{if } x = n \end{cases}$$

$$\text{and for } i \leq n-2, \qquad f_{x_i}(x) = \begin{cases} a & \text{if } x = a \\ b & \text{if } x = b \\ x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$$

Using the same terminology as in Example (ii), we say that this reduction is the sum of **a reduction of regular type and two simple elements.** This reduction is *not* faithful. To show this, we exhibit two different permutations with the same set of reductions.

Let $p = x_1, x_2, ..., x_{n-2}, a, b$ and let $q = x_1, x_2, ..., x_{n-2}, b, a$. We will show that $R(p) = R(q)$. To see this, we first note that $p \downarrow a = q \downarrow a = x_1, x_2, ..., x_{n-3}, a, b$ and $p \downarrow b = q \downarrow b = x_1, x_2, ..., x_{n-3}, b, a$. For $i \leq n-2$, we have $p \downarrow x_i = x_1, x_2, ..., x_{n-3}, a, b$ and $q \downarrow x_i = x_1, x_2, ..., x_{n-3}, b, a$. Let $u = x_1, x_2, ..., x_{n-3}, a, b$ and let $v = x_1, x_2, ..., x_{n-3}, b, a$. Thus $R(p)$ and $R(q)$ are the same two-element set: $R(p) = R(q) = \{u, v\}$.

(iv) Let $n \geq 5$ and let a, b and c be distinct integers such that $1 \leq a < n$, $1 \leq b < n$ and $1 \leq c < n$. Let $x_1, x_2, ..., x_{n-3}$ be any ordering of the set

$\{1,2,...,n\} - \{a,b,c\}$ such that $x_{n-3} = n$. We define

$$f_a(x) = \begin{cases} x & \text{if } x \notin \{n,a,b,c\} \\ a & \text{if } x = n \\ b & \text{if } x = c \\ c & \text{if } x = b \end{cases} \qquad f_b(x) = \begin{cases} x & \text{if } x \notin \{n,a,b,c\} \\ b & \text{if } x = n \\ a & \text{if } x = c \\ c & \text{if } x = a \end{cases}$$

$$f_c(x) = \begin{cases} x & \text{if } x \notin \{n,a,b,c\} \\ c & \text{if } x = n \\ a & \text{if } x = b \\ b & \text{if } x = a \end{cases}$$

and for $i \le n-3$, $f_{x_i}(x) = \begin{cases} x & \text{if } x \in \{a,b,c\} \\ x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$

This reduction is not faithful. To see this, let $p = x_1, x_2, ..., x_{n-3}, a, b, c$ and let $q = x_1, x_2, ..., x_{n-3}, a, c, b$. We find that $p \downarrow a = x_1, x_2, ..., x_{n-4}, a, c, b$ and $p \downarrow b = x_1, x_2, ..., x_{n-4}, b, c, a$ and $p \downarrow c = x_1, x_2, ..., x_{n-4}, c, b, a$.
For any $i \le n-3$ we have $p \downarrow x_i = x_1, x_2, ..., x_{n-4}, a, b, c$.
Similarly we find $q \downarrow a = x_1, x_2, ..., x_{n-4}, a, b, c$ and $q \downarrow b = x_1, x_2, ..., x_{n-4}, b, c, a$ and $q \downarrow c = x_1, x_2, ..., x_{n-4}, c, b, a$. And for any $i \le n-3$ we have $q \downarrow x_i = x_1, x_2, ..., x_{n-4}, a, c, b$. Thus p and q have the same set of four reductions. □

Our main interest here lies not just in reductions, but in reduction *systems*, wherein, for some positive integer n_0, we are given a reduction on S_n for every integer $n \ge n_0$. So we extend our definitions as follows.

Definition 1.4 (i) A **reduction system** for permutations is a set of functions $\{f_i^n : n \ge n_0, 1 \le i \le n\}$, where n_0 is a positive integer such that $n_0 \ge 2$, such that $\{f_i^n : 1 \le i \le n\}$ is a reduction on S_n for all $n \ge n_0$. The reduction $\{f_i^{n_0} : 1 \le i \le n_0\}$ is referred to as the **beginning** reduction.

(ii) Let $\{f_i^n : n \ge n_0, 1 \le i \le n\}$ be a reduction system. Suppose there is a function $\phi(n)$, defined for $n > n_0$, with positive integer values, such that $\phi(n) \le n$ for all $n \ge n_0$, and having the property that
$p \downarrow \phi(n) \downarrow i = p \downarrow i \downarrow n-1$ for all $n > n_0$, for all $i \le n-1$ and for all $p \in S_n$. Such a reduction system is called a **linked** reduction system.
The function ϕ is called the **linking function**.

(iii) A reduction system $\{f_i^n : n \ge n_0, 1 \le i \le n\}$ is said to be **faithful** if, for every $n \ge n_0$, $\{f_i^n : 1 \le i \le n\}$ is a faithful reduction on S_n.

Remarks: For a reduction system $\{f_i^n : n \geq n_0, 1 \leq i \leq n\}$, we will often denote the reduction $\{f_i^n : 1 \leq i \leq n\}$ by $\overline{f}_n$. In this notation, a reduction system is thus equivalently given by a sequence $\{\overline{f}_n : n \geq n_0\}$ where $\overline{f}_n$ is a reduction on S_n for every $n \geq n_0$.

A "finite-length" version of Definition 1.4 can also be formulated, in which one has a set of functions $\{f_i^n : n_0 \leq n \leq n_1, 1 \leq i \leq n\}$ where n_1 is a given integer such that $n_1 > n_0$. The modifications required are obvious. All of the concepts we will be discussing for reduction systems apply equally well to their finite-length versions. In particular, we may have $n_1 = n_0 + 1$, in which case such a reduction system (of length 2) consists of two reductions, one on S_{n_0} and one on S_{n_0+1}. To say that such a length 2 system is linked simply means that there is a positive integer $j \leq n_0 + 1$ such that $p \downarrow j \downarrow i = p \downarrow i \downarrow n_0$ for all $i \leq n_0$ and for all $p \in S_{n_0+1}$. In this case we will say that the two reductions are **linked by** j.

Looking back at the simple and regular reductions on S_n, defined for $n \geq 2$, we observe that both are linked reduction systems: the linking function is $\phi(n) = n$ for the regular reduction, and $\phi(n) = n - 1$ for the simple reduction. If we consider these for $n \geq 5$, both are faithful. (Neither is faithful for $n = 4$; two permutations with the same regular reduction set are 3142 and 2413, and two with the same simple reduction set are 1243 and 2143.)

Although it will not be treated here, we note that there is a weaker form of "faithfulness" for reductions that one can consider. Instead of forming the *set* of reductions of a permutation p, we can form the *multiset* consisting of these same reductions, in which each reduction $p \downarrow i$ occurs as many times as there are integers j for which $p \downarrow j = p \downarrow i$. Let $R'(p)$ denote this multiset. We then say that the reduction is **weakly faithful** if, for any permuations p and q, $p \neq q \rightarrow R'(p) \neq R'(q)$. A similar term can be applied to reduction sytems. Clearly any faithful reduction is weakly faithful.

