The critical group of $C_4 \times C_n$ *

Jian Wang, Yong-Liang Pan†
Department of Mathematics, University of Science and Technology of China
Hefei, Auhui 230026, The People's Republic of China

ABSTRACT

In this paper, the critical group structure of the Cartesian product graph $C_4 \times C_n$ is determined, where $n \geq 3$.

Keywords Graph; Laplacian matrix; Critical group; Invariant factor; Smith normal form; Tree number.

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1 Introduction

Let $G = (V, E)$ be a finite connected graph without self-loops, but with multiple edges allowed. Then the Laplacian matrix of G is the $|V| \times |V|$ matrix defined by

$$L(G)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ -a_{uv}, & \text{if } u \neq v, \end{cases} \tag{1.1}$$

where a_{uv} is the number of the edges joining u and v, and $d(u)$ is the degree of u.

Regarding $L(G)$ as representing an abelian group homomorphism: $Z^{|V|} \to Z^{|V|}$, its cokernel $\text{coker}(L(G)) = Z^{|V|}/\text{im}(L(G))$ is an abelian group, determined by the generators $x_1, \cdots, x_{|V|}$ and relation $L(G)X = 0$, where $x_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in Z^{|V|}$, whose unique nonzero 1 is in position i, and $X = (x_1, \cdots, x_{|V|})^t$. Note that the same symbol x_i denotes both an element of the group $\text{coker}(L(G))$ and a basis element of the free abelian group $Z^{|V|}$.

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†Corresponding author. Email: ylpan@ustc.edu.cn

The finitely generated abelian group coker$(L(G))$ can be described in terms of the Smith normal form (or simply SNF) of $L(G)$. Two integral matrices A and B of order n are equivalent (written by $A \sim B$) if there are unimodular matrices P and Q such that $B = PAQ$. Equivalently, B is obtainable from A by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by -1, (3) the addition of any integer times of one row (resp. column) to another row (resp. column). It is easy to see that $A \sim B$ implies that coker$(A) \cong$ coker(B). Given any $|V| \times |V|$ unimodular matrices P and Q and any integral matrix A with $PAQ = \mathrm{diag}(a_1, \cdots, a_{|V|})$, it is easy to see that $Z^{|V|}/\mathrm{im}(A) \cong (Z/a_1 Z) \oplus \cdots \oplus (Z/a_{|V|}Z)$. Here, the rank of $L(G)$ is $|V| - 1$, with kernel generated by the transpose of the vector $(1, \cdots, 1)$. Thus we can assume the SNF of $L(G)$ is $\mathrm{diag}(t_1, \cdots, t_{|V|-1}, 0)$, and it induces an isomorphism

$$\mathrm{coker}(L(G)) \cong K(G) \oplus Z. \tag{1.2}$$

where $K(G) = (Z/t_1 Z) \oplus (Z/t_2 Z) \oplus \cdots \oplus (Z/t_{|V|-1}Z)$.

In [1] and [5 (Chapter 14)], the finite abelian group $K(G)$ is defined to be the critical group of G. Its invariant factors $t_1, t_2, \cdots t_{|V|-1}$ can be computed in the following way: for $1 \le i < |V|$, $t_i = \Delta_i/\Delta_{i-1}$ where $\Delta_0 = 1$ and Δ_i is the i-th determinantal divisor of $L(G)$, defined as the greatest common divisor of all $i \times i$ minor subdeterminants of $L(G)$. From the well known Kirchhoff's Matrix-Tree Theorem [7, Theorem 13.2.1] we know that $t_1 \cdots t_{|V|-1}$ equals the number κ of spanning trees of G. It follows that the invariant factors of $K(G)$ can be used to distinguish pairs of non-isomorphic graphs which have the same κ, and so there is considerable interest in their properties. If G is a simple connected graph, the invariant factor t_1 of $K(G)$ must be equal to 1, however, most of them are not easy to be determined.

Compared to the number of the results on the spanning tree number κ, there are relatively few results describing the critical group structure of $K(G)$ in terms of the structure of G. There are also very few interesting infinite family of graphs for which the group structure has been complete determined (see [2, 3, 4, 6, 7, 8], and the references therein). In this paper, we describe the critical group structure of Cartesian product graph $C_4 \times C_n$ $(n \ge 3)$ completely, where C_n is the cycle on n vertices.

Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their Cartesian product is the graph $G_1 \times G_2$ whose vertex set is the cartesian product $V_1 \times V_2$. Suppose $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$. Then (u_1, v_1) is adjacent to (u_2, v_2) if and only if one of the following conditions satisfied: (i) $u_1 = u_2$ and $(v_1, v_2) \in E_2$, or (ii) $(u_1, u_2) \in E_1$ and $v_1 = v_2$. One may view $G_1 \times G_2$ as the graph obtained from G_2 by replacing each of its vertices with a copy of G_1, and each of its edges with $|V_1|$ edges joining corresponding vertices of G_1 in the two copies. From the definition of the Cartesian product of two graphs, it is easy to see that there are n layers of $C_4 \times C_n$, each of which is a copy of C_4. Let Z_n denote Z/nZ, then for $i \in Z_n$, $j \in Z_4$, let v_j^i denote the j-th vertex in the i-th layer of $C_4 \times C_n$. The vertex v_j^i is adjacent to vertices v_j^l and v_k^i, where $l = i \pm 1$, $k = j \pm 1($ mod 4) (see Fig. 1).

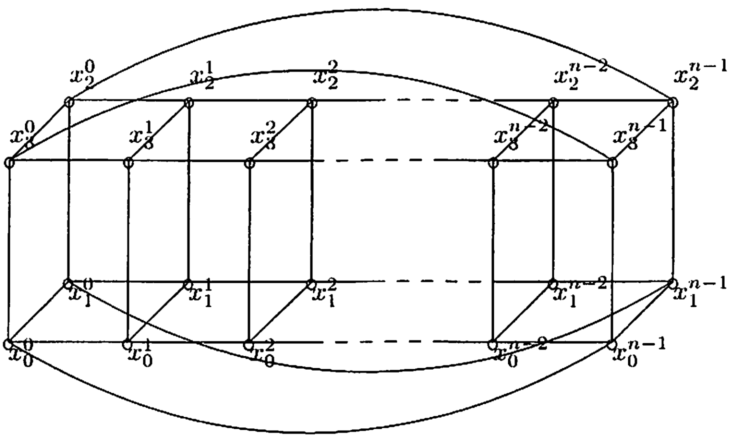

Fig. 1. Graph $C_4 \times C_n$.

