A function on bounds of the spectral radius of graphs

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Abstract: Let $G = (V, E)$ be a simple connected graph with n vertices. The degree of $v_i \in V$ and the average of degrees of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. The spectral radius of G is denoted by $\rho(G)$. In this paper, we introduce a parameter into an equation of adjacency matrix, and obtain two inequalities for upper and lower bounds of spectral radius. By assigning different values to this parameter, one can obtain some new and existing results on spectral radius. Specially, if G is a nonregular graph, then

$$\rho(G) \leq \max_{1\leq j<i\leq n}\{\frac{d_im_i - d_jm_j + \sqrt{(d_im_i - d_jm_j)^2 - 4d_id_j(d_i - d_j)(m_i - m_j)}}{2(d_i - d_j)}\},$$

and

$$\rho(G) \geq \min_{1\leq j<i\leq n}\{\frac{d_im_i - d_jm_j + \sqrt{(d_im_i - d_jm_j)^2 - 4d_id_j(d_i - d_j)(m_i - m_j)}}{2(d_i - d_j)}\}.$$

If G is a bidegreed graph whose vertices of same degree have equal average of degrees, then the equality holds.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and e edges. The vertex set of G is denoted by $V = \{v_1, v_2, ..., v_n\}$. Two vertices v_i and v_j being adjacent is denoted by $v_i \sim v_j$. For $v_i \in V$, the degree of v_i is denoted by d_i and the average of degrees of the vertices adjacent to v_i is denoted by m_i, that is, $m_i = \frac{1}{d_i}\sum_{v_j\sim v_i} d_j$. Let Δ and δ denote the maximum

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vertex degree and the minimum vertex degree of G, respectively.

Let $A(G)$ be the adjacency matrix of G. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G)$. The sequence of n eigenvalues is called the *spectrum* of G, the largest eigenvalue $\lambda_1(G)$ is often called the *spectral radius* of G, denoted by $\rho(G) = \lambda_1(G)$.

2. Some upper bounds for spectral radius

We now list some known upper bounds for the spectral radius $\rho(G)$.

(a) (Collatz and Sinogowitz [1]) If G is a connected graph of order n, then

$$\rho(G) \leq \rho(K_n) = n - 1. \tag{1}$$

The upper bound occurs only when G is the complete graph K_n.

(b) (Collatz and Sinogowitz [1]) If G is a tree of order n, then

$$\rho(G) \leq \rho(K_{1,n-1}) = \sqrt{n-1}. \tag{2}$$

The upper bound occurs only when G is the star $K_{1,n-1}$.

(c) (Hong [2]) If G is a connected unicyclic graph, then

$$\rho(G) \leq \rho(S_n^3), \tag{3}$$

where S_n^3 denoted the graph obtained by joining any two vertices of degree one of the star $K_{1,n-1}$ by an edge. The upper bound occurs only when G is the graph S_n^3.

(d) (Brualdi and Hoffman [3]) If $e = \binom{k}{2}$, then

$$\rho(G) \leq k - 1, \tag{4}$$

where the equality holds iff G is a disjoint union of the complete graph K_k and some isolated vertices.

(e) (Stanley [4])

$$\rho(G) \leq (-1 + \sqrt{1+8e})/2, \tag{5}$$

where the equality holds iff $e = \binom{k}{2}$ and G is a disjoint union of the complete graph K_k and some isolated vertices.

(f) (Hong [5]) If G is a connected graph, then

$$\rho(G) \leq \sqrt{2e - n + 1}, \tag{6}$$

where the equality holds iff G is one of the following graphs:

(I) the star $K_{1,n-1}$;

(II) the complete graph K_n.

(g) (Hong, Shu and Fang [6]) Let G be a simple graph, then

$$\rho(G) \leq \frac{\delta - 1 + \sqrt{(\delta+1)^2 + 4(2e - \delta n)}}{2}, \tag{7}$$

where the equality holds iff G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

(h) (Berman and Zhang [7]) If G is a connected graph, then

$$\rho(G) \leq max\{\sqrt{d_i d_j} : 1 \leq i, j \leq n, v_i v_j \in E\}, \tag{8}$$

where the equality holds iff G is a regular graph or bipartite semiregular graph.