One can also consider a somewhat more general linking condition than the one described above in part (ii) of Definition 1.4. One might replace the integer $n - 1$ in condition (ii) by an integer $\psi(n) < n$ for which $p \downarrow \phi(n) \downarrow i = p \downarrow i \downarrow \psi(n)$ for all $i \leq n - 1$ and for all $p \in S_n$. □

The two technical conditions in our next definition are also motivated by the properties of the two natural reduction systems described above, The regular and simple reduction systems both satisfy these conditions. The first holds with $k = n - 1$ in both cases and it is easy to check. The second condition requires a substantial argument for both reductions. We refer the reader to [3] for the regular reduction; a similar kind of argument can be formulated for simple reductions.

Definition 1.5 Let $\{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be a linked reduction system with linking function ϕ.

(i) The reduction system is said to be **amenable** if for every $n > n_0$ there is an integer $k < n$ such that, for all $p \in S_n$, $p \downarrow n \downarrow n-1 = p \downarrow k \downarrow n-1$.

(ii) The reduction system is said to have property **P** if for any $n > n_0$ and for any $p \in S_n$, the position of the entry $\phi(n)$ in p is determined by the set $R(p)$. That is, if p and q are any permutations in S_n for which $R(p) = R(q)$, then $\phi(n)$ has the same position in p as it does in q.

Generalizing the argument in [3] for regular reductions, we can establish the following theorem.

Theorem 1.2 Any linked reduction system which is amenable, has property **P**, and which begins with a faithful reduction, is faithful.

Proof: Let $\{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be a linked reduction system with linking function ϕ, which is amenable and which satisfies condition **P**, and for which $\{f_i^{n_0} : 1 \leq i \leq n_0\}$ is faithful. We show that $\{f_i^n : 1 \leq i \leq n\}$ is faithful for all $n \geq n_0$ by induction. Assuming that $\{f_i^n : 1 \leq i \leq n\}$ is faithful, let p and q be any elements of S_{n+1} for which $R(p) = R(q)$. Now, for all $i \leq n$ we have $p \downarrow \phi(n+1) \downarrow i = p \downarrow i \downarrow n$ and, since the system is amenable, for some $k \leq n$, $p \downarrow n+1 \downarrow n = p \downarrow k \downarrow n$. It follows that $R(p \downarrow \phi(n+1)) = \{r - \{n\} : r \in R(p)\}$, and similarly for q. Since $R(p) = R(q)$, it follows that $R(p \downarrow \phi(n+1)) = R(q \downarrow \phi(n+1))$. Since the reduction is faithful on S_n, we deduce that $p \downarrow \phi(n+1) = q \downarrow \phi(n+1)$. Therefore the n-permutations $p - \{\phi(n+1)\}$ and $q - \{\phi(n+1)\}$ must be identical. Moreover, property **P** implies that $\phi(n)$ has the same position in p as it does in q. It follows that $p = q$. □

A closer look at the proof of Theorem 1.2 shows that property **P** can actually be replaced by a slightly weaker condition as follows.

Corollary 1.3 Let $\mathcal{F} = \{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be an amenable linked reduction system with linking function ϕ, and assume that the beginning reduction$\{f_i^{n_0} : 1 \leq i \leq n_0\}$ is faithful. Then $\mathcal{F}$ is faithful if the following condition holds for all $n > n_0$:
for any $p, q \in S_n$, if $p - \{\phi(n)\} = q - \{\phi(n)\}$ and $R(p) = R(q)$ then $p = q$.

Proof: The proof of Theorem 1.2 applies verbatim since, in the induction step, once we have deduced that $p \downarrow \phi(n+1) = q \downarrow \phi(n+1)$, this implies that $p - \{\phi(n+1)\} = q - \{\phi(n+1)\}$. □

It will be convenient to denote the condition in Corollary 1.3 by $\mathbf{P_1}$. Thus $\mathbf{P_1}$ is the following statement: for any $n > n_0$ and for any $p, q \in S_n$, if $p - \{\phi(n)\} = q - \{\phi(n)\}$ and $R(p) = R(q)$ then $p = q$.

With obvious modifications, we also speak of the properties $\mathbf{P}$ and $\mathbf{P_1}$ for reduction systems of finite length. It is clear that Theorem 1.2 and Corollary 1.3 hold for finite length systems as well.

With a view of applying Theorem 1.2 and Corollary 1.3 to obtain faithful reductions, we now set out to investigate linked reduction systems in general.

2. Linked Reduction Systems

Lemma 2.1 Let $n_0 \geq 4$ and let $\{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be a linked reduction system with linking function ϕ. Then for all $n > n_0$ and for all $i < n$ such that $i \neq \phi(n)$ we have $f_i^n(\phi(n)) = n - 1$.

Proof: For contradiction, suppose we have a counterexample n and i. Let $j = \phi(n)$. So we have $f_i^n(j) \neq n - 1$. Now, let r be the unique integer for which $f_i^n(j) = f_i^{n-1} \circ f_j^n(r)$, and choose any integer $x \leq n$ such that $f_j^n(x) \neq i$ and such that x is not equal to any of i, j or r. Let p be the permutation in S_n whose first two entries are j, x, followed by all the other elements of the set $\{1, 2, ..., n\}$ in their natural order. The first entry of $p \downarrow i \downarrow n-1$ is $f_i^n(j)$, and the first entry of $p \downarrow j \downarrow i$ is $f_i^{n-1} \circ f_j^n(x)$. These two first entries are different because of our choice of x. But, since the system is linked, we must have $p \downarrow j \downarrow i = p \downarrow i \downarrow n-1$ for every p in S_n. This is a contradiction. □

Our next lemma shows that, for any amenable linked reduction system, the possible values for the linking function are extremely restricted.

Lemma 2.2 Let $n_0 \geq 4$ and let $\{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be an amenable linked reduction system with linking function ϕ, Then, for all $n > n_0$, we have either $\phi(n) = n$ or $\phi(n) = n - 1$, and $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$ for all $p \in S_n$.

Proof: Let $n > n_0$. Let $j = \phi(n)$. Since the system is amenable, there is a positive integer $k \leq n-1$ such that $p \downarrow n \downarrow n-1 = p \downarrow k \downarrow n-1$ for all $p \in S_n$. There is a unique integer $r \leq n$ for which $f_j^n(r) = k$. We of course have $r \neq j$. We consider two cases.

Case 1: Suppose $j < r$. Now, we have $p \downarrow n \downarrow n-1 = p \downarrow k \downarrow n-1 = p \downarrow j \downarrow k$. Applying this to the identity permutation $p = e_n$, we note that $p \downarrow j \downarrow k = f_k^{n-1} \circ f_j^n : [1, 2, ..., \hat{j}, ..., \hat{r}, ..., n]$ and $p \downarrow n \downarrow n-1 = 123 \cdots n-2$. It follows that $f_k^{n-1} \circ f_j^n(x) = x$ for $x < j$, $f_k^{n-1} \circ f_j^n(x) = x-1$ for $j < x < r$, and $f_k^{n-1} \circ f_j^n(x) = x-2$ for $x > r$. Now, if there is any integer $x \neq n-1$ for which either $j < x < r$ or $r < x \leq n-2$, let p be the permutation in S_n beginning with x and followed by all the other integers in their natural order. The first element of $p \downarrow j \downarrow k$ is $f_k^{n-1} \circ f_j^n(x)$, which is either $x-1$ or $x-2$. however, the first entry of $p \downarrow n \downarrow n-1$ is clearly x. Therefore no such x can exist. It follows that the possible values for (r, j) are $(n-2, n-3), (n-1, n-2), (n, n-1)$ and $(n, n-2)$. Let us see which of these possibilities we can rule out.

