GLOBAL BEHAVIOR OF A HIGHER ORDER DIFFERENCE EQUATION

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ABSTRACT

In this paper we study the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{Ax_{n-2l}}{B+C\prod_{i=0}^{2k} x_{n-i}}, \quad n = 0, 1, ...,$$

where A, B, C are nonnegative parameters, initial conditions are non-negative real numbers and k, l are nonnegative integers, $l \leq k$. Also we derive solutions of some special cases of this equation.

Keywords: Difference Equation, Globally Asymptotically, Periodicity, Oscilation.

1. INTRODUCTION

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Rational difference equations is an important class of difference equations where they have many applications in real life for example the difference equation $x_{n+1} = \frac{a+bx_n}{c+x_n}$ which is known by Riccati Difference Equation has an applications in Optics and Mathematical Biology (see[14]). Many researchers have investigated the behavior of the solution of rational difference equations. For example see Refs. [1-18].

Aloqeili [2] studied the solutions, stability character, semi-cycle behavior of the difference equation $x_{n+1} = \frac{x_{n-1}}{a-x_{n-1}x_n}$ and gave the following formulation

$$x_n = \begin{cases} x_0 \prod_{i=1}^{\frac{n}{2}} \frac{a^{2i-1}(1-a)-(1-a^{2i-1})x_{-1}x_0}{a^{2i}(1-a)-(1-a^{2i})x_{-1}x_0}, & n \ even, \\ x_{-1} \prod_{i=0}^{\frac{n+1}{2}} \frac{a^{2i-1}(1-a)-(1-a^{2i})x_{-1}x_0}{a^{2i+1}(1-a)-(1-a^{2i+1})x_{-1}x_0}, & n \ odd. \end{cases}$$

Andruch et al. [3] studied the asymtotic behavior of solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{b+cx_n x_{n-1}}$.

Cinar [5] investigated the global asymptotic stability of all positive solutions of the rational difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$.

Also, Cinar [6] investigated the positive solutions of the rational difference equation $x_{n+1} = \frac{ax_{n-1}}{-1+bx_n x_{n-1}}$.

Yalcinkaya [17] investigated the global behaviour of the rational difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$.

Karatas et al. [12] obtained the solution of the difference equation $x_{n+1} = \frac{ax_{n-(2k+2)}}{-a+\prod_{i=0}^{2k+2} x_{n-i}}$.

El-Owaidy et al. [8] studied the dynamics of the recurcive sequence $x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^p}$.

Battaloğlu [4] discussed the global asymptotic behavior and periodicity character of the following difference equation $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma x_{n-(k+1)}^p}$ by generalizing the results due to El-Owaidy et al.

Hamza et al. [10] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation $x_{n+1} = \frac{Ax_{n-1}}{B+C\prod_{i=l}^{k} x_{n-2i}}$.

Gibbons et al. [11] investigated the global asymptotic behavior of the difference equation $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{\alpha+x_n}$.

Elsayed [9] investigated the qualitative behavior of the solution of the difference equation $x_{n+1} = ax_n + \frac{bx_n^2}{cx_n+dx_{n-1}}$.

Our aim in this paper is to investigate the dynamics of the solution of the difference equation

$$(1.1) \qquad x_{n+1} = \frac{Ax_{n-2l}}{B + C\prod_{i=0}^{2k} x_{n-i}}, \quad n = 0, 1, ...$$

where A, B, C are nonnegative real numbers, initial conditions are nonnegative and l, k are nonnegative integers, $l \leq k$. Also we obtained solutions of some special cases of Eq.(1.1).

The following special cases can be obtained:

1. When $A = 0$, Eq.(1.1) reduces to the equation $x_{n+1} = 0$.

2. When $B = 0$, Eq.(1.1) can be reduced to a linear difference equation, by the change of variables $x_n = e^{y_n}$.

3. When $C = 0$, Eq.(1.1) reduces to the equation $x_{n+1} = \frac{A}{B} x_{n-2l}$, which is a linear equation.

For various values of k, we can get more equations.