2 Preliminaries

Let m be a positive integer. Denote $\alpha(m) = \frac{m+2+\sqrt{m^2+4m}}{2}$, $\beta(m) = \frac{m+2-\sqrt{m^2+4m}}{2}$, $u_p(m) = \frac{1}{\alpha(m)-\beta(m)}(\alpha^p(m) - \beta^p(m))$, $v_p(m) = \alpha^p(m) + \beta^p(m)$, for $p \in R$.

By the following proposition 2.1, it is easy to see that for every integer $p \geq 0$, $u_p(m)$ and $v_p(m)$ are integral. The propositions 2.1 and 2.2 can be easily proved by induction.

Proposition 2.1. *If p is integral, then*

$$\begin{cases} u_p(m) = (m+2)u_{p-1}(m) - u_{p-2}(m), \\ v_p(m) = (m+2)v_{p-1}(m) - v_{p-2}(m), \end{cases} \tag{2.1}$$

with initial values

$$\begin{cases} u_0(m) = 0, & u_1(m) = 1, \\ v_0(m) = 2, & v_1(m) = m+2. \end{cases} \tag{2.2}$$

And if $q \geq 0$ is another integer, then $u_{pq}(m) = v_{p(q-1)}(m)u_p(m)+u_{p(q-2)}(m)$.

Proposition 2.2. *If p is a nonnegative integer, then*

- $u_p(m) \equiv p \pmod{m}, \quad v_p(m) \equiv 2 \pmod{m};$ (2.3)
- $v_{2p}(m) = m(m+4)u_p^2(m) + 2;$ (2.4)
- $u_{pq}(m) = \begin{cases} V_q(m)u_p(m), & \text{if } q \text{ is even,} \\ V_q'(m)u_p(m), & \text{if } q \text{ is odd,} \end{cases}$ (2.5)

where

$$V_q(m) = \sum_{0<2i\leq q} v_{p(q+1-2i)}(m), \quad V_q'(m) = \left(\sum_{0<2i\leq q+1} v_{p(q+1-2i)}(m) \right) - 1. \tag{2.6}$$

If n is a positive integer of the form $p_1^{t_1} \cdots p_k^{t_k}$ where the p_i's are distinct primes, then let $T_{p_i}(n)$ denote t_i. Let $e_n = u_n(2)$, $f_n = u_n(4)$.

Proposition 2.3. *Let $T_2(n) = t_2$, $T_3(n) = t_3$, for $n \geq 2$. Then we have* $T_2(e_n) = \begin{cases} 0, & \text{if } t_2 = 0, \\ t_2+1, & \text{if } t_2 > 0; \end{cases}$ $T_2(f_n) = t_2$; $T_3(e_n) = t_3$; *and* $T_3(f_n) = \begin{cases} 0, & \text{if } t_2 = 0, \\ t_3+1, & \text{if } t_2 > 0. \end{cases}$

Proof. Let $n = 2^{t_2}q$, where q is odd.

By (2.5), $e_n = V_q'(2)e_{2^{t_2}}$ and $f_n = V_q'(4)f_{2^{t_2}}$. By (2.3), $v_p(2)$ and $v_p(4)$ are even for every p and then from (2.6) we have that $V_q'(2)$ and $V_q'(4)$ are odd. Thus $T_2(e_n) = T_2(e_{2^{t_2}})$ and $T_2(f_n) = T_2(f_{2^{t_2}})$. If $t_2 = 0$, then $T_2(e_{2^{t_2}}) = T_2(e_1) = 0$ and $T_2(f_{2^{t_2}}) = T_2(f_1) = 0$. Now we prove by induction on $t_2 > 0$ that $T_2(e_{2^{t_2}}) = t_2+1$ and $T_2(f_{2^{t_2}}) = t_2$. This is valid if $t_2 = 1$. Since from (2.4), (2.5) and (2.6) it follows that $e_{2^{t_2}} = v_{2^{t_2-1}}(2)e_{2^{t_2-1}} = (12e_{2^{t_2-2}}^2 + 2)e_{2^{t_2-1}}$ and $f_{2^{t_2}} = v_{2^{t_2-1}}(4)f_{2^{t_2-1}} = (32f_{2^{t_2-2}}^2 + 2)f_{2^{t_2-1}}$, then by the induction hypothesis we have that $T_2(e_{2^{t_2}}) = T_2(12e_{2^{t_2-2}}^2 + 2) + T_2(e_{2^{t_2-1}}) = 1+t_2$ and $T_2(f_{2^{t_2}}) = T_2(32f_{2^{t_2-2}}^2+2)+T_2(f_{2^{t_2-1}}) = 1+t_2-1 = t_2$. Thus $T_2(e_n) = t_2+1$ and $T_2(f_n) = t_2$.

Let $n = 3^{t_3}\gamma$, where $3 \nmid \gamma$.

By (2.5), $e_n = \begin{cases} V_\gamma'(2)e_{3^{t_3}}, & \text{if } 2 \nmid \gamma, \\ V_\gamma(2)e_{3^{t_3}}, & \text{if } 2 \mid \gamma. \end{cases}$ Note that $v_n(2) = 4v_{n-1}(2) - v_{n-2}(2) \equiv v_{n-1}(2)-v_{n-2}(2) \equiv -v_{n-3} \pmod 3$, $v_0(2) = 2$, $v_1(2) = 4 \equiv 1 \pmod 3$, $v_2(2) = 14 \equiv 2 \pmod 3$, $v_3(2) = 52 \equiv 1 \pmod 3$, $v_4(2) = 194 \equiv 2 \pmod 3$, $v_5(2) = 724 \equiv 1 \pmod 3$. Then it is not difficult to see that if $2 \nmid \gamma$ then $v_{3^{t_3}(\gamma+1-2i)}(2) \equiv 2 \pmod 3$; if $2 \mid \gamma$ then $v_{3^{t_3}(\gamma+1-2i)}(2) \equiv 1 \pmod 3$. Hence, if $2 \nmid \gamma$, then $V_\gamma'(2) \equiv 2 \times \frac{\gamma+1}{2} - 1 = \gamma \pmod 3$; if $2 \mid \gamma$, then $V_\gamma(2) \equiv \frac{\gamma}{2} \pmod 3$. It follows that neither $V_\gamma(2)$ (γ is even) nor $V_\gamma'(2)$ (γ is odd) contains the divisor 3, and hence $T_3(e_n) = T_3(e_{3^{t_3}})$. Now we prove by induction on t_3 that $T_3(e_{3^{t_3}}) = t_3$. It is valid if $t_3 = 0$, or 1. Since from (2.4) and (2.5) we have that $e_{3^{t_3}} = (v_{3^{t_3-1}\cdot 2}(2) + v_0(2) - 1)e_{3^{t_3-1}} = (12e^2_{3^{t_3-1}} + 3)e_{3^{t_3-1}}$. So by the induction hypothesis we have $T_3(e_{3^{t_3}}) = T_3(12e^2_{3^{t_3-1}}+3)+T_3(e_{3^{t_3-1}}) = 1+t_3-1 = t_3$. Thus $T_3(e_n) = t_3$.