Suppose we have $(r, j) = (n, n-2)$. As above, we know that $f_k^{n-1} \circ f_{n-2}^n(x) = x$ for $x \leq n-3$, and $f_k^{n-1} \circ f_{n-2}^n(n-1) = n-2$. Consider the permutation $q = n-1, 1, 2, 3, ..., n-2, n$. The first entry of $q \downarrow n-2 \downarrow k$ is $f_k^{n-1} \circ f_{n-2}^n(n-1) = n-2$. However, the first entry of $q \downarrow n \downarrow n-1$ (just delete n and $n-1$) is 1. Since we must have $q \downarrow n-2 \downarrow k = q \downarrow n \downarrow n-1$, this is impossible. A similar argument, using $q = n, 1, 2, ..., n-1$, rules out $(n-2, n-3)$ and $(n-1, n-2)$ as possibilities for (r, j). Therefore we must have $j = n-1$ and $r = n$.

Next we turn to the value of k. We will prove that $k = n-1$. We argue by contradiction. Suppose $k < n-1$. Now, we can apply the relation $p \downarrow n \downarrow n-1 = p \downarrow k \downarrow n-1$ to the identity $p = e_n$. Since $j = n-1$, by Lemma 2.1, we have $f_k^n(n-1) = n-1$. It follows that $p \downarrow k \downarrow n-1 = f_k^n : [1, 2, ..., \hat{k}, ..., \widehat{n-1}, n]$. Since $p \downarrow n \downarrow n-1 = 123...n-2$, it follows that

$$f_k^n(x) = \begin{cases} x & \text{if } x < k \\ x-1 & \text{if } k < x < n-1 \\ n-2 & \text{if } x = n \end{cases}$$

Now, let us apply the relation $p \downarrow n \downarrow n-1 = p \downarrow k \downarrow n-1$ to $p = n123...n-1$. The first entry of $p \downarrow n \downarrow n-1$ is 1. By Lemma 2.1, we have $f_k^n(n-1) = n-1$, and so the first entry of $p \downarrow k \downarrow n-1$ is $f_k^n(n)$. But this latter element is $n-2$, as shown above. Therefore we must have $k = n-1$, which gives the desired result for Case 1.

Case 2: Suppose $j > r$. The argument is entirely similar to that for Case 1, and we deduce that $r = n-1, j = n$ and $k = n-1$. We omit the details. □

Lemma 2.2 shows the significant impact that amenability has for a linked reduction system. An explanation is in order as to why it is natural to require this condition: the main idea of the proof of Theorem 1.2 is that a set of reductions $R(p)$ of a permutation p in S_n is to be converted into another set of reductions $R(p \downarrow \phi(n))$ (at the next level down) by deleting the entry $n-1$ from each of the elements of $R(p)$. Since this set of deletions includes $p \downarrow n \downarrow n-1$, this has to be accounted for as one of the elements $p \downarrow \phi(n) \downarrow k$ of the set $R(p \downarrow \phi(n))$, for some $k \leq n-1$. Since, in a linked system we have $p \downarrow \phi(n) \downarrow k = p \downarrow k \downarrow n-1$, we are led directly to the amenability condition. As Lemma 2.2 shows, for any linked reduction system satisfying this condition, the value of k must in fact be $n-1$ for every $n > n_0$. It is a simple matter to describe exactly when this will occur, as follows.

Lemma 2.3 Let $n_0 \geq 4$ and let $\mathcal{F} = \{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be a linked reduction system. The following statements for this system are equivalent:

(i) $\mathcal{F}$ is amenable

(ii) For every $n > n_0$, f_{n-1}^n is given by $f_{n-1}^n(x) = \begin{cases} x & \text{if } x < n-1 \\ n-1 & \text{if } x = n \end{cases}$

Proof: Suppose the system is amenable. Let $n > n_0$. By Lemma 2.2, we have $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$ for all $p \in S_n$. Now, let k be the integer for which $f_{n-1}^n(k) = n-1$. We claim that $k = n$. Suppose this is not the case. Then $k < n-1$. Applying the relation $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$ to $p = e_n = 123...n$, we find (as in the proof of Lemma 2.2) that

$$f_{n-1}^n(x) = \begin{cases} x & \text{if } x < k \\ x-1 & \text{if } k < x < n-1 \\ n-1 & \text{if } x = k \\ n-2 & \text{if } x = n \end{cases}$$

Now, let $p = n123...n-1$. The first entry of $p \downarrow n \downarrow n-1$ (wherein we just delete n and then $n-1$) is 1. However the first entry of $p \downarrow n-1 \downarrow n-1$ is clearly $f_{n-1}^n(n) = n-2$. Therefore we must have $k = n$. Looking again at the result of applying the relation $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$ to $p = e_n$, we see that (ii) holds.

Conversely, assume that the formula in (ii) holds for every $n > n_0$. Let $n > n_0$ and let $p \in S_n$. The reduction $p \downarrow n-1$ is found by deleting $n-1$ from p and then applying the function f_{n-1}^n in place to each of the other entries. When we do so, by (ii), all entries are unchanged except n, which is changed to $n-1$. We then delete $n-1$ from $p \downarrow n-1$ to get $p \downarrow n-1 \downarrow n-1$. The result is clearly the same as simply deleting n and $n-1$ from p, and so $p \downarrow n \downarrow n-1 = p \downarrow n-1 \downarrow n-1$. Therefore the system is amenable. □

We next consider the extent to which the reduction at one level of a linked reduction system determines the reduction at the next level. To do so, we first describe two ways in which any reduction can be "lifted" to a reduction at the next level.

Definition 2.1 Let $n \geq 2$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n.

(i) Let $\overline{g} = \{g_i : 1 \leq i \leq n+1\}$ be the reduction on S_{n+1} defined as follows:

$$g_{n+1}(x) = x \text{ for } x \leq n \text{ and, for } i \leq n, \quad g_i(x) = \begin{cases} f_i(x) & \text{if } x \leq n \text{ and } x \neq i \\ n & \text{if } x = n+1 \end{cases}$$

We say that $\overline{g}$ is obtained from $\overline{f}$ by a **lifting of type 1** and we write $\overline{g} = L_1(\overline{f})$.

(ii) Let $\overline{h} = \{h_i : 1 \leq i \leq n+1\}$ be the reduction on S_{n+1} defined as follows:

$$h_{n+1}(x) = x \text{ for all } x \leq n \quad \text{and} \quad h_n(x) = \begin{cases} x & \text{if } x < n \\ n & \text{if } x = n+1 \end{cases}$$

$$\text{and for all } i < n, \qquad h_i(x) = \begin{cases} f_i(x) & \text{if } x < n \text{ and } x \neq i \\ n & \text{if } x = n \\ f_i(n) & \text{if } x = n+1 \end{cases}$$

We say that $\overline{h}$ is obtained from $\overline{f}$ by a **lifting of type 2** and we write $\overline{h} = L_2(\overline{f})$.