2. Preliminaries

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-(k+1)}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. *An equilibrium point for Eq.(2.1) is a point $\overline{x} \in I$ such that $\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$.*

Definition 2. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

Definition 3. *(i) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-(k-1)} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta$, we have $|x_n - \overline{x}| < \varepsilon$ for all $n \geq -k$.*

(ii) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2.1) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-(k-1)} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma$, we have $\lim_{n \to \infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq.(2.1) is global attractor if for all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$, we have $\lim_{n \to \infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of Eq.(2.1) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2.1).

(v) The equilibrium point $\overline{x}$ of Eq.(2.1) is unstable if $\overline{x}$ is not locally stable.

The linearized equation associated with Eq.(2.1) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, ..., \overline{x}) \, y_{n-i}, \quad n = 0, 1, ... \tag{2.2}$$

The characteristic equation associated with Eq.(2.2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, ..., \overline{x}) \lambda^{k-i} = 0. \tag{2.3}$$

Definition 4. *A positive semicycle of a solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.*(2.1) *consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$, *all greater than or equal to equilibrium* $\overline{x}$ *with* $l \geq -k$ *and* $m \leq \infty$ *such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} < \overline{x}$ *and either* $m = \infty$ *or* $m \leq \infty$ *and* $x_{m+1} < \overline{x}$.

A negative semicycle of a solution $\{x_n\}_{n=-k)}^{\infty}$ *of Eq.*(2.1) *consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$ *all less than* $\overline{x}$ *with* $l \geq -k$ *and* $m \leq \infty$ *such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} \geq \overline{x}$ *and either* $m = \infty$ *or* $m \leq \infty$ *and* $x_{m+1} \geq \overline{x}$.

Definition 5. *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.*(2.1) *is called nonoscillatory if there exists* $N \geq -k$ *such that either* $x_n > \overline{x}$ *for* $\forall n \geq N$ *or* $x_n < \overline{x}$ *for* $\forall n \geq N$,

and it is called oscillatory if it is not nonoscillatory.

Theorem 1. [18]*Assume that* f *is a* C^1 *function and let* $\overline{x}$ *be an equilibrium point of Eq.*(2.1). *Then the following statements are true.*

(i) If all roots of Eq.(2.3) *lie in open disk* $|\lambda| < 1$, *then* $\overline{x}$ *is locally asymptotically stable.*

(ii) If at least one root of Eq.(2.3) *has absolute value greater than one, then* $\overline{x}$ *is unstable.*

3. Dynamics of Eq.(1.1)

In this section, we investigate the dynamics of Eq.(1.1) under the assumptions that all parameters are nonnegative, the initial conditions are nonnegative and l, k are nonnegative.

The change of variables $x_n = \sqrt[2k+1]{\frac{B}{C}} y_n$ reduces Eq.(1.1) to the difference equation

$$y_{n+1} = \frac{\alpha y_{n-2l}}{1 + \prod_{i=0}^{2k} y_{n-i}}, \quad n = 0, 1, ..., \tag{3.1}$$

where $\alpha = \frac{A}{B}$. We can see that $\overline{y}_1 = 0$ is always an equilibrium point of Eq.(3.1). When $\alpha > 1$, Eq.(3.1) also possesses the unique positive equilibrium $\overline{y}_2 = \sqrt[2k+1]{\alpha - 1}$.

Theorem 2. *The following statements are true:*

(i) If $\alpha < 1$, *then the equilibrium point* $\overline{y}_1 = 0$ *of Eq.*(3.1) *is locally asymptotically stable,*

(ii) If $\alpha > 1$, then the equilibrium points $\overline{y}_1 = 0$ and $\overline{y}_2 = \sqrt[2k+1]{\alpha - 1}$ are unstable.