If $t_2 = 0$, namely n is odd, then we have $f_n = 6f_{n-1} - f_{n-2} \equiv -f_{n-2} \equiv \cdots \equiv (-1)^{\frac{n-1}{2}} f_1 \pmod 3$. Note that $f_1 = 1$, so $T_3(f_n) = 0$.

If $t_2 > 0$, namely n is even, then we can write $n = 3^{t_3} \cdot 2\epsilon$, where $3 \nmid \epsilon$. By (2.5), $f_n = \begin{cases} V'_\epsilon(4) f_{2\cdot 3^{t_3}}, & \text{if} 2 \nmid \epsilon, \\ V_\epsilon(4) f_{2\cdot 3^{t_3}}, & \text{if} 2 \mid \epsilon. \end{cases}$ By (2.1), $v_n(4) = 6v_{n-1}(4) - v_{n-2}(4) \equiv -v_{n-2}(4) \equiv \cdots \equiv (-1)^{\frac{n}{2}} v_0(4) = (-1)^{\frac{n}{2}} 2 \pmod 3$. Then from (2.6) we have that if $2 \nmid \epsilon, V'_\epsilon(4) \equiv 2 \times \frac{\epsilon+1}{2} - 1 = \epsilon \pmod 3$; if $2 \mid \epsilon$, then $V_\epsilon(4) \equiv (-2) \times \frac{\epsilon}{2} = -\epsilon \pmod 3$. Thus neither $V'_\epsilon(4)$ (ϵ is odd) nor $V_\epsilon(4)$ (ϵ is even) is divisible by 3. So $T_3(f_n) = T_3(f_{2\cdot 3^{t_3}})$. Now we prove by induction on t_3 that $T_3(f_{2\cdot 3^{t_3}}) = t_3 + 1$. If $t_3 = 0$ or 1, we have $f_2 = 6$ and $f_6 = 6930$ respectively, so it is valid. Since from (2.4), (2.5) and (2.6) it follows that $f_{2\cdot 3^{t_3}} = (v_{2\cdot 3^{t_3-1}\cdot 2}(4) + v_0(4) - 1) f_{2\cdot 3^{t_3-1}} = (32 f^2_{2\cdot 3^{t_3-1}} + 3) f_{2\cdot 3^{t_3-1}}$, then by the induction hypothesis we have $T_3(f_{2\cdot 3^{t_3}}) = T_3(32 f^2_{2\cdot 3^{t_3-1}} + 3) + T_3(f_{2\cdot 3^{t_3-1}}) = 1 + t_3$. □

3 System of relations for the cokernel of the Laplacian on $C_4 \times C_n$

Now we work on the system of relations of the cokernel of the Laplacian of $C_4 \times C_n$. Let $x^i_j = (0, \cdots, 0, 1, 0, \cdots, 0) \in Z^{4n}$, whose unique nonzero 1 is in the position corresponding to vertex v^i_j. It follows from the relations of coker$L(C_4 \times C_n)$ that we can get the system of equations:

$$4x^i_j - (x^i_{j+1} + x^i_{j-1}) - x^{i+1}_j - x^{i-1}_j = 0, \quad j \in Z_4,\ i \in Z_n. \tag{3.1}$$

Lemma 3.1. *There are three sequences of integral numbers $(a_i)_{i\geq 0}$, $(b_i)_{i\geq 0}$, $(c_i)_{i\geq 0}$ such that*

$$x^i_j = a_i x^1_j + b_i(x^1_{j+1} + x^1_{j-1}) + c_i x^1_{j+2} - a_{i-1} x^0_j - b_{i-1}(x^0_{j+1} + x^0_{j-1}) - c_{i-1} x^0_{j+2}, \tag{3.2}$$

where $j \in Z_4$, $1 \leq i \leq n$. Moreover, the numbers in the above sequences have recurrence relations and initial conditions as follows

$$\begin{cases} a_i = \frac{1}{4}(i + u_i(4) + 2u_i(2)), & (i \geq 0), \\ b_i = \frac{1}{4}(i - u_i(4)), & (i \geq 0), \\ c_i = \frac{1}{4}(i + u_i(4) - 2u_i(2)), & (i \geq 0). \end{cases} \tag{3.3}$$

Proof. From (3.1), it follows that

$$x^{i+1}_j = 4x^i_j - (x^i_{j+1} + x^i_{j-1}) - x^{i-1}_j, \text{ for } j \in Z_4,\ 2 \leq i \leq n-1. \tag{3.4}$$

This lemma is valid for the cases of $i = 1, 2$. Suppose that $x_j^l = a_l x_j^1 + b_l(x_{j+1}^1 + x_{j-1}^1) + c_l x_{j+2}^1 - a_{l-1}x_j^0 - b_{l-1}(x_{j+1}^0 + x_{j-1}^0) - c_{l-1}x_{j+2}^0$, for $l \leq h$. Then from the induction assumption and equation (3.4), it follows that