It is clear that $\overline{g}$ and $\overline{h}$ are indeed reductions in Definition 2.1. We also note that, if $\overline{\sigma}_n$ and $\overline{\rho}_n$ respectively denote the simple and regular reductions on S_n. then we have $\overline{\sigma}_{n+1} = L_2(\overline{\sigma}_n)$ and $\overline{\rho}_{n+1} = L_1(\overline{\rho}_n)$.

Lemma 2.4 (i) Let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n and let $\overline{g} = L_1(\overline{f})$. Then, with respect to these two reductions, we have
$p \downarrow n+1 \downarrow i = p \downarrow i \downarrow n$ for all $p \in S_{n+1}$ and for all $i \leq n$, and
$p \downarrow n+1 \downarrow n = p \downarrow n \downarrow n$ for all $p \in S_{n+1}$.

(ii) Let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n and let $\overline{h} = L_2(\overline{f})$. Then, with respect to these two reductions, we have
$p \downarrow n \downarrow i = p \downarrow i \downarrow n$ for all $p \in S_{n+1}$ and for all $i \leq n$, and
$p \downarrow n+1 \downarrow n = p \downarrow n \downarrow n$ for all $p \in S_{n+1}$.

Proof: Since f_n is the identity function on $\{1, 2, ..., n-1\}$, the function $g_n(x)$ is given by exactly the same formula as $h_n(x)$. As shown in the proof that (ii) implies (i) in Lemma 2.3, this implies, in both cases, that $p \downarrow n+1 \downarrow n = p \downarrow n \downarrow n$ for all $p \in S_{n+1}$. As for the linking conditions, we give the argument for (ii); part (i) is done in the same way. So, let us show, for $\overline{f}$ and $\overline{h}$, that $p \downarrow n \downarrow i = p \downarrow i \downarrow n$ for all $p \in S_{n+1}$ and for all $i \leq n$. This is trivial for $i = n$, so we assume that $i < n$. Let $p = p_1, p_2, ..., p_{n+1}$ be any permutation in S_{n+1}. Let r, s and t respectively be the integers for which $p_r = i, p_s = n$ and $p_t = n + 1$. In our argument we will assume that $r < s < t$. It is easy to see that the same kind of argument will work no matter what the relative order of r, s and t. So, suppose that $r < s < t$. To obtain the reduction $p \downarrow n$, we delete n from p and then apply the function h_n to the other elements in place. Since, by definition, h_n fixes every element except $n + 1$, which is changed to n, the result is

$$p \downarrow n = p_1, p_2, ..., p_r, ..., p_{s-1}, p_{s+1}, ..., p_{t-1}, n, p_{t+1}, ..., p_{n+1}.$$

Now, to get $p \downarrow n \downarrow i$, we delete i from $p \downarrow n$ and then apply the function f_i in place to the other entries. The result is $f_i(p_1), f_i(p_2), ...$

$$..., f_i(p_{r-1}), f_i(p_{r+1}), ..., f_i(p_{s-1}), f_i(p_{s+1}), ..., f_i(p_{t-1}), f_i(n), f_i(p_{t+1}), ..., f_i(p_n$$

On the other hand, to obtain $p \downarrow i$, we delete i from p and then apply the function h_i in place to the other entries. By definition, the value of $h_i(x)$ is the same as $f_i(x)$, except for $x = n$, where it is n, and for $n + 1$, where it is $f_i(n)$. Thus $p \downarrow i$ is equal to $f_i(p_1), f_i(p_2), ...$

$$..., f_i(p_{r-1}), f_i(p_{r+1}), ..., f_i(p_{s-1}), n, f_i(p_{s+1}), ..., f_i(p_{t-1}), f_i(n), f_i(p_{t+1}), ..., f_i($$

To obtain $p \downarrow i \downarrow n$, we just delete n from $p \downarrow i$. We get $f_i(p_1), f_i(p_2), ...$

$$..., f_i(p_{r-1}), f_i(p_{r+1}), ..., f_i(p_{s-1}), f_i(p_{s+1}), ..., f_i(p_{t-1}), f_i(n), f_i(p_{t+1}), ..., f_i(p_n$$

This is identical to $p \downarrow n \downarrow i$. □

Another way to state the conclusions in Lemma 2.4 is as follows. In (i), $\overline{f}$ and $\overline{g}$ form an amenable reduction system of length 2 which is linked by $n + 1$, and in (ii), $\overline{f}$ and $\overline{h}$ form an amenable reduction system of length 2 which is linked by n.

Using the two types of liftings described in Definition 2.1, we can now give a complete characterization of amenable linked reduction systems. Begging the reader's indulgence, we will employ one more piece of notation. We will denote a reduction system $\{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ by a script letter such as $\mathcal{F}$, and correspondingly, for $n \geq n_0$, denote the reduction $\{f_i^n : 1 \leq i \leq n\}$ on S_n by $\mathcal{F}_n$.

Theorem 2.5 Let $n_0 \geq 4$ and let $\mathcal{F} = \{f_i^n : n \geq n_0, 1 \leq i \leq n\}$ be a reduction system, and for any $n \geq n_0$, let $\mathcal{F}_n = \{f_i^n : 1 \leq i \leq n\}$. Then the following are equivalent:

(i) $\mathcal{F}$ is an amenable linked reduction system

(ii) For every $n > n_0$, either $\mathcal{F}_n = L_1(\mathcal{F}_{n-1})$ or $\mathcal{F}_n = L_2(\mathcal{F}_{n-1})$

Proof: We can deduce the implication (ii) $\rightarrow$ (i) directly from Lemma 2.4. Assuming (ii), we define $\phi(n) = n$ if $\mathcal{F}_n = L_1(\mathcal{F}_{n-1})$ and $\phi(n) = n-1$ if $\mathcal{F}_n = L_2(\mathcal{F}_{n-1})$. The first parts of statements (i) and (ii) in Lemma 2.4 imply that ϕ is a linking function for the system. The second parts of (i) and (ii) imply that the system is amenable.

In the other direction, let $\mathcal{F}$ be any amenable linked reduction system with linking function ϕ. By Lemma 2.2, for all $n > n_0$, we have either $\phi(n) = n$ or $\phi(n) = n-1$, and by Lemma 2.3, for every $n > n_0$ the function f_{n-1}^n is given by $f_{n-1}^n(x) = \begin{cases} x & \text{if } x < n-1 \\ n-1 & \text{if } x = n \end{cases}$

Let $n > n_0$. We show that if $\phi(n) = n$ then $\mathcal{F}_n = L_1(\mathcal{F}_{n-1})$, and if $\phi(n) = n-1$ then $\mathcal{F}_n = L_2(\mathcal{F}_{n-1})$. The two cases are done similarly, so we will only present the argument for the second. So, let $n > n_0$ and suppose we have $\phi(n) = n-1$. This means that $p \downarrow n-1 \downarrow i = p \downarrow i \downarrow n-1$ for all $i \leq n-1$ and for all $p \in S_n$. Let i be any integer with $1 \leq i < n-1$, and let $p = e_n = 1, 2, 3, ..., n$. The formula for $f_{n-1}^n(x)$ implies that

$p \downarrow n-1 = f_{n-1}^n : [1, 2, 3, ..., \widehat{n-1}, n] = [1, 2, 3, ..., n-1]$ and so

$p \downarrow n-1 \downarrow i = f_i^{n-1} : [1, 2, ..., \hat{i}, ..., n-1]$.