(i) (Favaron and et al. [8])

(I) For any graph G without isolated vertices

$$\rho(G) \leq max\{m_i : v_i \in V\}. \tag{9}$$

(II) For any graph G

$$\rho(G) \leq max\{\sqrt{d_i m_i} : v_i \in V\}. \tag{10}$$

(j) (Das and Kumar [9]) If G is a simple connected graph, then

$$\rho(G) \leq max\{\sqrt{m_i m_j} : v_i v_j \in E\}, \tag{11}$$

where the equality holds iff G is either a graph with all the vertices of equal average degree or a bipartite graph with vertices of same set having equal average degree.

(k) (Das and Kumar [9]) Let G be a simple connected graph,then

$$\rho(G) \leq \sqrt{2e - (n-1)\delta + (\delta - 1)\Delta}, \tag{12}$$

where the equality holds iff G is a regular graph or a star graph.

(l) (Shu and Wu [10]) Let G be a simple connected graph with degree sequence $\Delta = d_1 \geq d_2 \geq ... \geq d_n = \delta$, then

$$\rho(G) \leq \min_{1 \leq i \leq n} \{\frac{d_i - 1 + \sqrt{(d_i+1)^2 + 4(i-1)(d_1 - d_i)}}{2}\}, \tag{13}$$

where $1 \leq i \leq n$. If i=1, the equality holds iff G is a regular graph. If $2 \leq i \leq n$, the equality holds iff G is either a regular graph or a bidegreed graph in which $d_1 = d_2 = ... = d_{i-1} = n-1$ and $d_i = ... = d_n = \delta$.

(m) (Shu and Wu [10]) Let G be a simple connected graph with second largest degree Δ'. If there are p vertices with degree Δ, then

$$\rho(G) \leq \frac{\Delta' - 1 + \sqrt{(\Delta'+1)^2 + 4p(\Delta - \Delta')}}{2}. \tag{14}$$

The equality holds if and only if G is a Δ-regular graph, or $G \cong K_p + H$, where H is a $(\Delta' - p)$-regular graph with $n - p$ vertices (The join $K_p + H$ of disjoint graph K_p and H is the graph obtained from $K_p \cup H$ by joining each vertex of K_p to each vertex of H).

(n) (Stevanovic [11]) If G is connected and non regular, then

$$\rho(G) < \Delta - \frac{1}{2n(n\Delta - 1)\Delta^2}. \tag{15}$$

The upper bounds from (1) to (4) are applied to some particular graphs. Hong [12] has pointed out that the upper bound in (6) is an improvement on the upper bound (5). Das and Kumar [9] has pointed out that the upper bound (10) is better than the upper bound (6).

3. Some lower bounds for spectral radius

We now list some known lower bounds for the spectral radius $\rho(G)$.

(o) (Collatz and Sinogowitz [1]) If G is a connected graph of order n, then

$$\rho(G) \geq \lambda_1(P_n) = 2cos(\pi/(n+1)). \tag{16}$$

The lower bound occurs only G is the path P_n.

(p) (Hong [2]) If G is a connected unicyclic graph, then

$$\rho(G) \geq \lambda_1(C_n) = 2, \tag{17}$$

where C_n denotes the cycle on n vertices. The lower bound occurs only G is the cycle C_n.

(q) (Favaron and et al. [8]) For any simple graph,

$$\rho(G) \geq \sqrt{\Delta}. \tag{18}$$

The lower bound occurs only G is the star $K_{1,\Delta}$.

(r) (Das and Kumar [9]) If G is a simple graph with at least one edge and d_1 is the highest degree of G. Then

$$\rho(G) \geq \sqrt{\frac{(d_1 + d_j - 1) + \sqrt{(d_1 + d_j - 1)^2 - 4(d_1 - 1)(d_j - 1) + 4c_{1j}^2 + 8c_{1j}\sqrt{d_1}}}{2}}, \tag{19}$$

where $d_j = max\{d_k : v_1 v_k \in E\}$ and c_{1j} is the cardinality of the common neighbors of v_1 and v_j.

