Walk Regular Digraphs *

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Abstract

A strongly connected digraph Γ is said to be *walk regular* if for any nonnegative integer l and any vertex u of Γ, the number of circuits of length l containing u depends only on l. This family of digraphs is a directed version of walk regular graphs. In this paper, we discuss some basic properties of walk regular digraphs.

1 Introduction

Let $\Gamma = (V, E)$ be a digraph with the vertex set V and the arc set E. If $(u, v) \in E$, we say that u dominates v. The set of vertices of Γ dominated by u is said to be the *out-neighbors* of u, denoted by $\Gamma_+(u)$. The set of vertices of Γ dominating u is said to be the *in-neighbors* of u, denoted by $\Gamma_-(u)$. A digraph Γ is said to be *regular* of valency k if $|\Gamma_+(u)| = |\Gamma_-(u)| = k$ for any vertex u of Γ. A *walk* of length l in Γ is a sequence $(u_0, u_1, ..., u_l)$ of vertices such that $(u_{i-1}, u_i) \in E$, $i = 1, 2, ..., l$. If u_l dominates u_0, the walk $(u_0, u_1, ..., u_l)$ is said to be a *circuit*. The *girth* of Γ is the length of a shortest circuit. If a digraph contains an edge, its girth is 2. The number of arcs traversed in a shortest walk from u to v is called the *distance* from u to v in Γ, denoted by $\partial(u, v)$. The maximum value of the distance function in Γ is called the *diameter* of Γ. A digraph is said to be *strongly connected* if, for any two distinct vertices u and v, there is a walk from u to v.

Definition 1.1 *A digraph is said to be walk regular if for any given non-negative integer l and any vertex u of Γ, the number of circuits of length l containing u depends only on l.*

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Let Γ be a digraph with diameter D. For $0 \leq k \leq D$, the *distance-k matrix* A_k, is defined by

$$(A_k)_{uv} := \begin{cases} 1 & \text{if } \partial(u,v) = k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $A_0 = I$, and $A_1 = A$, which is the adjacency matrix of Γ. If $AA^T = A^T A$, then Γ is said to be normal. About normal matrices, there are the following properties:

Proposition 1.1 *([1]) Let A be an $n \times n$ complex matrix with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. Then A is normal if and only if any of the following assertions holds:*
(a) $U^* A U = D$ *for some matrix U such that* $UU^* = I$*,and* $D = \mathrm{diag}(\lambda_0, \lambda_1, \ldots$
(b) $A^* = p(A)$ *for some polynomial* $p \in \mathbb{C}[x]$.
(c) $\mathrm{tr}(AA^*) = \sum_{i=0}^{n-1} |\lambda_i|^2$.

Let A be a normal matrix with $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$ with multiplicities $m_0, m_1, \cdots, m_d$ and $m(x) = (x-\lambda_0)(x-\lambda_1)\cdots(x-\lambda_d)$ is the minimal polynomial of A. Let $\mathcal{A}(\Gamma)$ be the adjacency algebra of Γ. It is known that $\{I, A, \cdots, A^d\}$ is a basis of $\mathcal{A}(\Gamma)$ and $d \geq D$ since the powers $I, A, A^2, \cdots, A^D$ are linearly independent. From Proposition 1.1(a), we know that the eigenvectors of a normal $n \times n$ square matrix constitute an orthogonal basis of the vector space $\mathbb{C}^n$, with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* \mathbf{v}$. For each polynomial $p \in \mathbb{C}[x]$ we define p operates on the vector $\mathbf{v} \in \mathbb{C}^n$ by $p\mathbf{v} = p(A)\mathbf{v}$. For each λ_i, let U_i be the matrix whose columns form an orthonormal basis of the eigenspace $V_i := Ker(A - \lambda_i I)$. Then the orthogonal projection onto V_i is represented by the matrix $E_i = U_i U_i^*$, or alternatively, $E_i = \frac{1}{\phi_i} \prod_{j=0, j\neq i}^{d} (A - \lambda_j I)$, where $\phi_i = \prod_{j=0,j\neq i}^{d} (\lambda_i - \lambda_j)$. These matrices are called the principal idempotents of A and satisfy the following properties: $E_i E_j = \delta_{ij} E_i$, $AE_i = \lambda_i E_i$. Also $\{E_0, E_1, \cdots, E_d\}$ is a basis of $\mathcal{A}(\Gamma)$.

Then we can give the orthogonal decomposition of the unitary vector $\mathbf{e}_u$ of $\mathbb{C}^n$, represented vertex u as follow:

$$\mathbf{e}_u = \mathbf{z}_u^0 + \mathbf{z}_u^1 + \ldots + \mathbf{z}_u^d, \tag{1}$$

where $\mathbf{z}_u^i = E_i \mathbf{e}_u, i = 0, 1, ..., d$.