On the other hand, $p \downarrow i = f_i^n : [1, 2, ..., \hat{i}, ..., n-1, n]$. Now, by Lemma 2.1, we know that $f_i^n(\phi(n)) = n-1$, that is, $f_i^n(n-1) = n-1$. Since $p \downarrow i \downarrow n-1$ results by simply deleting $n-1$ from $p \downarrow i$, we get

$p \downarrow i \downarrow n-1 = f_i^n : [1, 2, ..., \hat{i}, ..., \widehat{n-1}, n] = f_i^n : [1, 2, ..., \hat{i}, ..., n-2, n]$.

Comparing the two lists for $p \downarrow i \downarrow n-1$ and $p \downarrow n-1 \downarrow i$, we see that we must have $f_i^n(x) = f_i^{n-1}(x)$ for all $x \neq i$, except for $x = n$, for which $f_i^n(n) = f_i^{n-1}(n-1)$. This is exactly what is needed to conclude that $\mathcal{F}_n = L_2(\mathcal{F}_{n-1})$ $\square$.

Theorem 2.5 says that, on the one hand, to construct an amenable linked reduction system, we can choose (for any $n_0 \geq 4$) any beginning reduction on S_{n_0}. We then iteratively form reductions on S_n for $n > n_0$ by lifting the reduction from S_{n-1} using either a type 1 or type 2 lifting. The choice of lifting, type 1 or type 2, is arbitrary at each step. Furthermore, every amenable linked reduction system is constuctible in this way.

We note that the regular reduction system, for $n \geq n_0$, results by starting with the regular reduction on S_{n_0} and then applying liftings of type 1 repeatedly. The simple reduction system results by starting with the simple reduction on S_{n_0} and then applying liftings of type 2 repeatedly.

3. Faithfulness

We now turn to the goal of obtaining faithful reduction systems. By Corollary 1.3, an amenable linked reduction system will be faithful if it begins with a faithful reduction and if it satisties property $\mathbf{P_1}$. In light of Theorem 2.5, we should now determine whether (beginning with a faithful reduction) property $\mathbf{P_1}$ results by applying a sequence of type 1 or type 2 liftings, and to what extent the beginning reduction plays a role.

To do so, we will first establish three lemmas which give much sharper focus to property $\mathbf{P_1}$.

Lemma 3.1 Let $n \geq 3$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n and assume that $f_{n-1}(n) = n-1$. Suppose a is either equal to n or $n-1$, and that we have $f_i(a) = n-1$ for all $i < n$ such that $i \neq a$. Let $p = x_1, x_2, ..., x_{n-1}$ be a permutation of the set $\{1, 2, ..., n\} - \{a\}$. Let j and k be integers such that $1 \leq j < j+1 < k \leq n$. Let p_1 be the permutation on the set $\{1, 2, ..., n\}$ obtained by inserting a into p so that it occupies position j: $p_1 = x_1, x_2, ..., x_{j-1}, a, x_j, x_{j+1}, ..., x_{n-1}$. Let p_2 be obtained by inserting a into p so that it occupies position k. Then $R(p_1) \neq R(p_2)$.

Proof: Let us first assume that $a = n$. Let s be the integer for which $x_s = n-1$. We of course have $p_2 = x_1, x_2, ..., x_{k-1}, a, x_k, x_{k+1}, ..., x_{n-1}$. (Our notation for p_1 and p_2 is modified in the obvious way if $j = 1$ or $k = n$.) Let us define sets $Z(p_1)$ and $Z(p_2)$ as follows:
$Z(p_1) = \{r$: there is a permutation $q \in R(p_1)$ such that $n-1$ occupies position r in $q\}$
$Z(p_2) = \{r$: there is a permutation $q \in R(p_2)$ such that $n-1$ occupies position r in $q\}$
We will show that $Z(p_1) \neq Z(p_2)$ from which our conclusion follows.

Case 1: $s < j$. In this case we note that, in the reduction $p_1 \downarrow n$, where we simply delete n, the entry $n-1$ occurs in position s. If i is any one of the integers $x_1, x_2, ..., x_{j-1}$ then, in the reduction $p_1 \downarrow i$, we delete i and then apply f_i. Under this function, because of our assumption in the statement of the lemma, a becomes $n-1$. Therefore $n-1$ occupies position $j-1$ in $p_1 \downarrow i$. In a similar way we see that, if i is any of the integers $x_j, x_2, ..., x_{n-1}$, then $n-1$ occupies position j in $p_1 \downarrow i$. It follows that $Z(p_1) = \{s, j-1, j\}$. Because $j < k$, it similarly follows that $Z(p_2) = \{s, k-1, k\}$ if $k < n$ and $Z(p_2) = \{s, k-1\}$ if $k = n$. Since $s < j$ and $j+1 < k$, the integer j does not belong to the set $Z(p_2)$. Since j does belong to $Z(p_1)$, we conclude that $Z(p_1) \neq Z(p_2)$.

Case 2: $j \leq s < k$. As in Case 1, in this case we again have $Z(p_2) = \{s, k-1, k\}$ if $k < n$ and $Z(p_2) = \{s, k-1\}$ if $k = n$. And similarly we have $Z(p_1) = \{j-1, j, s\}$ if $j > 1$ and $Z(p_1) = \{j, s\}$ if $j = 1$. Now $j-1$ is smaller than all of the integers $s, k-1, k$, so $j-1$ does not belong to the set $Z(p_2)$. So the only possible way we could have $Z(p_1) = Z(p_2)$ is if $Z(p_1) = \{j, s\}$ where $j = 1$, that is $Z(p_1) = \{1, s\}$. The smallest element of the set $Z(p_2)$ is s, and so if $Z(p_1) = Z(p_2)$ we would have $s = 1$. But this would imply that $Z(p_1) = \{1\}$; this set clearly cannot be equal to $Z(p_2)$.

Case 3: $k \leq s$. In this case, we again note that $Z(p_2) = \{s, k-1, k\}$. Since $j+1 < k \leq s$, the integer j does not belong to the set $Z(p_2)$. However j does belong to $Z(p_1)$. Therefore $Z(p_1) \neq Z(p_2)$.