4. Main results

Lemma 1 [13]. Let G be a simple connected graph with n vertices and A its adjacency matrix. Let P be any polynomial and $S_i(P(A))$ is the row sums of $P(A)$ corresponding to vertex v_i. Then

$$minS_i(P(A)) \leq P(\rho(A)) \leq maxS_i(P(A)).$$

The equality holds if and only if the row sums of $P(A)$ are all equal.
Lemma 2. For any simple graph G with n vertices, there exists at least a vertex $v_i \in V$, such that $d_i \geq m_i$, where d_i and m_i are the degree of the vertex v_i and the average of degrees of the vertices adjacent to v_i, respectively.
Proof.

$$d_i m_i = \sum_{v_j \sim v_i} d_j \Rightarrow \sum_{i=1}^{n} d_i m_i = \sum_{i=1}^{n} \sum_{v_j \sim v_i} d_j = \sum_{i=1}^{n} d_i^2.$$

Assume for all $i \in \{1, 2, ..., n\}$, we have $d_i < m_i$, then

$$\sum_{i=1}^{n} d_i^2 < \sum_{i=1}^{n} \sum_{v_j \sim v_i} d_j = \sum_{i=1}^{n} d_i^2.$$

This is a contradiction.

□

Theorem 3. Let G be a simple connected graph with n vertices and $\rho(G)$ be the spectral radius of G. The degree of the vertex v_i and the average of degrees of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. Then

$$\min_{1\leq i\leq n}\{\frac{-x+\sqrt{x^2+4d_ix+4d_im_i}}{2}\} \leq \rho(G) \leq \max_{1\leq i\leq n}\{\frac{-x+\sqrt{x^2+4d_ix+4d_im_i}}{2}\}, \quad (20)$$

where $x \in \mathcal{D} = \{x | x \geq \max_{1\leq i\leq n}\{-2d_i + 2\sqrt{d_i(d_i - m_i)}\}, d_i \geq m_i\}$.

If G is a regular graph, or G is a bidegreed graph whose vertices of same degree have equal average of degrees, that is for all $d_i = d_j, i, j \in \{1, 2, ..., n\}$, have $m_i = m_j$, then the equality holds.
Proof. Note that $S_i(A^k)$ is exactly the number of walks of length k in G which begin from v_i. In particular

$$S_i(A) = d_i,$$

and

$$S_i(A^2) = \sum_{v_j \sim v_i} d_j = d_i m_i. \quad (A)$$

For $\forall x \in R$, we have

$$S_i(A^2) + xS_i(A) = d_i m_i + x d_i.$$

Let

$$\max_{1\le i\le n}\{S_i(A^2) + xS_i(A)\} = \max_{1\le i\le n}\{d_i m_i + x d_i\} = M;$$

and

$$\min_{1\le i\le n}\{S_i(A^2) + xS_i(A)\} = \min_{1\le i\le n}\{d_i m_i + x d_i\} = m.$$

By Lemma 1

$$m \le \rho(G)^2 + x\rho(G) \le M.$$

Solving the quadratic inequality, we obtain

$$\frac{-x+\sqrt{x^2+4m}}{2} \le \rho(G) \le \frac{-x+\sqrt{x^2+4M}}{2}, \qquad (B)$$

or

$$\frac{-x-\sqrt{x^2+4M}}{2} \le \rho(G) \le \frac{-x-\sqrt{x^2+4m}}{2}. \qquad (C)$$

For (B), we get

$$\min_{1\le i\le n}\{\frac{-x+\sqrt{x^2+4d_i x+4d_i m_i}}{2}\} \le \rho(G) \le \max_{1\le i\le n}\{\frac{-x+\sqrt{x^2+4d_i x+4d_i m_i}}{2}\}.$$

Since $x^2 + 4d_i x + 4d_i m_i \ge 0, i \in \{1, 2, ..., n\}$. For $d_i \ge m_i$, we have

$$x \in (-\infty, (-2d_i - 2\sqrt{d_i(d_i - m_i)})] \cup [(-2d_i + 2\sqrt{d_i(d_i - m_i)}), +\infty).$$

Since

$$\lim_{x\to-\infty} \frac{-x+\sqrt{x^2+4d_i x+4d_i m_i}}{2} = +\infty,$$

when $x \in (-\infty, (-2d_i - 2\sqrt{d_i(d_i - m_i)})]$, the inequalities (20) is meaningless, thus we consider $x \in \mathcal{D}$.