$$\begin{aligned}
x_j^{h+1} &= 4x_j^h - (x_{j+1}^h + x_{j-1}^h) - x_j^{h-1} \\
&= 4\Big(a_h x_j^1 + b_h(x_{j+1}^1 + x_{j-1}^1) + c_h x_{j+2}^1 - a_{h-1}x_j^0 - b_{h-1}(x_{j+1}^0 + x_{j-1}^0) - c_{h-1}x_{j+2}^0\Big) - \Big(a_h x_{j+1}^1 + b_h(x_{j+2}^1 + x_j^1) + c_h x_{j-1}^1 - a_{h-1}x_{j+1}^0 - b_{h-1}(x_{j+2}^0 + x_j^0) - c_{h-1}x_{j-1}^0\Big) - \Big(a_h x_{j-1}^1 + b_h(x_j^1 + x_{j-2}^1) + c_h x_{j+1}^1 - a_{h-1}x_{j-1}^0 - b_{h-1}(x_j^0 + x_{j-2}^0) - c_{h-1}x_{j+1}^0\Big) - \Big(a_{h-1}x_j^1 + b_{h-1}(x_{j+1}^1 + x_{j-1}^1) + c_{h-1}x_{j+2}^1 - a_{h-2}x_j^0 - b_{h-2}(x_{j+1}^0 + x_{j-1}^0) - c_{h-2}x_{j+2}^0\Big) \\
&= \Big(4a_h - 2b_h - a_{h-1}\Big)x_j^1 + \Big(4b_h - a_h - c_h - b_{h-1}\Big)\Big(x_{j+1}^1 + x_{j-1}^1\Big) + \Big(4c_h - 2b_h - c_{h-1}\Big)x_{j+2}^1 - \Big(4a_{h-1} - 2b_{h-1} - a_{h-2}\Big)x_j^0 - \Big(4b_{h-1} - a_{h-1} - c_{h-1} - b_{h-2}\Big)\Big(x_{j+1}^0 + x_{j-1}^0\Big) - \Big(4c_{h-1} - 2b_{h-1} - c_{h-2}\Big)x_{j+2}^0 \\
&= a_{h+1}x_j^1 + b_{h+1}(x_{j+1}^1 + x_{j-1}^1) + c_{h+1}x_{j+2}^1 - a_h x_j^0 - b_h(x_{j+1}^0 + x_{j-1}^0) - c_h x_{j+2}^0.
\end{aligned}$$

Thus (3.2) holds by induction.

From the process of induction just now, it is easy to see that

$$\begin{cases} a_{i+1} = 4a_i - 2b_i - a_{i-1}, \\ b_{i+1} = 4b_i - (a_i + c_i) - b_{i-1}, \\ c_{i+1} = 4c_i - 2b_i - c_{i-1}, \end{cases} \tag{3.5}$$

for $i \geq 1$. Let $\tau_i = a_i + c_i$ and $\eta_i = a_i - c_i$. After a short calculation, we can get

$$\begin{cases} \eta_{i+1} = 4\eta_i - \eta_{i-1}, \\ \eta_0 = 0, \quad \eta_1 = 1; \\ \tau_{i+2} - 8\tau_{i+1} + 14\tau_i - 8\tau_{i-1} + \tau_{i-2} = 0, \\ \tau_0 = 0, \quad \tau_1 = 1. \end{cases}$$

By proposition 2.1, we have $\eta_i = u_i(2) = e_i$. Let $\phi_i = 2\tau_i - i$, then one can verify that $\phi_{i+2} = 6\phi_{i+1} - \phi_i$, with $\phi_0 = 0$ and $\phi_1 = 1$. Immediately, $\phi_i = u_i(4) = f_i$, and then $\tau_i = \frac{1}{2}(i + u_i(4))$. Now the equalities in (3.3) can been verified directly. □

We know from lemma 3.1 that the systerm of equation (3.2) has at most 8 generators, i.e., each x_j^i can be expressed in terms of $x_0^0, x_1^0, x_2^0, x_3^0, x_0^1, x_1^1, x_2^1, x_3^1$. So there are at least $4n-8$ diagonal entries of the Smith normal form of $L(G)$ are equal to 1, however the remaining invariant factors of coker$(C_4 \times C_n)$ hide inside the relations matrix induced by $x_0^0, x_1^0, x_2^0, x_3^0, x_0^1, x_1^1, x_2^1, x_3^1$.

Let $Y = (x_0^1, x_1^1, x_2^1, x_3^1, x_0^0, x_1^0, x_2^0, x_3^0)^t$, $A_n = \begin{pmatrix} a_n & b_n & c_n & b_n \\ b_n & a_n & b_n & c_n \\ c_n & b_n & a_n & b_n \\ b_n & c_n & b_n & a_n \end{pmatrix}$

and $M = \begin{pmatrix} A_{n+1} & -A_n \\ A_n & -A_{n-1} \end{pmatrix}$. From (3.2) and the cyclic structure of $C_4 \times C_n$, we have

$$\begin{cases} x_j^0 = x_j^n = a_n x_j^1 + b_n(x_{j+1}^1 + x_{j-1}^1) + c_n x_{j+2}^1 - a_{n-1}x_j^0 \\ \qquad -b_{n-1}(x_{j+1}^0 + x_{j-1}^0) - c_{n-1}x_{j+2}^0, \\ x_j^1 = x_j^{n+1} = a_{n+1}x_j^1 + b_{n+1}(x_{j+1}^1 + x_{j-1}^1) + c_{n+1}x_{j+2}^1 - a_n x_j^0 \\ \qquad -b_n(x_{j+1}^0 + x_{j-1}^0) - c_n x_{j+2}^0, \end{cases}$$

where $0 \le j \le 3$. Therefore

$$(M - I)Y = 0. \tag{3.6}$$

From the argument above, we know that one can reduce $L(G)$ to $I_{4n-8} \oplus (M - I)$ by performing some row and column operations up to equivalence. Now we only need to evaluate the SNF of $M - I$.

4 Analysis of the coefficients of the Smith normal form of $M - I$

If we multiply the last 4 rows of $M - I$ by -1, then we have that

$$\begin{pmatrix} A_{n+1} - I_4 & -A_n \\ A_n & -A_{n-1} - I_4 \end{pmatrix} \sim \begin{pmatrix} A_{n+1} - I_4 & -A_n \\ -A_n & A_{n-1} + I_4 \end{pmatrix}. \tag{4.1}$$

From lemma 3.1, one can verify that $a_{i+1}+c_{i+1}+2b_{i+1} = a_i+c_i+2b_i+1$, for each $i \in N$, and it results that each line sum of the right matrix of (4.1) is equal to 0. Immediately, we have the following lemma.