The same kind of argument gives the result when $a = n-1$. We let s be the integer for which $x_s = n$. We reverse the roles of n and $n-1$ from the first half of the proof, and again consider three cases. Let us just include the details for one of these, say when $s < j$: this implies that the position of $n-1$ in $p_1 \downarrow n$ is $j-1$. The same position for $n-1$ occurs in $p_1 \downarrow i$ for any of the other integers i among $x_1, x_2, ..., x_{j-1}$, because $f_i(n-1) = n-1$. Since $f_{n-1}(n) = n-1$, in the reduction $p_1 \downarrow n-1$ it is position s which $n-1$ occupies. If i is equal to any of $x_j, x_{j+1}, ..., x_{n-1}$ then $n-1$ has position j in $p_1 \downarrow i$. We thus have $Z(p_1) = \{s, j-1, j\}$. In a similar way we find that $Z(p_2) = \{s, k-1, k\}$ if $k < n$ and $Z(p_2) = \{s, k-1\}$ if $k = n$. But j does not belong to the set $\{s, k-1, k\}$, since $j+1 < k$ and $j > s$. So $Z(p_2)$ cannot be equal to $Z(p_1)$ □.

Lemma 3.2 Let $n \geq 3$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n and assume that $f_{n-1}(n) = n-1$. Suppose a is either equal to n or $n-1$, and that we have $f_i(a) = n-1$ for all $i < n$ such that $i \neq a$. Let $p = x_1, x_2, ..., x_{n-1}$ be a permutation of the set $\{1, 2, ..., n\} - \{a\}$. Let j and k be integers such that $1 \leq j < k \leq n$. Let p_1 be the permutation on the set $\{1, 2, ..., n\}$ obtained by inserting a into p so that it occupies position j and let p_2 be obtained by inserting a into p so that it occupies position k. If $\{j, k\} \neq \{1, 2\}$ and $\{j, k\} \neq \{n-1, n\}$ then $R(p_1) \neq R(p_2)$.

Proof: We suppose $a = n$; the details are similar when $a = n-1$. By Lemma 3.1, it is enough to consider the case when $2 \leq j \leq n-2$ and $k = j+1$. Let s be the integer for which $x_s = n-1$, and define $Z(p_1)$

and $Z(p_2)$ as in the proof of Lemma 3.1. As in that argument, we find that $Z(p_1) = \{s, j-1, j\}$ and $Z(p_2) = \{s, j, j+1\}$. These two sets are distinct. □

Lemma 3.3 Let $n \geq 3$ and let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ be a reduction on S_n and assume that $f_{n-1}(n) = n-1$. Suppose a is either equal to n or $n-1$, and that we have $f_i(a) = n-1$ for all $i < n$ such that $i \neq a$. Suppose p_1 and p_2 are two different permutations in S_n such that $p_1 - \{a\} = p_2 - \{a\}$ and $R(p_1) = R(p_2)$.

(i) If $a = n$, there is an integer $r < n-1$ and a permutation $x_1, x_2, ..., x_{n-3}$ of the set $\{1, 2, ..., n\} - \{n, n-1, r\}$ such that either

$$p_1 = x_1, x_2, ..., x_{n-3}, n-1, r, n \quad \text{and} \quad p_2 = x_1, x_2, ..., x_{n-3}, n-1, n, r \quad \text{or}$$

$$p_1 = n, r, n-1, x_1, x_2, ..., x_{n-3} \quad \text{and} \quad p_2 = r, n, n-1, x_1, x_2, ..., x_{n-3}$$

(ii) If $a = n-1$, there is an integer $r < n-1$ and a permutation $x_1, x_2, ..., x_{n-3}$ of the set $\{1, 2, ..., n\} - \{n, n-1, r\}$ such that either

$$p_1 = x_1, x_2, ..., x_{n-3}, n, r, n-1 \quad \text{and} \quad p_2 = x_1, x_2, ..., x_{n-3}, n, n-1, r \quad \text{or}$$

$$p_1 = n-1, r, n, x_1, x_2, ..., x_{n-3} \quad \text{and} \quad p_2 = r, n-1, n, x_1, x_2, ..., x_{n-3}$$

Proof: We will do (i) and leave (ii) for the reader. So, suppose that $a = n$. By the preceding lemma, we know that the positions of n in p_1 and p_2 are consecutive and are either $\{1, 2\}$ or $\{n-1, n\}$. These two cases are opposites. We will show that the first pair of equations holds for p_1 and p_2 in the case of $\{n-1, n\}$. The opposite case then follows directly from this using Lemma 1.1. So we then suppose that n is last in p_1 and second last in p_2. So we have $p_1 = x_1, x_2, ..., x_{n-3}, x_{n-2}, x_{n-1}, n$ and $p_2 = x_1, x_2, ..., x_{n-3}, x_{n-2}, n, x_{n-1}$. What we need to show is that $x_{n-2} = n-1$. To do so, let s be the integer for which $x_s = n-1$. With $Z(p_1)$ and $Z(p_2)$ defined as above, we have $Z(p_1) = \{s, n-1\}$ and $Z(p_2) = \{n-2, n-1, s\}$. Since $R(p_1) = R(p_2)$, we also have $Z(p_1) = Z(p_2)$. This implies that $n-2 = s$, as desired. □

The relevance of the preceding lemmas to the condition $\mathbf{P}_1$ is clear: if $\overline{f}_n$ is the nth reduction in an amenable linked system then, by Lemmas 2.2 and 2.3, $\overline{f}_n$ satisfies the conditions on $\overline{f}$ stated in Lemma 3.3, with $a = \phi(n)$. Therefore, in attempting to determine whether condition $\mathbf{P}_1$ holds for $\overline{f}_n$, all we have to do is consider whether it is possible that $R(p_1) = R(p_2)$ for two permutations p_1 and p_2 given by the equations in (i) or (ii) in Lemma 3.3, depending on whether $\overline{f}_n$ has been obtained from $\overline{f}_{n-1}$ by a type 1 or type 2 lifting. As we will see next, these possibilities can be explicitly characterized.

Lemma 3.4 Let $n \geq 3$. Let $\overline{f} = \{f_i : 1 \leq i \leq n-1\}$ be a faithful reduction on S_{n-1} and let $\overline{g}$ be a reduction on S_n which has been obtained from $\overline{f}$ by applying a lifting of type 1 or 2. Then $\overline{g}$ does not satisfy property $\mathbf{P_1}$ if and only if the following condition holds:

there is an integer $r < n-1$ and an ordering $x_1, x_2, ..., x_{n-3}, x_{n-2}$ of the set $\{1, 2, ..., n-1\} - \{r\}$ such that $x_{n-2} = n-1$ and such that

$$f_r(x) = \begin{cases} x & \text{if } x \neq r \text{ and } x \neq n-1 \\ r & \text{if } x = n-1 \end{cases}$$ and, for all $i \leq n-2$,

$$f_{x_i}(x) = \begin{cases} r & \text{if } x = r \\ x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$$

Proof: Notice that these equations entirely define the reduction $\overline{f}$. We recognize the description for $\overline{f}$ from Example 1.1(ii) in Section 1 above. Using the terminology of that example, we can paraphrase this lemma as follows: if $\overline{f}$ is faithful and if either $L_1(\overline{f})$ or $L_2(\overline{f})$ is not faithful, then $\overline{f}$ is a reduction of regular type plus one simple element.