(i) If G is a regular graph, then $m_i = d_i, i = 1, 2, ..., n$. For $x \in \mathcal{D}$,

$$\max_{1\le i\le n}\{\frac{-x+\sqrt{x^2+4d_i x+4d_i m_i}}{2}\} = \max_{1\le i\le n} d_i = \Delta,$$

and

$$\min_{1\le i\le n}\{\frac{-x+\sqrt{x^2+4d_i x+4d_i m_i}}{2}\} = \min_{1\le i\le n} d_i = \Delta,$$

the equality holds.
(ii) Let G be a bidegreed graph with vertices degree Δ and δ. For all $d_i = d_j, i, j \in \{1, 2, ..., n\}$, we have $m_i = m_j$. Then

$$\{\frac{-x+\sqrt{x^2+4d_ix+4d_im_i}}{2}, i = 1, 2, ..., n\}$$

$$= \{\frac{-x+\sqrt{x^2+4\Delta x+4\Delta m_\Delta}}{2}, \frac{-x+\sqrt{x^2+4\delta x+4\delta m_\delta}}{2}\},$$

where m_Δ and m_δ are denoted by the average of degrees of the vertices adjacent to $d_i = \Delta$ and $d_j = \delta(i, j \in \{1, 2, ..., n\})$, respectively. Let

$$\frac{-x+\sqrt{x^2+4\Delta x+4\Delta m_\Delta}}{2} = \frac{-x+\sqrt{x^2+4\delta x+4\delta m_\delta}}{2}.$$

Solving this equation we have

$$x_0 = -\frac{\Delta m_\Delta - \delta m_\delta}{\Delta - \delta}.$$

For this x_0, the row sums of $P(A)$ are all equal. By Lemma 1

$$\begin{aligned} \rho(G) &= \max_{1\le i\le n}\{\frac{-x+\sqrt{x^2+4d_ix+4d_im_i}}{2}\} \\ &= \min_{1\le i\le n}\{\frac{-x+\sqrt{x^2+4d_ix+4d_im_i}}{2}\} \\ &= f_\Delta(-\frac{\Delta m_\Delta - \delta m_\delta}{\Delta - \delta}) = f_\delta(-\frac{\Delta m_\Delta - \delta m_\delta}{\Delta - \delta}) \\ &= \frac{\Delta m_\Delta - \delta m_\delta + \sqrt{(\Delta m_\Delta - \delta m_\delta)^2 - 4\Delta\delta(\Delta-\delta)(m_\Delta - m_\delta)}}{2(\Delta-\delta)}, \end{aligned} \tag{21}$$

where we denote $f_\Delta(x) = f_i(x)$ when $d_i = \Delta$ and $f_\delta(x) = f_j(x)$ when $d_j = \delta$. $i, j \in \{1, 2, ..., n\}$.

□

Remark. We have similar inequalities about (C). For $x \ll 0$, we can use the inequalities to obtain some similar results to the Corollary 11 and the Corollary 12.
Corollary 4 [8].

$$\rho(G) \le \max_{1\le i\le n}\sqrt{d_im_i}.$$

Proof. Since $0 \in \mathcal{D}$, the result follows by x=0 and Theorem 3.

□

Corollary 5.

$$\rho(G) \geq \min_{1\leq i\leq n} \sqrt{d_i m_i}. \tag{22}$$

Proof. The result follows by x=0 and Theorem 3.

□

Corollary 6.

$$\delta \leq \rho(G) \leq \Delta. \tag{23}$$

The equality holds if and only if G is a regular graph.
Proof. Let

$$f_i(x) = \frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2}, i = 1, 2, ..., n.$$

Differentiating $f_i(x)$ we obtain

$$f_i'(x) = \frac{1}{2}(-1 + \frac{x + 2d_i}{\sqrt{x^2 + 4d_i x + 4d_i m_i}}).$$

(i) If $d_i \geq m_i$, then $f_i'(x) \geq 0$, $f_i(x)$ is an increasing function of x, and

$$x \geq -2d_i + 2\sqrt{d_i^2 - d_i m_i} \quad \text{or} \quad x \leq -2d_i - 2\sqrt{d_i^2 - d_i m_i}.$$