Lemma 4.1. *$M - I \sim (0) \oplus M_1$, where M_1 is the submatrix of $M - I$ resulting from the deletion of the first row and column.*

Let $h_n = e_n + e_{n+1}$, $g_n = f_n + f_{n+1}$, $p_i = e_i + e_{n-i}$, $q_i = f_i + f_{n-i}$, and let

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then one can check that L_1 and R_1 are unimodular matrices and

$$L_1 M_1 R_1 = \begin{pmatrix} 0 & 0 & 0 & n & n & 0 & 0 \\ 0 & p_{-1} & p_0 & 0 & 0 & 0 & 0 \\ 0 & p_0 & p_1 & 0 & 0 & 0 & 0 \\ \frac{q_{-1}+q_0}{2} & \frac{p_{-1}+q_{-1}}{2} & \frac{p_0+q_0}{2} & \frac{n-q_{-1}}{4} & \frac{n-q_0}{4} & 0 & 0 \\ \frac{q_0+q_1}{2} & \frac{p_0+q_0}{2} & \frac{p_1+q_1}{2} & \frac{n-q_0}{4} & \frac{n-q_1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{n+p_{-1}}{2} & \frac{n+p_0}{2} & p_{-1} & p_0 \\ 0 & 0 & 0 & \frac{n+p_0}{2} & \frac{n+p_1}{2} & p_0 & p_1 \end{pmatrix}.$$

Putting $m = 2$ and 4, then it follows from proposition 2.1 that

$$\begin{cases} p_{i+1} = 4p_i - p_{i-1}, \\ q_{i+1} = 6q_i - q_{i-1}. \end{cases} \tag{4.2}$$

Let $M_2 = L_1 M_1 R_1$ and $U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}$.

Then by (4.2) we have $U^i M_2 =$

$$\begin{pmatrix} 0 & 0 & 0 & n & n & 0 & 0 \\ 0 & p_{i-1} & p_i & 0 & 0 & 0 & 0 \\ 0 & p_i & p_{i-1} & 0 & 0 & 0 & 0 \\ \frac{q_{i-1}+q_i}{2} & \frac{p_{i-1}+q_{i-1}}{2} & \frac{p_i+q_i}{2} & \frac{n-q_{i-1}}{4} & \frac{n-q_i}{4} & 0 & 0 \\ \frac{q_i+q_{i+1}}{2} & \frac{p_i+q_i}{2} & \frac{p_{i+1}+q_{i+1}}{2} & \frac{n-q_i}{4} & \frac{n-q_{i+1}}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{n+p_{i-1}}{2} & \frac{n+p_i}{2} & p_{i-1} & p_i \\ 0 & 0 & 0 & \frac{n+p_0}{2} & \frac{n+p_{i+1}}{2} & p_i & p_{i+1} \end{pmatrix}. \tag{4.3}$$

Now we distinguish two cases.

Case 1 $n = 2s + 1$ odd.

In this case, by (4.2) one can verify that

$$\begin{pmatrix} p_s & p_{s+1} \\ p_{s+1} & p_{s+2} \end{pmatrix} = \begin{pmatrix} h_s & h_s \\ h_s & 3h_s \end{pmatrix}, \quad \begin{pmatrix} q_s & q_{s+1} \\ q_{s+1} & q_{s+2} \end{pmatrix} = \begin{pmatrix} g_s & g_s \\ g_s & 5g_s \end{pmatrix}. \tag{4.4}$$

Let $i = s+1$ in (4.3), then by (4.4) we have

$$U^{s+1}M_2 = \begin{pmatrix} 0 & 0 & 0 & n & n & 0 & 0 \\ 0 & h_s & h_s & 0 & 0 & 0 & 0 \\ 0 & h_s & 3h_s & 0 & 0 & 0 & 0 \\ g_s & \frac{g_s+h_s}{2} & \frac{g_s+h_s}{2} & \frac{n-g_s}{4} & \frac{n-g_s}{4} & 0 & 0 \\ 3g_s & \frac{g_s+h_s}{2} & \frac{5g_s+3h_s}{2} & \frac{n-g_s}{4} & \frac{n-g_s}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{n+h_s}{2} & \frac{n+h_s}{2} & h_s & h_s \\ 0 & 0 & 0 & \frac{n+h_s}{2} & \frac{n+3h_s}{2} & h_s & 3h_s \end{pmatrix}.$$

Let

$$L_2 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 & -2 \\ -1 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is clear that L_2 and R_2 are unimodular matrices. By a direct calculation, we get

$$L_2U^{s+1}M_2R_2 = X \oplus Y, \tag{4.5}$$

where $X = \begin{pmatrix} 0 & 2h_s & 0 \\ h_s & 0 & 2h_s \\ g_s & h_s & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & h_s & 0 & 0 \\ \frac{n+h_s}{2} & 0 & h_s & 0 \\ \frac{n-g_s}{4} & \frac{h_s+g_s}{2} & 0 & g_s \end{pmatrix}$.

Using the standard method for calculating the determinant factors, we have $\mathrm{SNF}(X) = \mathrm{diag}\left((h_s, g_s), h_s, \frac{4h_sg_s}{(h_s,\ g_s)}\right)$ and $\mathrm{SNF}(Y) = \mathrm{diag}\left((n,\ h_s,\ g_s), \frac{(n,\ h_s)(h_s,\ g_s)}{(n,\ h_s,\ g_s)}, \frac{h_s(nh_s, ng_s, h_sg_s)}{(n,h_s)(h_s,g_s)}, \frac{nh_sg_s}{(nh_s,\ ng_s,\ h_sg_s)}\right)$.

Now, it is easy to see $\mathrm{SNF}(M_1) = \mathrm{SNF}(M_2) = \mathrm{diag}\Big((n,\ h_s,\ g_s), (h_s,\ g_s), \frac{(n,h_s)(h_s,g_s)}{(n,h_s,g_s)}, h_s, \frac{h_s(nh_s,\ ng_s, h_sg_s)}{(n,\ h_s)(h_s,\ g_s)}, \frac{h_sg_s}{(h_s,\ g_s)}, \frac{4nh_sg_s}{(nh_s,\ ng_s,\ h_sg_s)}\Big)$.

Case 2 $n = 2s$ **even.**

In this case, by (4.2) one can verify that

$$\begin{pmatrix} p_s & p_{s+1} \\ p_{s+1} & p_{s+2} \end{pmatrix} = \begin{pmatrix} 2e_s & 4e_s \\ 4e_s & 14e_s \end{pmatrix}, \quad \begin{pmatrix} q_s & q_{s+1} \\ q_{s+1} & q_{s+2} \end{pmatrix} = \begin{pmatrix} 2f_s & 6f_s \\ 6f_s & 34f_s \end{pmatrix}.$$

Apply (4.3), we have

$$U^{s+1}M_2 = \begin{pmatrix} 0 & 0 & 0 & 2s & 2s & 0 & 0 \\ 0 & 2e_s & 4e_s & 0 & 0 & 0 & 0 \\ 0 & 4e_s & 14e_s & 0 & 0 & 0 & 0 \\ 4f_s & f_s + e_s & 3f_s + 2e_s & \frac{s-f_s}{2} & \frac{s-3f_s}{2} & 0 & 0 \\ 20f_s & 3f_s + 2e_s & 17f_s + 7e_s & \frac{s-3f_s}{2} & \frac{s-17f_s}{2} & 0 & 0 \\ 0 & 0 & 0 & s + e_s & s + 2e_s & 2e_s & 4e_s \\ 0 & 0 & 0 & s + 2e_s & s + 7e_s & 4e_S & 14e_s \end{pmatrix}.$$