Let us do type 2 liftings first. We apply Lemma 3.3 to $\overline{g} = L_2(\overline{f})$ with $a = \phi(n) = n-1$. We need to show that, if there are permutations p_1 and p_2 given by either of the pairs of equations in (ii) of Lemma 3.3, and if $R(p_1) = R(p_2)$, then $\overline{f}$ must be given by the above equations. It will become clear that both pairs of equations lead to the same description for $\overline{f}$, so we will only need to give the details for one of these. So let us suppose that we have an integer $r < n-1$ such that

$$p_1 = x_1, x_2, ..., x_{n-3}, n, r, n-1 \quad \text{and} \quad p_2 = x_1, x_2, ..., x_{n-3}, n, n-1, r$$

and for which $R(p_1) = R(p_2)$. We observe that the set $R(p_2)$ has only one element whose last entry is $n-1$, namely $p_2 \downarrow r$. Since $p_1 \downarrow n$ is such an element in $R(p_1)$, it follows that $p_2 \downarrow r = p_1 \downarrow n = x_1, x_2, ..., x_{n-3}, r, n-1$. Since $p_2 \downarrow r = g_r : [x_1, x_2, ..., x_{n-3}, n, n-1]$, the latter equality implies that

$$g_r(x) = \begin{cases} x & \text{if } x \neq r \text{ and } x \neq n \\ r & \text{if } x = n \end{cases}$$ Now, by definition of the type 2 lifting, we have $$g_r(x) = \begin{cases} f_r(x) & \text{if } x < n-1 \text{ and } x \neq r \\ n-1 & \text{if } x = n-1 \\ f_r(n-1) & \text{if } x = n \end{cases}$$

From this it is clear that f_r is given by the formula stated in the lemma.

Now, for any $i \leq n-3$, the last entry of the reduction $p_1 \downarrow x_i$ is $n-1$.

Since, as already observed, $R(p_2)$ has only one element whose last element is $n-1$, all of these reductions $p_1 \downarrow x_i$ must be equal to $p_1 \downarrow n = x_1, x_2, ..., x_{n-3}, r, n-1$. But we have

$p_1 \downarrow x_i = g_{x_i} : [x_1, x_2, ..., \widehat{x_i}, ..., x_{n-3}, n, r, n-1]$. Therefore we get

$$g_{x_i}(x) = \begin{cases} x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \\ x_{n-3} & \text{if } x = n \\ r & \text{if } x = r \\ n-1 & \text{if } x = n-1 \end{cases}$$

But again, by definition of the type 2 lifting, we have

$$g_{x_i}(x) = \begin{cases} f_{x_i}(x) & \text{if } x < n-1 \text{ and } x \neq x_i \\ n-1 & \text{if } x = n-1 \\ f_{x_i}(n-1) & \text{if } x = n \end{cases}$$

From this it follows that f_{x_i} is given by the formula stated in the lemma.

In the other direction, if the functions f_{x_i} and f_r are given by the formulas stated in the lemma, then g_{x_i} and g_r are also given as above. Looking back at Example 1.1(iii) in Section 1 we see that $\overline{g}$ is a regular reduction plus 2 simple elements, namely r and $n-1$. As verified in that example, the permutations $p_1 = x_1, x_2, ..., x_{n-2}, r, n-1$ and $p_2 = x_1, x_2, ..., x_{n-2}, n-1, r$ have the same set of reductions, and so property $\mathbf{P_1}$ fails for $\overline{g}$.

An obvious modification of the preceding argument shows that we are led to exactly the same description for the functions f_{x_i} and f_r if the second pair of equations in lemma 3.3 applies; we just consider the elements of the reduction sets whose first entry is $n-1$, rather than those whose last element is $n-1$.

The argument is similar for type 1 liftings. Let $\overline{g} = L_1(\overline{f})$ and suppose $\overline{g}$ does not satisfy property $\mathbf{P_1}$. As in the proof for type 2 liftings, we will only consider one of the pairs of equations in part (i) of Lemma 3.3, the other leading to the same set of equations. We thus assume that we have an integer $r < n-1$ and permutations $p_1 = x_1, x_2, ..., x_{n-3}, n-1, r, n$ and $p_2 = x_1, x_2, ..., x_{n-3}, n-1, n, r$ for which $R(p_1) = R(p_2)$. We observe that $n-1$ is last in every one of the reductions of p_1 except for $p_1 \downarrow n$. So $R(p_1)$ has exactly one member in which $n-1$ is not last. So the same must be true for p_2. It follows that $p_2 \downarrow x_i = p_2 \downarrow n-1 = p_2 \downarrow n = p_1 \downarrow n$. Denote $n-1$ by x_{n-2}. Now we have

$p_2 \downarrow n-1 = g_{n-1} : [x_1, x_2, ..., x_{n-3}, n, r] = p_1 \downarrow n = [x_1, x_2, ..., x_{n-3}, n-1, r]$

From this we deduce that $g_{n-1}(x) = \begin{cases} x & \text{if } x \neq n-1 \text{ and } x \neq n \\ n-1 & \text{if } x = n \end{cases}$

We also note that $R(p_2)$ has exactly one member for which $n-1$ is the last entry, namely $p_2 \downarrow r$. So the same must be true for the set $R(p_1)$. Since $p_1 \downarrow n-1$ is a member of $R(p_1)$ which ends with $n-1$, we must have $p_1 \downarrow n-1 = p_2 \downarrow r$. Thus we have

$g_{n-1} : [x_1, x_2, ..., x_{n-3}, r, n] = g_r : [x_1, x_2, ..., x_{n-3}, n-1, n]$.

Since the former is equal to $[x_1, x_2, ..., x_{n-3}, r, n-1]$, we deduce that

$$g_r(x) = \begin{cases} x & \text{if } x \neq n \text{ and } x \neq n-1 \\ r & \text{if } x = n-1 \\ n-1 & \text{if } x = n \end{cases}$$

Similarly, since $R(p_1)$ only has one member, $p_1 \downarrow n$ where $n-1$ is not last, we must have
$p_2 \downarrow x_i = p_1 \downarrow n$ for all $i \leq n-3$. From this we get

$g_{x_i} : [x_1, x_2, ..., \widehat{x_i}, ..., x_{n-3}, n-1, n, r] = [x_1, x_2, ..., x_{n-3}, n-1, r]$

and so we find $g_{x_i}(x) = \begin{cases} x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \\ r & \text{if } x = r \\ n-1 & \text{if } x = n \end{cases}$

Now, because this is a type 1 lifting, we have, for all $t < n$,

$g_t(x) = \begin{cases} f_t(x) & \text{if } x \neq t \text{ and } x \neq n \\ n-1 & \text{if } x = n \end{cases}$ Therefore we can deduce the stated formulas for $\overline{f}$. Conversely, we can easily show that if the conditions on $\overline{f}$ hold, then $\overline{g}$ does not have property $\mathbf{P_1}$: the two permutations p_1 and p_2 described in the proof have the same reduction set. □

Using the terminology of Example 1.1(ii) in Section 1, we can express the conclusion of Lemma 3.4 in an equivalent way as follows.