(ii) If $d_i < m_i$, then $f_i'(x) < 0$, $f_i(x)$ is a decreasing function of x, and $x \in R$.
Either $d_i \geq m_i$ or $d_i < m_i$

$$\lim_{x\to+\infty} f_i(x) = \lim_{x\to+\infty} \frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2} = d_i.$$

If we denote that $d_1 = \Delta$ and $d_n = \delta$, then
(i) $d_1 \geq m_1$, $f_1(x)$ is an increasing function of x and

$$\lim_{x\to+\infty} f_1(x) = d_1 = \Delta.$$

(ii) $d_n \leq m_n$, $f_n(x)$ is a decreasing function of x and

$$\lim_{x\to+\infty} f_n(x) = d_n = \delta.$$

So

$$\delta = \min_{1\leq i\leq n} d_i \leq \rho(G) \leq \max_{1\leq i\leq n} d_i = \Delta.$$

Clearly, The equality holds if and only if G is a regular graph.

□

Corollary 7. Let

$$R = \inf_{x \in \mathcal{D}} \{ \max_{1 \le i \le n} \{ \frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2} \} \},$$

and

$$r = \sup_{x \in \mathcal{D}} \{ \min_{1 \le i \le n} \{ \frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2} \} \}.$$

Then

$$r \le \rho(G) \le R. \tag{24}$$

Proof. Since for all $i \in \{1, 2, ..., n\}$, $f_i(x)(x \in \mathcal{D})$ are continuous monotone functions, the function $\max_{1 \le i \le n} \{f_i(x)\}$ and the function $\min_{1 \le i \le n} \{f_i(x)\}$ are also continuous functions. $\exists x_1 \in \mathcal{D}$, such that

$$\max_{1 \le i \le n} \{f_i(x_1)\} \le \max_{1 \le i \le n} \{f_i(0)\} = \max_{1 \le i \le n} \sqrt{d_i m_i},$$

and $\exists x_2 \in \mathcal{D}$, such that

$$\min_{1 \le i \le n} \{f_i(x_2)\} \ge \min_{1 \le i \le n} \{f_i(0)\} = \min_{1 \le i \le n} \sqrt{d_i m_i}.$$

Since

$$\min_{1 \le i \le n} \{f_i(x)\} \le \max_{1 \le i \le n} \{f_i(x_1)\} \le \max_{1 \le i \le n} \sqrt{d_i m_i},$$

$\max_{1 \le i \le n} \sqrt{d_i m_i}$ is a upper bound of $\min_{1 \le i \le n} \{f_i(x)\}$, by the existence theorem of supremum, the supremum of $\min_{1 \le i \le n} \{f_i(x)\}$ is existent. Similarly, the infimum of $\max_{1 \le i \le n} \{f_i(x)\}$ is existent. Let

$$r = \sup_{x \in \mathcal{D}} \{ \min_{1 \le i \le n} \{f_i(x)\} \} = \sup_{x \in \mathcal{D}} \{ \min_{1 \le i \le n} \{ \frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2} \} \},$$

and

$$R = \inf_{x \in \mathcal{D}} \{ \max_{1 \le i \le n} \{f_i(x)\} \} = \inf_{x \in \mathcal{D}} \{ \min_{1 \le i \le n} \{ \frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2} \} \},$$

clearly, $r \le \rho(G) \le R$.

□

Clearly, the upper and lower bound (24) is better than the upper and lower bound (20), respectively. Since

$$\min_{1\le i\le n}\sqrt{d_i m_i}\le r\le R\le \max_{1\le i\le n}\sqrt{d_i m_i},$$

the upper and lower bound (24) is also better than the upper bound (10) and the lower bound (22), respectively.

Corollary 8. If G is a nonregular graph, then

$$\rho(G)\le \max_{1\le j<i\le n}\{\frac{d_i m_i - d_j m_j + \sqrt{(d_i m_i - d_j m_j)^2 - 4d_i d_j(d_i - d_j)(m_i - m_j)}}{2(d_i - d_j)}\}. \quad (25)$$

If G is a bidegreed graph whose vertices of same degree have equal average of degrees, then the equality holds.