Proof. The linearized equation associated with Eq.(3.1) about $\overline{y}$ is

$$z_{n+1}+\frac{\alpha\overline{y}^{2k+1}}{\left(1+\overline{y}^{2k+1}\right)^2}\left(\sum_{i=0}^{2k} z_{n-i} - z_{n-2l}\right)-\frac{\alpha}{\left(1+\overline{y}^{2k+1}\right)^2} z_{n-2l} = 0, \; n = 0, 1, \ldots.$$

The characteristic equation associated with this equation is

$$\lambda^{2k+1} + \alpha\frac{\overline{y}^{2k+1}}{\left(1+\overline{y}^{2k+1}\right)^2}\left(\sum_{i=0}^{2k} \lambda^i - \lambda^{2k-2l}\right) - \alpha\frac{1}{\left(1+\overline{y}^{2k+1}\right)^2}\lambda^{2k-2l} = 0.$$

Then the linearized equation of Eq.(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} - \alpha z_{n-2l} = 0, \quad n = 0, 1, \ldots.$$

The characteristic equation of Eq.(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$\lambda^{2k-2l}\left(\lambda^{2l+1} - \alpha\right) = 0.$$

So

$$\lambda = 0 \text{ and } \lambda = \sqrt[2l+1]{\alpha}.$$

In view of Theorem 1:

If $\alpha < 1$, then $|\lambda| < 1$ for all roots and the equilibrium point $\overline{y}_1 = 0$ is locally asymptotically stable.

If $\alpha > 1$, it follows that the equilibrium point $\overline{y}_1 = 0$ is unstable.

The linearized equation of E q.(3.1) about the equilibrium point $\overline{y}_2 = \sqrt[2k+1]{\alpha - 1}$ becomes

$$z_{n+1} + \left(1 - \frac{1}{\alpha}\right)\left(\sum_{i=0}^{2k} z_{n-i} - z_{n-2l}\right) - \frac{1}{\alpha} z_{n-2l} = 0, \; n = 0, 1, \ldots.$$

The characteristic equation of E q.(3.1) about the equilibrium point $\overline{y}_2 = \sqrt[2k+1]{\alpha - 1}$ is,

$$\lambda^{2k+1} + \left(1 - \frac{1}{\alpha}\right)\left(\sum_{i=0}^{2k} \lambda^i - \lambda^{2k-2l}\right) - \frac{1}{\alpha}\lambda^{2k-2l} = 0.$$

It is clear that this equation has a root in the interval $(-\infty, -1)$. Then the equilibrium point $\overline{y}_2 = \sqrt[2k+1]{\alpha - 1}$ is unstable.

Theorem 3. *Assume that $\alpha < 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(3.1) is globally asymptotically stable.*

□

Proof. Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of Eq.(3.1). From Theorem 2 we know that the equilibrium point $\overline{y}_1 = 0$ of Eq.(3.1) is locally asymptotically stable. So it is sufficed to show that

$\lim_{n\to\infty} y_n = 0.$

Since

$y_{n+1} = \frac{\alpha y_{n-2l}}{1+\prod_{i=0}^{2k} y_{n-i}} \leq \alpha y_{n-2l}.$

We obtain

$y_{n+1} \leq \alpha y_{n-2l}.$

Then it can be written for $t = 0, 1, ...$

$y_{t(2l+1)+1} \leq \alpha^{t+1} y_{-2l},\ y_{t(2l+1)+2} \leq \alpha^{t+1} y_{-(2l-1)}, ..., y_{t(2l+1)+2l+1} \leq \alpha^{t+1} y_0.$

If $\alpha < 1$, then $\lim_{t\to\infty} \alpha^{(t+1)} = 0$

and

$$\lim_{n\to\infty} y_n = 0.$$

The proof is complete. □

Theorem 4. *A necessary and sufficient condition for Eq.(3.1) to have a prime period $(2l+1)$ solution is that $\alpha = 1$. In this case the prime period $(2l+1)$ is of the form $...0, 0, \varphi, 0, \varphi, 0, \varphi, ..., 0, \varphi...$ where $\varphi > 0$ and the number of the pairs $0, \varphi$ is $l-1$.*

Proof. Sufficiency: Let $\alpha = 1$, then for every $\varphi > 0$, it is obvious that $0, 0, \varphi, 0, \varphi, 0, \varphi, ..., 0, \varphi$...from Eq.(3.1).

Necessity: Assume that Eq.(3.1) has a prime period $(2l+1)$ solution $...a, a, \varphi, a, \varphi, a, \varphi, ..., a, \varphi$....We have $a = \frac{\alpha a}{1+a^{2k-l}\varphi^{l+1}}$ and $\varphi = \frac{\alpha\varphi}{1+a^{2k+1-l}\varphi^{l}}$.