Let

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -4 & 1 & 7 & -1 & 0 & 0 \\ 0 & 5 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and

$$R_3 = \begin{pmatrix} 0 & -2 & 0 & 1 & -2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 5 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 2 & 6 & 0 & -4 & 6 & 1 & 2 \\ -1 & -6 & 0 & 4 & -1 & -1 & -2 \\ 0 & -3 & 2 & 2 & -3 & 1 & -1 \\ 0 & 3 & -1 & -2 & 3 & 0 & 1 \end{pmatrix}.$$

Then we have $L_3 U^{s+1} M_2 R_3 =$

$$\begin{pmatrix} 2s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2e_s & 0 & 0 & 0 & 0 & 0 \\ 3e_s & 0 & 6e_s & 0 & 0 & 0 & 0 \\ s - 2f_s & e_s + 4f_s & 0 & 8f_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6e_s & 0 & 0 \\ s & 0 & 0 & 0 & 0 & e_s & 0 \\ \frac{1}{2}(f_s + s) & f_s & 0 & 0 & 3e_s & f_s & 2f_s \end{pmatrix}. \qquad (4.6)$$

Let M_3 denote the matrix on the right side of (4.6). If we can further reduce M_3 to the direct product of some small matrices as in the above case of n being odd, then the calculation will become easier. Unfortunately, we can not achieve it.

Let

$$M_3' = \begin{pmatrix} s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_s & 0 & 0 & 0 & 0 & 0 \\ 3e_s & 0 & 3e_s & 0 & 0 & 0 & 0 \\ s-2f_s & e_s+4f_s & 0 & f_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3e_s & 0 & 0 \\ s & 0 & 0 & 0 & 0 & e_s & 0 \\ \frac{1}{2}(f_s+s) & f_s & 0 & 0 & 3e_s & f_s & f_s \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

It is clear that L_4 and R_4 are unimodular matrices and $L_4 M_3' R_4 = E \oplus F$, where

$$E = \begin{pmatrix} s & 0 & 0 & 0 \\ \frac{1}{2}(s+f_s) & f_s & 0 & 0 \\ 0 & 0 & e_s & 0 \\ 0 & 0 & 0 & 3e_s \end{pmatrix}, \quad F = \begin{pmatrix} f_s & 0 & 0 \\ 0 & e_s & 0 \\ 0 & 0 & 3e_s \end{pmatrix}. \tag{4.7}$$

Now we can compute the determinantal divisors of E and F and furthermore obtain the SNF of M_3'. Here we directly give the result and omit the details of computation. However we must say that proposition 2.3 plays an important role in this computation.

Note that $\mathrm{SNF}(E) =$

$$\begin{cases} \mathrm{diag}\left((s,e_s,f_s), \frac{(s,e_s)(e_s,f_s)}{(s,e_s,f_s)}, \frac{e_s(se_s,sf_s,e_sf_s)}{(s,e_s)(e_s,f_s)}, \frac{3se_sf_s}{(se_s,sf_s,e_sf_s)}\right), & \text{if } 2\nmid s, \\ \mathrm{diag}\left((s,e_s,f_s), \frac{(s,e_s)(e_s,f_s)}{(s,e_s,f_s)}, \frac{3e_s(se_s,sf_s,e_sf_s)}{(s,e_s)(e_s,f_s)}, \frac{se_sf_s}{(se_s,sf_s,e_sf_s)}\right), & \text{if } 2\mid s; \end{cases}$$

and

$$\mathrm{SNF}(F) = \begin{cases} \mathrm{diag}\left((e_s,f_s), e_s, \frac{3e_sf_s}{(e_s,f_s)}\right), & \text{if } 2\nmid s, \\ \mathrm{diag}\left((e_s,f_s), 3e_s, \frac{e_sf_s}{(e_s,f_s)}\right), & \text{if } 2\mid s. \end{cases}$$

Note that we further have $\mathrm{SNF}(M_3') =$

$$\begin{cases} \mathrm{diag}\left((s,e_s,f_s),(e_s,f_s), \frac{(s,e_s)(e_s,f_s)}{(s,e_s,f_s)}, e_s, \frac{e_s(se_s,ef_s,e_sf_s)}{(s,e_s)(e_s,f_s)}, \frac{3e_sf_s}{(e_s,\ f_s)}, \frac{3se_sf_s}{(se_s,ef_s,e_sf_s)}\right), \\ \qquad \text{if } \ 2\nmid s, \\ \mathrm{diag}\left((s,e_s,f_s),(e_s,f_s), \frac{(s,e_s)(e_s,f_s)}{(s,e_s,f_s)}, 3e_s, \frac{3e_s(se_s,sf_s,e_sf_s)}{(s,e_s)(e_s,f_s)}, \frac{e_sf_s}{(e_s,\ f_s)}, \frac{se_sf_s}{(se_s,ef_s,e_sf_s)}\right), \\ \qquad \text{if } \ 2\mid s. \end{cases}$$

Note that M_3 is obtained from M_3' by multiplying its rows 1, 2 , 5 by 2, columns 3, 7 by 2, column 4 by 8. Then we have that there are integers t_i such that $S_i(M_3) = 2^{t_i} S_i(M_3')$, for $1 \le i \le 7$.

- $n = 2s$ with s odd.

It follows from proposition 2.3 that $2 \nmid e_s$ and $2 \nmid f_s$. Moreover, $\Delta_i(M_3')$ is odd and hence $S_i(M_3')$ is odd. Since $\det(M_3[3,4,6,7|1,2,5,6]) = -9e_s^3(e_s + 4f_s)$ is odd, where $M_3[3,4,6,7|1,2,5,6]$ is the submatrix that lies in the rows 3, 4, 6, 7 and columns 1, 2, 5, 6 of M_3. Thus $t_1 = t_2 = t_3 = t_4 = 0$. Note that every nonzero element in rows 1, 2, 5, columns 3, 4, 7 of M_3 is even and on the main diagonal, so every 5×5 submatrix of M_3 must contain at least one row and at least one column of them. Thus $2^2 \mid \Delta_5(M_3)$. Since $\det(M_3[1,3,4,6,7|1,2,3,5,6]) = 36se_s^3(e_s + 4f_s)$, then 2^3 is not its divisor. Thus $t_5 = 2$. As above, $2^4 \mid \Delta_6(M_3)$, but $\det(M_3[1,3,4,5,6,7|1,2,3,5,6,7]) = -144se_s^3 f_s(e_s + 4f_s)$, which is not divisible by 2^5. So $t_6 = 4 - 2 = 2$. Finally, it is easy to see that $t_7 = 8 - 4 = 4$. Thus the SNF of M_3 here is

$$\operatorname{diag}\left((s, e_s, f_s), (e_s, f_s), \frac{(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}, e_s, \frac{4e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{12e_s f_s}{(e_s, f_s)}, \frac{48se_s f_s}{(se_s, sf_s, e_s f_s)}\right)$$

- $n = 2s$ with s even.