From the decomposition (1), we call Γ *spectrally regular* if $m_u(\lambda_i) := (E_i)_{uu}$ does not depend on u, where the notion $m_u(\lambda_i)$, is analogous to that of u-local multiplicity of eigenvalue λ_i, introduced by Fiol, Garriga and Yebra[7] for undirected graphs. It is easy to see that $\sum_{u \in V} m_u(\lambda_i) = m_i, 0 \leq i \leq d$ and $\sum_{i=0}^{d} m_u(\lambda_i) = 1$.

In this paper, we first discover the relationship between walk regular digraphs and spectrally regular digraphs, then discuss some properties of normal walk regular digraphs. Finally, we prove a special class of walk regular digraphs is distance-regular.

2 Main Results

Lemma 2.1 *If A is normal, then*

$$m_u(\lambda_i) = \|\mathbf{z}_u^i\|^2.$$

Proof. Since $E_i^* = (U_iU_i^*)^* = U_iU_i^* = E_i$, then

$$\begin{aligned}\|\mathbf{z}_u^i\|^2 &= \langle \mathbf{z}_u^i, \mathbf{z}_u^i \rangle = (\mathbf{z}_u^i)^* \mathbf{z}_u^i \\ &= (E_i\mathbf{e}_u)^* E_i\mathbf{e}_u = \mathbf{e}_u^T E_i^* E_i \mathbf{e}_u \\ &= \mathbf{e}_u^T E_i \mathbf{e}_u = (E_i)_{uu}. \quad \square\end{aligned}$$

Let Γ be a walk regular digraph. Then for each nonnegative integer l, $(A^l)_{uu}$ is the number of circuits of length l containing u. The following result is a natural extension of the corresponding result for walk-regular graphs, given by Fiol and Garriga [5] and Delorme and Tillich [2].

Theorem 2.2 *Let Γ be a normal digraph. Then the following conditions are equivalent:*
(i) Γ *is walk regular;*
(ii) Γ *is spectrally regular.*

Proof. If Γ is spectrally regular, suppose $A^l = \alpha_0 E_0 + \alpha_1 E_1 + \cdots + \alpha_d E_d$. Then for any vertex $u \in V$,

$$(A^l)_{uu} = (\alpha_0 E_0 + \alpha_1 E_1 + \cdots + \alpha_d E_d)_{uu} = \alpha_0 (E_0)_{uu} + \alpha_1 (E_1)_{uu} + \cdots + \alpha_d (E_d)_{uu}.$$

For each k, $(E_k)_{uu}$ is independent of u, hence A^l_{uu} does not depend on u. Γ is walk regular.

Conversly suppose Γ is walk regular. Let $E_k = \beta_0 I + \beta_1 A + \cdots + \beta_d A^d$. Then

$$m_u(\lambda_k) = (E_k)_{uu} = (\beta_0 I + \beta_1 A + \cdots + \beta_d A^d)_{uu} = \beta_0 (I)_{uu} + \beta_1 (A)_{uu} + \cdots + \beta_d (A^d)_{uu},$$

which implies that $m_u(\lambda_k) = \frac{m_k}{n}$, as desired. $\square$

Assume that A has $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$ with $|\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_d|$. By the Perron-Frobenius theorem, λ_0 is simple and has a positive eigenvector $\mathbf{v}$, if Γ is k-regular, then we may pick $\mathbf{v} = \mathbf{j}$, where $\mathbf{j}$ denotes the all 1- vector, and $\lambda_0 = k$.

Proposition 2.3 *Let Γ be a normal walk regular digraph. Then Γ is regular.*

Proof. By Lemma 2.1, $m_u(\lambda_0) = \|\mathbf{z}_u^0\|^2 = \nu_u^2/\|\nu\|^2$. By Theorem 2.2 $\nu_u = \nu_v$ for any distinct $u, v \in V$; and so $\nu = \nu_1(1, 1, \cdots, 1)^T$. Hence Γ is λ_0- regular. □

For a given digraph Γ with adjacency matrix A, we consider the following scalar product in $\mathbb{C}[x]$:

$$\langle p, q \rangle = \frac{1}{n} tr(p(A)q(A)^*)$$

It is obvious that the product is well defined in the quotient ring $\mathbb{C}[x]/(m(x$ Notice that $1, x, x^2, \cdots, x^d$ are linear independent in $\mathbb{C}_d[x]$, then by using the Gram-Schmidt method and normalizing appropriately, it is immediate to prove the existance and the uniqueness of an orthogonal system of polynomials $\{p_k\}_{0 \le k \le d}$ called predistance polynomials introduced by Fiol and Garriga in [4], which, for any $0 \le h, k \le d$, satisfy:

(1) $deg(p_k) = k$;
(2) $\langle p_h, p_k \rangle = 0$, if $h \neq k$;
(3) $\| p_k \|^2 = p_k(\lambda_0)$.