Then

$\varphi + \varphi^{l+1} a^{2k+1-l} - a - a^{2k-l+1}\varphi^{l+1} = \alpha(\varphi - \alpha)$, this implies that $\alpha = 1$.

The proof is complete. □

4. The Solutions Form of Some Special Cases of Eq.(1.1)

Our aim in this section is to find a specific form of the solutions of some special cases of Eq.(1.1).

4.1. On the Difference Equation $x_{n+1} = \frac{(-1)^n x_{n-2k}}{1+(-1)^n \prod_{i=0}^{2k} x_{n-i}}$.

When we take $l = k, A = C = (-1)^n$ and $B = 1$ in Eq.(1.1), we obtain the equation

$$x_{n+1} = \frac{(-1)^n x_{n-2k}}{1 + (-1)^n \prod_{i=0}^{2k} x_{n-i}}, \quad n = 0, 1, ... \tag{4.1}$$

where initial conditions are non zero real numbers, k is an odd positive number and $\prod_{i=0}^{2k} x_{-i} \neq \mp 1$.

We shall give a few lemmas which will be useful in investigation of the solutions of Eq.(4.1).

Lemma 1. *Let* $\prod_{i=0}^{2k} x_{-i} = p$ *and* $p \neq \mp 1$. *Then for some solutions of Eq.*(4.1), *the following equalities are true*

$$x_1 = \frac{x_{-2k}}{1+p}, x_2 = -x_{-(2k-1)}(1+p), x_3 = \frac{x_{-(2k-2)}}{1-p}, x_4 = -x_{-(2k-3)}(1-p),$$

$$...$$

$$x_{2k-2} = \frac{x_{-3}}{1+p}, x_{2k-1} = -x_{-2}(1+p), x_{2k} = \frac{x_{-1}}{1-p}, x_{2k+1} = -x_0(1-p).$$

Also, $\prod_{i=0}^{2k-3} x_{i+4} = \prod_{i=0}^{2k-3} x_{-i}$.

Proof. When we have $n = 0, 1, ..., 2k$ in Eq.(4.1), it is easily obtain by iteration method. □

Lemma 2. *Let* $\prod_{i=0}^{2k} x_{-i} = p$ *and* $p \neq \mp 1$. *Then for some solutions of Eq.*(4.1), *the following equalities are true*

$$x_{2k+5} = -x_{-(2k-3)}, x_{2k+6} = \frac{x_{-(2k-4)}(-1+p)}{1+p}, x_{2k+7} = -x_{-(2k-5)},$$

$$x_{2k+8} = \frac{x_{-(2k-6)}(1+p)}{-1+p}, x_{2k+9} = -x_{-(2k-7)}, ..., x_{4k-1} = -x_{-3},$$

$$x_{4k} = \frac{x_{-2}(-1+p)}{1+p}, x_{4k+1} = -x_{-1}, x_{4k+2} = \frac{x_0(1+p)}{-1+p}.$$

Also, $\prod_{i=2k+1}^{4k-2} x_{i+4} = \prod_{i=0}^{2k-3} x_{-i}$.

Proof. When we have $n = 2k+4, 2k+5, ..., 4k+1$ in Eq.(4.1) and from Lemma 1, it is easily obtain by iteration method. □

Lemma 3. *Let* $\prod_{i=0}^{2k} x_{-i} = p$ *and* $p \neq \mp 1$. *Then for some solutions of Eq.*(4.1), *the following equalities are true*

$x_{4k+6}=x_{-(2k-3)}(1+p), x_{4k+7}=\frac{-x_{-(2k-4)}}{1+p}, x_{4k+8}=-x_{-(2k-5)}(-1+p),$

$x_{4k+9}=\frac{x_{-(2k-6)}}{-1+p}, x_{4k+10}=x_{-(2k-7)}(1+p),\ldots,x_{6k}=x_{-3}(1+p),$

$x_{6k+1}=\frac{-x_{-2}}{1+p}, x_{6k+2}=-x_{-1}(-1+p), x_{6k+3}=\frac{x_0}{-1+p}.$