Let $t = T_2(s)$, then from proposition 2.3, it follows that $T_2(e_s) = t + 1$ and $T_2(f_s) = t$. It is clear that $S_1(M_3) = S_1(M_3')$, so $t_1 = 0$. Since $T_2(\det(M_3[6,7|1,2])) = T_2(sf_s) = 2t = T_2(\Delta_2(M_3')) = 2t$, then clearly $t_2 = 0$. It is not hard to see that the maximal power of 2 contained in each of the 3×3 minor subdeterminants of M_3 is at least $3t + 2$, and then we can conclude that $T_2(\Delta_3(M_3)) = 3t + 2$, since $\det(M_3[4,6,7|1,2,7]) = -2sf_s(e_s + 4f_s)$ is not divisible by 2^{3t+3}. Then $T_2(S_3(M_3)) = T_2(\Delta_3(M_3)) - T_2(\Delta_2(M_3)) = (3t + 2) - 2t = t + 2$. So $t_3 = T_2(S_3(M_3)) - T_2(S_3(M_3')) = (t + 2) - t = 2$. All the 4×4 minor subdeterminants of M_3 contain the divisor 2^{4t+4}, and then we can say that $T_2(\Delta_4(M_3)) = 4t + 4$, since $\det(M_3[3,4,6,7|1,2,3,7]]) = -12se_s f_s(e_s + 4f_s)$ is not divisible by 2^{4t+5}. Then $T_2(S_4(M_3)) = T_2(\Delta_4(M_3)) - T_2(\Delta_3(M_3) = (4t + 4) - (3t + 2) = t + 2$. So $t_4 = T_2(S_4(M_3)) - T_2(S_4(M_3')) = (t + 2) - (t + 1) = 1$. Go on in this way, we obtain that $t_5 = t_6 = 1$ and $t_7 = 3$. Thus we get that SNF of M_3 here is

$$\operatorname{diag}\left((s, e_s, f_s), (e_s, f_s), \frac{4(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}, 6e_s, \frac{6e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{2e_s f_s}{(e_s, f_s)}, \frac{8se_s f_s}{(se_s, sf_s, e_s f_s)}\right)$$

5 Conclusion

Now we can give the main result as follows.

Theorem 5.1. *If $n = 2s + 1$, then the critical group of $C_4 \times C_n$ $(n \ge 3)$ is* $Z_{(n, h_s, g_s)} \oplus Z_{(h_s, g_s)} \oplus Z_{\frac{(n, h_s)(h_s, g_s)}{(n, h_s, g_s)}} \oplus Z_{h_s} \oplus Z_{\frac{h_s(nh_s, ng_s, h_s g_s)}{(n, h_s)(h_s, g_s)}} \oplus Z_{\frac{h_s g_s}{(h_s, g_s)}} \oplus$

$Z_{\frac{4nh_sg_s}{(nh_s,ng_s,h_sg_s)}}$.

If $n = 2s$ with s odd, then the critical group of $C_4 \times C_n$ $(n \geq 3)$ is $Z_{(s,e_s,f_s)} \oplus Z_{(e_s,f_s)} \oplus Z_{\frac{(s,e_s)(e_s,f_s)}{(s,e_s,f_s)}} \oplus Z_{e_s} \oplus Z_{\frac{4e_s(se_s,sf_s,e_sf_s)}{(s,e_s)(e_s,f_s)}} \oplus Z_{\frac{12e_sf_s}{(e_s,f_s)}} \oplus Z_{\frac{48se_sf_s}{(se_s,sf_s,e_sf_s)}}$.

If $n = 2s$ with s even, then the critical group of $C_4 \times C_n$ $(n \geq 3)$ is $Z_{(s,e_s,f_s)} \oplus Z_{(e_s,f_s)} \oplus Z_{\frac{4(s,e_s)(e_s,f_s)}{(s,e_s,f_s)}} \oplus Z_{6e_s} \oplus Z_{\frac{6e_s(se_s,sf_s,e_sf_s)}{(s,e_s)(e_s,f_s)}} \oplus Z_{\frac{2e_sf_s}{(e_s,f_s)}} \oplus Z_{\frac{8se_sf_s}{(se_s,sf_s,e_sf_s)}}$.

Example 5.1. *To give an illustration of theorem 5.1, we consider the three graphs $C_4 \times C_4$, $C_4 \times C_5$ and $C_4 \times C_6$. Note that $e_0 = 0$, $e_1 = 1$, $e_2 = 4$, $e_3 = 15$, $e_4 = 56$, $e_5 = 209$, $e_6 = 780$, $f_0 = 0$, $f_1 = 1$, $f_2 = 6$, $f_3 = 35$, $f_4 = 204$, $f_5 = 1189$, $f_6 = 6930$. Then by theorem 5.1 we have that $K(C_4 \times C_4) = (Z_2)^2 \oplus Z_8 \oplus (Z_{24})^3 \oplus Z_{96}$; $K(C_4 \times C_5) = (Z_{19})^2 \oplus Z_{779} \oplus Z_{15580}$ and $K(C_4 \times C_6) = Z_5 \oplus (Z_{15})^2 \oplus Z_{60} \oplus Z_{1260} \oplus Z_{5040}$. Maple gives the identical result.*

Let $H_n(m) = u_n(m) + u_{n+1}(m)$. Clearly, $H_n(2) = h_n$ and $H_n(4) = g_n$.

Theorem 5.2. *If $n_1 \mid n_2$, then $K(C_4 \times C_{n_1})$ is a subgroup of $K(C_4 \times C_{n_2})$.*

Proof. We only need to prove that every invariant factor of $K(C_4 \times C_{n_1})$ is a divisor of the corresponding one of $K(C_4 \times C_{n_2})$. We distinguish three cases.