Proof. If G is a nonregular graph , then $\exists i, j \in \{1, 2, ..., n\}$, such that $d_i \neq d_j$. We solve the simultaneous equations for x

$$\begin{cases} f_i(x) = 0 \\ f_j(x) = 0, \end{cases}$$

we obtain

$$x = -\frac{d_i m_i - d_j m_j}{d_i - d_j}.$$

Clearly

$$\min_{1\le j<i\le n} f(-\frac{d_i m_i - d_j m_j}{d_i - d_j}) \le \rho(G) \le \max_{1\le j<i\le n} f(-\frac{d_i m_i - d_j m_j}{d_i - d_j}),$$

where $d_i \neq d_j$.

Since

$$f(-\frac{d_i m_i - d_j m_j}{d_i - d_j}) = \frac{d_i m_i - d_j m_j + \sqrt{(d_i m_i - d_j m_j)^2 - 4d_i d_j(d_i - d_j)(m_i - m_j)}}{2(d_i - d_j)}$$

The result follows by Theorem 3.

If G is a bidegreed graph whose vertices of same degree have equal average of degrees, let $d_i = \Delta$ and $d_j = \delta$, $i, j \in \{1, 2, ..., n\}$. Then we have that

$$\max_{1\le j<i\le n}\{\frac{d_i m_i - d_j m_j + \sqrt{(d_i m_i - d_j m_j)^2 - 4d_i d_j(d_i - d_j)(m_i - m_j)}}{2(d_i - d_j)}\}$$

$$= \frac{\Delta m_\Delta - \delta m_\delta + \sqrt{(\Delta m_\Delta - \delta m_\delta)^2 - 4\Delta\delta(\Delta - \delta)(m_\Delta - m_\delta)}}{2(\Delta - \delta)}.$$

By (21), the equality holds.

□

Corollary 9. If G is a nonregular graph, then

$$\rho(G) \geq \min_{1\leq j<i\leq n} \{\frac{d_i m_i - d_j m_j + \sqrt{(d_i m_i - d_j m_j)^2 - 4d_i d_j (d_i - d_j)(m_i - m_j)}}{2(d_i - d_j)}\}. \quad (26)$$

If G is a bidegreed graph whose vertices of same degree have equal average of degrees, then the equality holds.

Proof. Similar to the proof of Corollary 8.

Theorem 10. Let G be a simple graph with n vertices and e edges. Let Δ be the maximum degree of vertices of G and $\rho(G)$ be the spectral radius of G. Then

$$\rho(G) \geq \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 - 4(\Delta n - 2e)}}{2}. \quad (27)$$

Proof. We use the symbol $v_j \bar{\sim} v_i$ to denote two vertices v_j and v_i are not adjacent. From (A) we have that

$$\begin{aligned} S_i(A^2) &= \sum_{v_j \sim v_i} d_j = d_i m_i \\ &= 2e - d_i - \sum_{v_j \bar{\sim} v_i} d_j \\ &\geq 2e - d_i - (n - d_i - 1)\Delta \\ &= 2e + (\Delta - 1)d_i - \Delta(n-1). \end{aligned} \quad (D)$$

Hence

$$S_i(A^2) - (\Delta - 1)S_i(A) \geq 2e - \Delta(n-1).$$

As this holds for every vertex $v_i \in V$. Lemma 1 implies that

$$\rho(G)^2 - (\Delta - 1)\rho(G) \geq 2e - \Delta(n-1).$$

Solving the quadratic inequality, we obtain

$$\rho(G) \geq \frac{\Delta - 1 + \sqrt{(\Delta+1)^2 - 4(\Delta n - 2e)}}{2}.$$

In order to the equality to hold, all inequalities in the above argument must be equalities, from (D) we have that

$$\sum_{v_j \bar{\sim} v_i} d_j = (n - d_i - 1)\Delta.$$

It implies that G is a regular graph.

Conversely, if G is a regular graph the equality is satisfied.

□

For (20), by assigning different values to the parameter x, we can obtain some existing results on spectral radius.

Corollary 11. For $v_i \in V$ of satisfy the condition $d_i \geq m_i$, if $m_i \geq 2 - \frac{1}{d_i}$, then

$$\rho(G) \leq \max_{1 \leq i \leq n} \{1 + \sqrt{1 + d_i m_i - 2d_i}\}, \tag{28}$$

Proof. Let $x = -2$,

$$\frac{-x + \sqrt{x^2 + 4d_i x + 4d_i m_i}}{2} = 1 + \sqrt{1 + d_i m_i - 2d_i},$$

when $d_i \geq m_i$, $m_i \geq 2 - \frac{1}{d_i}$, $1 + d_i m_i - 2d_i \geq 0$, $-2 \in \mathcal{D}$, the result follows by $x = -2$ and theorem 2.