Also, $\prod_{i=4k+2}^{6k-1} x_{i+4}=\prod_{i=0}^{2k-3} x_{-i}.$

Proof. When we have $n=4k+5,4k+6,\ldots,6k+2$ in Eq.(4.1) and from Lemma 2, it is easily obtain by iteration method. □

Theorem 5. *Assume that* $\prod_{i=0}^{2k} x_{-i}=p, p\neq\mp1, k\equiv1(\bmod 2)$ *and* $1\leq m\leq 2k+1$. *Let* $\{x_n\}_{n=-2k}^{\infty}$ *be a solution of Eq.*(4.1). *Then for* $n=0,1,\ldots$ *all solutions of Eq.*(4.1) *are of the form*

for $m\equiv1(\bmod 4)$

$$x_{(2k+1)n+m}=\begin{cases}\frac{x_{-[2k-(m-1)]}}{1+p} & n\equiv0\ (\bmod 4)\\ \frac{x_{-[2k-(m-1)]}(-1+p)}{1+p} & n\equiv1\ (\bmod 4)\\ -\frac{x_{-[2k-(m-1)]}}{1+p} & n\equiv2\ (\bmod 4)\\ x_{-[2k-(m-1)]} & n\equiv3\ (\bmod 4)\end{cases}, \tag{4.2}$$

for $m\equiv2(\bmod 4)$

$$x_{(2k+1)n+m}=\begin{cases}-x_{-[2k-(m-1)]}(1+p) & n\equiv0\ (\bmod 4)\\ -x_{-[2k-(m-1)]} & n\equiv1\ (\bmod 4)\\ -x_{-[2k-(m-1)]}(-1+p) & n\equiv2\ (\bmod 4)\\ x_{-[2k-(m-1)]} & n\equiv3\ (\bmod 4)\end{cases}, \tag{4.3}$$

for $m\equiv3(\bmod 4)$

$$x_{(2k+1)n+m}=\begin{cases}\frac{-x_{-[2k-(m-1)]}}{-1+p} & n\equiv0\ (\bmod 4)\\ \frac{x_{-[2k-(m-1)]}(1+p)}{-1+p} & n\equiv1\ (\bmod 4)\\ \frac{x_{-[2k-(m-1)]}}{-1+p} & n\equiv2\ (\bmod 4)\\ x_{-[2k-(m-1)]} & n\equiv3\ (\bmod 4)\end{cases}, \tag{4.4}$$

for $m\equiv0(\bmod 4)$

$$x_{(2k+1)n+m}=\begin{cases}x_{-[2k-(m-1)]}(-1+p) & n\equiv0\ (\bmod 4)\\ -x_{-[2k-(m-1)]} & n\equiv1\ (\bmod 4)\\ x_{-[2k-(m-1)]}(1+p) & n\equiv2\ (\bmod 4)\\ x_{-[2k-(m-1)]} & n\equiv3\ (\bmod 4)\end{cases} \tag{4.5}$$

Proof. In view of Lemma 1, it is obvious that our assumption is true for $n = 0$ and $m = 1, 2, 3, 4$.

Now we show that the result holds for $n = 1$ and $m = 1, 2, 3, 4$.

From Eq.(4.1), we can write

$$x_{(2k+1)+1} = \frac{-x_1}{1 - \prod_{i=0}^{2k} x_{2k+1-i}}.$$

From Lemma 1, we obtain

$$x_{2k+2} = \frac{x_{-2k}(-1+p)}{1+p}.$$

If it is taken $n = 2k + 2$ in Eq.(4.1), we have

$$x_{(2k+2)+1} = \frac{-x_2}{1 + \prod_{i=0}^{2k} x_{2k+2-i}}.$$

Then from previous equality and Lemma 1,

$$x_{2k+3} = \frac{-x_{(2k-1)}(1+p)}{1+p}.$$

If it is taken $n = 2k + 3$ in Eq.(4.1), we have

$$x_{(2k+3)+1} = \frac{-x_3}{1 + \prod_{i=0}^{2k} x_{2k+3-i}}.$$

Then

$$x_{2k+4} = \frac{x_{-(2k-2)}(1+p)}{-1+p}.$$

Finally for $n = 1$ and $m = 4$ from Lemma 2, we get

$$x_{2k+5} = -x_{-(2k-3)}.$$

Similarly it can be show that our assumption is true for $n = 2, 3$ and $m = 1, 2, 3, 4$ using Lemma 2 and Lemma 3.