Definition 2.1 *([1]) A digraph Γ of diameter D is weakly distance-regular if, for each nonnegative integer $l \le D$, the number a_{uv}^l of walks of length l from vertex u to vertex v only depends on their distance $\partial(u, v) = k$, for any $l = 0, 1, \cdots, D$. In this case we write $a_{uv}^l = a_k^l, 0 \le k, l \le D$.*

Recall that, in a weakly distance-regular digraph, we have $D = d$([1],Theor 2.2) and such polynomials satisfy $p_k(A) = A_k$, $0 \le k \le d$.

Theorem 2.4 *Let Γ be a normal digraph with predistance polynomials $p_0, p_1, \cdots, p_d$. Then the following statements are equivalent.*

(i) Γ is walk regular;
(ii) The matrices $p_k(A)$, $1 \le k \le d$, have null diagonals.

Proof. Suppose Γ is walk regular . For $1 \le k \le d$, let $p_k(x) = \sum_{i=0}^k \gamma_i x^i$. Then $p_k(A) = \sum_{i=0}^k \gamma_i A^i$. For each vertex u,

$$(p_k(A))_{uu} = (\sum_{i=0}^{k} \gamma_i A^i)_{uu} = \sum_{i=0}^{k} \gamma_i a_0^i.$$

For each $1 \le k \le d$, we have

$$0 = \langle p_k, p_0 \rangle = \frac{1}{n} tr(p_k(A)) = \frac{1}{n} \cdot n \sum_{i=0}^{k} \gamma_i a_0^i = \sum_{i=0}^{k} \gamma_i a_0^i.$$

Hence $p_k(A)$, $1 \le k \le d$, have null diagonals.

Conversly, suppose $p_k(A)$ have null diagonals for each $1 \le k \le d$. Let $x^l = \sum_{k=0}^{l} \alpha_{lk} p_k$. Then $A^l = \sum_{k=0}^{l} \alpha_{lk} p_k(A)$ and

$$(A^l)_{uu} = \sum_{k=0}^{l} \alpha_{lk}(p_k(A))_{uu} = \alpha_{l0}(p_0(A))_{uu} = \alpha_{l0}.$$

Therefore $A^l_{uu} = \alpha_{l0}$, which is independent of u and Γ is walk regular. □

Let Γ be a normal walk regular digraph with the adjacency matrix A. Then by Proposition 1.1(b), $A^T = p(A)$ for some polynomial $p \in \mathbb{C}[x]$, so we may assume that $A^T = \mu_0 I + \mu_1 A + \cdots + \mu_d A^d$. Hence for any two vertices u and v with $(u, v) \in E$, the number of walks of length l from v to u is a constant. In particular, if the digraph Γ satisfies the very strong condition (Δ): For any two vertices $u, v \in V$ with $\partial(u, v) = k$, $0 \le k \le D$, the number $b^l_k(u, v)$ of walks of length l from v to u is independent of the choice of u and v. The following Theorem tells us that a digraph satisfying (Δ) is nothing but the distance-regular digraph.

A digraph $\Gamma = (V, E)$ with girth g is called stable if $partial(u, v) + partial(v, u) = g$ for any pair of vertices $u, v \in V$ at distance $0 < partial(u, v) < g$.

Theorem 2.5 *Let Γ be a strongly connected regular digraph with girth $g \ge 3$. Then Γ is distance-regular if and only if Γ satisfies the condition* (Δ).

Proof. Firstly we conclude that if Γ satisfying the condition (Δ), then Γ is stable . Let $(u = u_0, u_1, \cdots\cdots, u_{g-1})$ be a circuit of length g. It is obvious that for all $0 < t < g$, $partial(u, u_t) = t$. It follows that $b^{g-t}_t > 0$. Now let u, v be two vertices with $\partial(u, v) = t$, $0 < t < g$. Then there exists a walk of length $g - t$ from v to u by $b^{g-t}_t > 0$, which implies that $partial(v, u) \le g - t$; and so $partial(v, u) = g - t$. By [3], $D = g$ or $D = g - 1$.

For any two vertices u and v with $partial(u, v) = k$, $0 \le k \le D$, we consider the number $a^l_k(u, v)$ of walks of length l from v to u. Assume that Γ satisfying the condition (Δ). If $k = 0$, A^l_{uu} is just b^l_0 and if $k = D = g$, $a^l_D(u, v) = b^l_D(u, v) = b^l_D$. If $0 < k < g$, we have $a^l_k(u, v) = b^l_{g-k}(u, v) = b^l_{g-k}$ since Γ is stable. It follows that the numbers a^l_k, $0 \le k \le D$, are constants. Hence Γ is a stable weakly distance-regular digraph, which is a distance-regular digraph by [1].

The converse is obvious. □

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