□

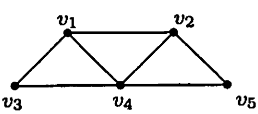
Corollary 12. For $v_i \in V$ of satisfy the condition $d_i \geq m_i$, if $m_i \geq \frac{3}{2} - \frac{9}{16d_i}$, then

$$\rho(G) \geq \min_{1 \leq i \leq n} \frac{3 + \sqrt{9 - 24d_i + 16d_i m_i}}{4}. \tag{29}$$

Proof. Similarly, x = -1.5 $\in \mathcal{D}$, the result follows by x = -1.5 and theorem 2.

□

Example. Fig. 1.



G

*Fig.*1.

$\rho(G) \approx 2.9354$ [14, p.273, fig.1.16].

$$f_1(x) = f_2(x) = \frac{-x + \sqrt{x^2 + 12x + 36}}{2} = \begin{cases} 3 & (x \geq -6) \\ -x - 3 & (x < -6) \end{cases};$$

$$f_3(x) = f_5(x) = \frac{-x + \sqrt{x^2 + 8x + 28}}{2}; f_4(x) = \frac{-x + \sqrt{x^2 + 16x + 40}}{2}.$$

$\mathcal{D} = [-8 + 2\sqrt{6}, +\infty)$.

$f_i(x)$	$f_1(x), f_2(x)$	$f_3(x), f_5(x)$	$f_4(x)$
x	$x \in R$	$x \in R$	$x \geq -8+2\sqrt{6}$ or $x \leq -8-2\sqrt{6}$
$f_i'(x)$	≤ 0	<0	>0 $(x \geq -8+2\sqrt{6})$
$lim_{x\to+\infty} f_i(x)$	$d_1 = d_2 = 3$	$d_3 = d_5 = 2$	$d_4 = 4$

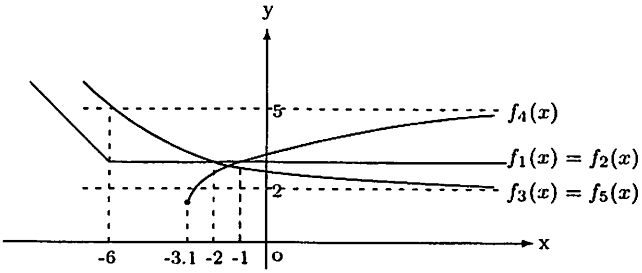


Fig.2.

In Fig.2, when $-2 \leq x \leq -1$,

$$R = \inf_{x\in\mathcal{D}}\{max\{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x)\}\} = 3.$$

Hence

$$\rho(G) \leq 3.$$

Let

$$f_3(x) = f_4(x),$$

we obtain $x_0 = -1.5$,

$$\begin{aligned} r &= \sup_{x\in\mathcal{D}}\{min\{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x)\}\} \\ &= f_3(-1.5) = f_4(-1.5) \approx 2.886. \end{aligned}$$

Hence

$$\rho(G) \geq 2.886.$$

Since $x = -2 \in \mathcal{D}$, by (28), we have

$$\rho(G) \leq max\{1+\sqrt{1+9-6}, 1+\sqrt{1+7-4}, 1+\sqrt{1+10-8}\} = 3,$$

and $x = -1.5 \in \mathcal{D}$, by (29), we have

$$\rho(G) \geq min\{\frac{3+\sqrt{9-72+144}}{4}, \frac{3+\sqrt{9-48+112}}{4}, \frac{3+\sqrt{9-96+160}}{4}\} \approx 2.886.$$

The various mentioned upper bounds for the graph shown in Fig.1 give the following results:

(7),(24),(25),(28)	(8)	(9)	(10),(12)	(11)	(13),(14)	(15)	(23)
3	3.464	3.5	3.162	3.24	3.236	3.9997	4

The various mentioned lower bounds for the graph shown in Fig.1 give the following results:

(18)	(19)	(24),(26),(29)	(22)	(23),(27)
2	2.622	2.886	2.646	2

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