Now suppose that our assumption holds for $(n - 1)$. We shall show that the result holds for n. From our assumption for $(n - 1)$ we have for $m \equiv 1 (\mathrm{mod}\, 4)$

(4.6) $$x_{(2k+1)n-(2k+1-m)} = \begin{cases} \frac{x_{-[2k-(m-1)]}}{1+p} & n \equiv 1 \pmod 4 \\ \frac{x_{-[2k-(m-1)]}(-1+p)}{1+p} & n \equiv 2 \pmod 4 \\ -\frac{x_{-[2k-(m-1)]}}{1+p} & n \equiv 3 \pmod 4 \\ x_{-[2k-(m-1)]} & n \equiv 0 \pmod 4 \end{cases},$$

for $m \equiv 2 (\mathrm{mod}\, 4)$

(4.7)
$$x_{(2k+1)n-(2k+1-m)} = \begin{cases} -x_{-[2k-(m-1)]}(1+p) & n \equiv 1 \pmod 4 \\ -x_{-[2k-(m-1)]} & n \equiv 2 \pmod 4 \\ -x_{-[2k-(m-1)]}(-1+p) & n \equiv 3 \pmod 4 \\ x_{-[2k-(m-1)]} & n \equiv 0 \pmod 4 \end{cases},$$

for $m \equiv 3 (\mathrm{mod}\, 4)$

(4.8) $$x_{(2k+1)n-(2k+1-m)} = \begin{cases} \frac{-x_{-[2k-(m-1)]}}{-1+p} & n \equiv 1 \pmod 4 \\ \frac{x_{-[2k-(m-1)]}(1+p)}{-1+p} & n \equiv 2 \pmod 4 \\ \frac{x_{-[2k-(m-1)]}}{-1+p} & n \equiv 3 \pmod 4 \\ x_{-[2k-(m-1)]} & n \equiv 0 \pmod 4 \end{cases},$$

for $m \equiv 0 (\mathrm{mod}\, 4)$

(4.9) $$x_{(2k+1)n-(2k+1-m)} = \begin{cases} x_{-[2k-(m-1)]}(-1+p) & n \equiv 1 \pmod 4 \\ -x_{-[2k-(m-1)]} & n \equiv 2 \pmod 4 \\ x_{-[2k-(m-1)]}(1+p) & n \equiv 3 \pmod 4 \\ x_{-[2k-(m-1)]} & n \equiv 0 \pmod 4 \end{cases}.$$

From Eq.(4.1), we write

$$x_{(2k+1)n+m} = \frac{(-1)^{(2k+1)n+(1-m)} x_{(2k+1)n-(2k+1-m)}}{1 + (-1)^{(2k+1)n+(1-m)} \prod_{i=0}^{2k} x_{(2k+1)n-(i+1-m)}}.$$

Then

$$x_{(2k+1)n+m} = \frac{-x_{(2k+1)n-(2k+1-m)}}{1 - \prod_{i=0}^{2k} x_{(2k+1)n-(i+1-m)}},$$

for $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$.

Since $m = 4s + 1$, we have
$x_{(2k+1)n+m} = \frac{-x_{(2k+1)n-(2k-4s)}}{1-x_{(2k+1)n-(-4s)}x_{(2k+1)n-(1-4s)}\cdots x_{(2k+1)n-(2k-4s)}}$.

From equalities (4.6), (4.7), (4.8), (4.9) and since $k \equiv 1(\bmod 2)$ and $m \leq 2k + 1$ we have

$$x_{(2k+1)n+m} = \frac{-\frac{x_{-[2k-(m-1)]}}{1+p}}{1-\frac{p}{-1+p}} = \frac{-\frac{x_{-[2k-(m-1)]}}{1+p}}{\frac{-1}{-1+p}}.$$

That is, we have

$$x_{(2k+1)n+m} = \frac{x_{-[2k-(m-1)]}(-1+p)}{1+p}. \tag{4.10}$$

Similarly, we have
$x_{(2k+1)n+m} = \frac{x_{(2k+1)n-(2k+1-m)}}{1+\prod_{i=0}^{2k} x_{(2k+1)n-(i+1-m)}}$

for $m \equiv 2(\bmod 4)$ and $n \equiv 1(\bmod 4)$.
Since $m = 4s + 2$, we have
$x_{(2k+1)n+m} = \frac{x_{(2k+1)n-(2k-1-4s)}}{1+x_{(2k+1)n+1-(-4s)}x_{(2k+1)n-(-4s)}\cdots x_{(2k+1)n-(2k-1-4s)}}$.