Corollary 3.5 Let $n \geq 3$ and let $\overline{f} = \{f_i : 1 \leq i \leq n-1\}$ be a faithful reduction on S_{n-1}. If $\overline{f}$ is not the sum of a reduction of regular type plus one simple element, then both liftings $L_1(\overline{f})$ and $L_2(\overline{f})$ are faithful.

Now, let us conclude by applying these results to reduction systems. Suppose we generate an amenable linked reduction system by choosing a faithful starting reduction $\mathcal{F}_{n_0}$ on S_{n_0} and then applying a sequence of type

1 and type 2 liftings. In order to use our results to deduce that the system is faithful, we need to know that none of the reductions generated is the sum of a regular reduction plus one simple element. It is not sufficient to assume this for $\mathcal{F}_{n_0}$. For example, if we take $\mathcal{F}_{n_0}$ to be the (canonical) regular reduction on $\mathcal{F}_{n_0}$ then $L_2(\mathcal{F}_{n_0})$ is equal to the sum of a reduction of regular type plus one simple element, the simple element being n_0. If we apply a second type 2 lifting, we obtain one of the reductions discussed in Example 1.1(iii) of Section 1, with two simple elements, which is not faithful. We can formulate a condition to avoid this type of problem. We need one more result.

Lemma 3.6 Let $n \geq 3$ and let $\overline{f}$ be a reduction on S_n. Then

(i) $L_2(\overline{f})$ is not a reduction of regular type.

(ii) If $L_2(\overline{f})$ is the sum of a reduction of regular type plus a simple element, then the simple element must be n and $\overline{f}$ is a reduction of regular type.

(iii) $L_1(\overline{f})$ is not equal to the sum of a reduction of regular type plus one simple element.

(iv) If $L_1(\overline{f})$ is a reduction of regular type so is $\overline{f}$.

(v) Let $n_0 \geq 4$ and let $\{\mathcal{F}_n : n \geq n_0\}$ be a linked amenable reduction sytem. If $\mathcal{F}_{n_0}$ is not equal to a reduction of regular type or the sum of a reduction of regular type plus one simple element, then the same is true for $\mathcal{F}_n$ for all $n \geq n_0$.

Proof: Suppose we let $\overline{f} = \{f_i : 1 \leq i \leq n\}$ and we let $\overline{g} = \{g_i : 1 \leq i \leq n+1\}$ denote the lifting $L_2(\overline{f})$ in (i) and (ii), and $L_1(\overline{f})$ in (iii) and (iv). To prove (i), we note that in the reduction $L_2(\overline{f})$, there is an element $a = n$ having the property that $f_a(n+1) = a$ and for all $x \neq a$, $f_x(a) = a$. A reduction of regular type has no such element.

For (iii) we note that a simple element a would have the property that $g_a(x) = \begin{cases} x & \text{if } x \neq a \text{ and } x \neq n+1 \\ a & \text{if } x = n+1 \end{cases}$ and $g_t(a) = a$ for any $t \neq a$. But, under a type 1 lifting, all the functions g_t, for $t \neq n+1$, map $n+1$ to n, so no such element a can exist.

For statements (ii) and (iv), the assumption gives us explicit formulas for all of the functions which comprise $\overline{g}$. By directly applying the definition of type 1 or 2 lifting as appropriate, we immediately get explicit formulas for all of the functions comprising $\overline{f}$. The conclusions are then made directly. We include the details for (iv). If $\overline{g}$ is a reduction of regular type there is an ordering $x_1, x_2, ..., x_{n+1}$ of the set $\{1, 2, ..., n+1\}$ such that

$x_{n+1} = n+1$ and such that $g_{x_i}(x) = \begin{cases} x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$

By definition of L_1, we have $g_{n+1}(x) = x$ for all $x \leq n$ and, for all $t \leq n$,

$$g_t(x) = \begin{cases} f_t(x) & \text{if } x \leq n \text{ and } x \neq t \\ n & \text{if } x = n+1 \end{cases}$$

It follows that $x_n = n$ and that $f_{x_i}(x) = \begin{cases} x_j & \text{if } x = x_j \text{ for some } j < i \\ x_{j-1} & \text{if } x = x_j \text{ for some } j > i \end{cases}$

and so $\overline{f}$ is a reduction of regular type with respect to the ordering $x_1, x_2, ..., x_n$.

For (v), we recall that each reduction in a linked system can be obtained from the previous one by applying a lifting of type 1 or 2. So (v) follows by induction using statements (i) through (iv). □

Corollary 3.5 and part (v) of Lemma 3.6 imply the following theorem.

Theorem 3.7 Let $n_0 \geq 4$ and let $\{\mathcal{F}_n : n \geq n_0\}$ be an amenable linked reduction sytem. Suppose $\mathcal{F}_{n_0}$ is faithful and is not equal to a reduction of regular type or the sum of a reduction of regular type plus one simple element. Then $\mathcal{F}_n$ is faithful for all $n \geq n_0$.

Corollary 3.8 Let $n_0 \geq 5$ and let $\{\mathcal{F}_n : n \geq n_0\}$ be an amenable linked reduction sytem which begins with the simple reduction on S_{n_0}. Then $\mathcal{F}_n$ is faithful for all $n \geq n_0$.

One can construct other suitable starting reductions in an ad hoc manner, by deliberately avoiding the structure of simple types, regular types and regular types plus a simple element. In such a case, we would also need to obtain and verify the faithfulness of that particular reduction.

It is also worth noting what happens if the beginning reduction is regular. It is easy to see that a type 1 lifting of a reduction of regular type is again of regular type. Using this fact together with the above results, it can be seen that, if we begin with a regular reduction and do not apply two type 2 liftings in succession, the resulting reduction system will be faithful. If two liftings of type 2 are ever applied in succession then the reduction obtained at that point will not be faithful.

4. Some directions for future work

As was mentioned in the remarks following Definition 1.4, one can also consider the notion of weakly faithful reductions, in which one considers the *multiset* of reductions $R'(p)$ for a permutation p: a reduction is weakly faithful if $p \neq q \rightarrow R'(p) \neq R'(q)$. To what extent can the above results be extended to weakly faithful reductions? How significant is the difference between faithfulness and weak faithfulness for permutation reductions?

We would like to suggest two other possible directions for further study. One was briefly hinted at in the remarks following Definition 1.4. It involves looking at a somewhat more general "linking condition" in the definition of a linked reduction system. Specifically, the condition that $p \downarrow \phi(n) \downarrow i = p \downarrow i \downarrow n-1$ for all $i \leq n-1$ might be replaced by the condition $p \downarrow \phi(n) \downarrow i = p \downarrow i \downarrow \psi(n)$ for all $i \leq n-1$, where $\psi(n)$ can be an arbitrary integer-valued function with $1 \leq \psi(n) \leq n-1$ for all n. To what extent can our results on linked systems and faithfulness be generalized? How can one construct interesting classes of examples, and for what types of functions $\psi(n)$?

Finally, it would be interesting to have additional constructive information with which to recognize unfaithful reductions. Is there a small collection of particular kinds of unfaithful reductions (including examples such as Example 1.1(iii) and (iv) above), having the property that any unfaithful reduction must obtain a copy of one of these? Can this at least be done for a broad class of reductions?

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