Case 1. $n_1 = 2s + 1$ and $n_2 = (2k + 1)(2s + 1)$.

Let $p = 2s + 1$, $q = 2k + 1$, then $H_{\lfloor \frac{n_2}{2} \rfloor}(m) = H_{pk+s}(m)$. Since $\alpha\beta = 1$, then from the definition we can directly verify that $u_{pk+s}(m) = v_{pk}u_s(m) + u_{pk-s}(m)$, $u_{pk+s+1}(m) = v_{pk}(m)u_{s+1}(m) + u_{pk-s-1}(m)$. Thus $H_{pk+s}(m) = v_{pk}(m)H_s(m) + H_{pk-s-1}(m) = v_{pk}(m)H_s(m) + H_{p(k-1)+s} = \cdots = \left(\sum_{i=1}^{k} v_{ip}(m) + 1\right) H_s(m)$. It means that $H_s(m) \mid H_{pk+s}(m)$ and hence $h_s \mid h_{pk+s}$, $g_s \mid g_{pk+s}$. So every invariant factor of $K(C_4 \times C_{2s+1})$ is a divisor of the corresponding one of $K(C_4 \times C_{(2k+1)(2s+1)})$.

Case 2. $n_1 = 2s + 1$ and $n_2 = 2k(2s + 1)$.

Since one can verify that $(u_n(m) + u_{n+1}(m))(u_n(m) - u_{n+1}(m)) = -u_{2n+1}(m)$ and $u_n(m) = v_p(m)u_{n-p}(m) - u_{n-2p}(m)$, we have that $H_n(m) \mid u_{2n+1}(m)$ and if $p \mid n$, then $u_p(m) \mid u_n(m)$. Thus $H_s(m) \mid u_{n_1}(m)$, and $u_{n_1}(m) \mid u_{kn_1}(m)$. Then $H_s(m) \mid u_{kn_1}(m)$. It means that $h_s \mid e_{kn_1}$ and $g_s \mid f_{kn_1}$. So every invariant factor of $K(C_4 \times C_{2s+1})$ is a divisor of the corresponding one of $K(C_4 \times C_{2k(2s+1)})$.

Case 3. $n_1 = 2s$ and $n_2 = 2ks$.

Since $u_s(m) \mid u_{ks}(m)$, then $e_s \mid e_{ks}$ and $f_s \mid f_{ks}$. So every invariant factor of $K(C_4 \times C_{2s})$ is a divisor of the corresponding one of $K(C_4 \times C_{2ks})$. □

Theorem 5.3. *The spanning tree number of $C_4 \times C_n$ $(n \geq 3)$ is $2^7 3^2 n e^4_{\frac{n}{2}} f^2_{\frac{n}{2}}$, i.e.,* $\frac{n}{4^{n+1}}\left((\sqrt{3}+1)^n - (\sqrt{3}-1)^n\right)^4 \cdot \left((\sqrt{2}+1)^n - (\sqrt{2}-1)^n\right)^2$.

Proof. We prove this theorem by distinguishing two cases.

Case 1: $n = 2s+1$.

Note that $h_s^4 = (e_s + e_{s+1})^4 = \frac{1}{4^{n+1}}\left((\sqrt{3}+1)^n - (\sqrt{3}-1)^n\right)^4$ and $g_s^2 = (f_s + f_{s+1})^2 = \frac{1}{4}\left((\sqrt{2}+1)^n - (\sqrt{2}-1)^n\right)^2$.

From (4.3), we know that the spanning tree number of $C_4 \times C_n$ of this case is $(\det X) \cdot (\det Y) = 4nh_s^4 g_s^2 = \frac{n}{4^{n+1}}\left((\sqrt{3}+1)^n - (\sqrt{3}-1)^n\right)^4 \cdot \left((\sqrt{2}+1)^n - (\sqrt{2}-1)^n\right)^2 = 2^7 3^2 n e^4_{\frac{n}{2}} f^2_{\frac{n}{2}}$.

Case 2: $n = 2s$.

From (4.4), we know that the spanning tree number of $C_4 \times C_n$ of this case is $\det(M_3) = 2^8 3^2 s e_s^4 f_s^2 = 2^7 3^2 n e^4_{\frac{n}{2}} f^2_{\frac{n}{2}}$. □

Corollary 5.1. *For every $n \geq 3$, we have that*

$$\prod_{j=1}^{n-1}\left(4 - 2\cos\frac{2\pi j}{n}\right)^2\left(6 - 2\cos\frac{2\pi j}{n}\right) = \frac{1}{4^{n+2}}\left((\sqrt{3}+1)^n - (\sqrt{3}-1)^n\right)^4 \cdot \left((\sqrt{2}+1)^n - (\sqrt{2}-1)^n\right)^2.$$

Proof. It is not difficult to know that the Laplacian eigenvalues of C_n are $(2 - 2\cos\frac{2\pi j}{n})$, $0 \leq j \leq n-1$. Then it follows from the argument of the second section of [7] that the Laplacian eigenvalues of $C_4 \times C_n$ are: $0, 2, 2, 4, 2 - 2\cos\frac{2\pi j}{n}$, $4 - 2\cos\frac{2\pi j}{n}$ (with multiplicity 2), $6 - 2\cos\frac{2\pi j}{n}$, where $1 \leq j \leq n-1$. Then by the well known Kirchhoff Matrix-Tree Theorem we know the spanning tree number of $C_4 \times C_n$ is $\frac{4}{n}\prod_{j=1}^{n-1}\left(2 - 2\cos\frac{2\pi j}{n}\right)\left(4 - 2\cos\frac{2\pi j}{n}\right)^2\left(6 - 2\cos\frac{2\pi j}{n}\right)$. Since C_n has n spanning trees, we have $\frac{1}{n}\prod_{j=1}^{n-1}(2-2\cos\frac{2\pi j}{n}) = n$. Thus the spanning tree number of $C_4 \times C_n$ equals $4n\prod_{j=1}^{n-1}(4-2\cos\frac{2\pi j}{n})^2(6-2\cos\frac{2\pi j}{n})$. Recall theorem 5.3, we have $4n\prod_{j=1}^{n-1}\left(4 - 2\cos\frac{2\pi j}{n}\right)^2\left(6 - 2\cos\frac{2\pi j}{n}\right) = \frac{n}{4^{n+1}}\left((\sqrt{3}+1)^n - (\sqrt{3}-1)^n\right)^4 \cdot \left((\sqrt{2}+1)^n - (\sqrt{2}-1)^n\right)^2$. So this corollary holds. □

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