From equalities (4.6), (4.7), (4.8), (4.9), (4.10) and since $k \equiv 1(\bmod 2)$ and $m \leq 2k + 1$ we have

$x_{(2k+1)n+m} = \frac{-x_{-[2k-(m-1)]}(1+p)}{1+p}$.

Then

$$x_{(2k+1)n+m} = -x_{-[2k-(m-1)]}. \tag{4.11}$$

Also, for $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, we obtain
$$x_{(2k+1)n+m} = \frac{-x_{(2k+1)n-(2k+1-m)}}{1-\prod_{i=0}^{2k} x_{(2k+1)n-(i+1-m)}}$$
$$= \frac{-x_{(2k+1)n-(2k-2-4s)}}{1-x_{(2k+1)n+2-(-4s)}x_{(2k+1)n+1-(-4s)}\cdots x_{(2k+1)n-(2k-2-4s)}}.$$

From equalities (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11), we have

$$x_{(2k+1)n+m} = \frac{-\frac{x_{-[2k-(m-1)]}}{-1+p}}{1-\frac{p}{1+p}}.$$

Then

(4.12)
$$x_{(2k+1)n+m} = \frac{-x_{-[2k-(m-1)]}(1+p)}{-1+p}.$$

The last, for $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$, we get

$$x_{(2k+1)n+m} = \frac{x_{(2k+1)n-(2k+1-m)}}{1+\prod_{i=0}^{2k} x_{(2k+1)n-(i+1-m)}}$$

$$= \frac{x_{(2k+1)n-(2k-3-4s)}}{1+x_{(2k+1)n+3-(-4s)}x_{(2k+1)n+2-(-4s)}\cdots x_{(2k+1)n-(2k-3-4s)}}.$$

From equalities (4.6), (4.7), (4.8), (4.9), (4.10), (4.11) and (4.13), we have

$$x_{(2k+1)n+m} = \frac{-x_{-[2k-(m-1)]}(1-p)}{1-p}.$$

Then

(4.13)
$$x_{(2k+1)n+m} = -x_{-[2k-(m-1)]}.$$

Thus, we have proved (4.2), (4.3), (4.4) and (4.5) for $n \equiv 1 \pmod 4$. Similarly one can prove (4.2), (4.3), (4.4) and (4.5) for $n = 4l+2, n = 4l+3$ and $n = 4l$ where $l \in Z^+$.

Also we give the following corollary in view of Theorem 5. □

Corollary 1. *Every solution of Eq.*(4.1) *is periodic with prime period* $8k+4$.

Theorem 6. *Assume that* $k > 1, x_{-2k}, x_{-(2k-1)}, \ldots, x_0 > 0$ *and* $\prod_{i=0}^{2k} x_{-i} > 1$. *Then a solution* $\{x_n\}_{n=-2k}^{\infty}$ *of Eq.*(4.1) *is oscillates about* $\overline{x} = 0$ *and for the solutions of Eq.*(4.1), *the following statements are true:*

(*i*) *Every positive semicycle consists of one, two or 2k+3 terms;*

(ii) *Every negative semicycle consists of one or two terms;*

(iii) *Every positive semicycle of length one is followed by a negative semicycle of length one or two;*

(iv) *Every positive semicycle of length two is followed by a negative semicycle of length two;*

(v) *Every positive semicycle of length 2k+3 is followed by a negative semicycle of length two;*

(vi) *Every negative semicycle of length one is followed by a positive semicycle of length one;*

(vii) *Every negative semicycle of length two is followed by a positive semicycle of length one, two or 2k+3.*

Proof. Since $x_{-2k}, x_{-(2k-1)}, ..., x_0 > \overline{x}$ and $\prod_{i=0}^{2k} x_{-i} > 1$, we have the following inequalities for $\overline{x} = 0$

$x_1, x_5, x_9, ..., x_{2k-1} > \overline{x},\ x_2, x_6, x_{10}, ..., x_{2k} < \overline{x},$
$x_3, x_7, x_{11}, ..., x_{2k+1} < \overline{x}, x_4, x_8, x_{12}, ..., x_{2k-2} > \overline{x},$
$x_{2k+2}, x_{2k+6}, x_{2k+10}, ..., x_{4k} > \overline{x}, x_{2k+3}, x_{2k+7}, x_{2k+11}, ..., x_{4k+1} < \overline{x},$
$x_{2k+4}, x_{2k+8}, x_{2k+12}, ..., x_{4k+2} > \overline{x}, x_{2k+5}, x_{2k+9}, x_{2k+13}, ..., x_{4k-1} < \overline{x},$
$x_{4k+3}, x_{4k+7}, x_{4k+11}, ..., x_{6k+1} < \overline{x}, x_{4k+4}, x_{4k+8}, x_{4k+12}, ..., x_{6k+2} < \overline{x},$
$x_{4k+5}, x_{4k+9}, x_{4k+13}, ..., x_{6k+3} > \overline{x}, x_{4k+6}, x_{4k+10}, x_{4k+14}, ..., x_{6k} > \overline{x},$
$x_{6k+4}, x_{6k+8}, x_{6k+12}, ..., x_{8k+2} > \overline{x}, x_{6k+5}, x_{6k+9}, x_{6k+13}, ..., x_{8k+3} > \overline{x},$
$x_{6k+6}, x_{6k+10}, x_{6k+14}, ..., x_{8k+4} > \overline{x}, x_{6k+7}, x_{6k+11}, x_{6k+15}, ..., x_{8k+1} > \overline{x},$

from (4.2), (4.3), (4.4) and (4.5). Since the solution $\{x_n\}_{n=-2k}^{\infty}$ of Eq.(4.1) is periodic with prime $8k+4$ and the above inequalities, the solution $\{x_n\}_{n=-2k}^{\infty}$ of Eq.(4.1) is oscillates about $\overline{x} = 0$. Also it is seen that (i), (ii), (iii), (iv), (v), (vi) and (vii) are true. □

4.2. On the Difference Equation $x_{n+1} = \frac{Ax_{n-2k}}{-A+\prod_{i=0}^{2k} x_{n-i}}$. When we take $l = k, B = -A$ and $C = 1$ in Eq.(1.1), we obtain the equation

$$x_{n+1} = \frac{Ax_{n-2k}}{-A + \prod_{i=0}^{2k} x_{n-i}}, \quad n = 0, 1, ..., \tag{4.14}$$

where k is a positive integer and initial conditions are non zero real numbers with $\prod_{i=0}^{2k} x_{-i} \neq A$.

Theorem 7. *Assume that* $\prod_{i=0}^{2k} x_{-i} = p$ *and* $p \neq A$. *Let* $\{x_n\}_{n=-2k}^{\infty}$ *be a solution of Eq.*(4.14). *Then for* $n = 0, 1, ...$

$x_{2(2k+1)n+1} = \frac{Ax_{-2k}}{-A+p}, x_{2(2k+1)n+2} = \frac{1}{A}x_{-(2k-1)}(-A+p),$

$x_{2(2k+1)n+3} = \frac{Ax_{-(2k-2)}}{-A+p}, x_{2(2k+1)n+4} = \frac{1}{A}x_{-(2k-3)}(-A+p),$

...

$x_{2(2k+1)n+2k} = \frac{1}{A}x_{-1}(-A+p), x_{2(2k+1)n+2k+1} = \frac{Ax_0}{-A+p},$

$x_{2(2k+1)n+2k+2} = x_{-2k}, x_{2(2k+1)n+2k+3} = x_{-(2k-1)},$

...

$$x_{2(2k+1)n+4k+1} = x_{-1}, x_{2(2k+1)n+4k+2} = x_0.$$

Proof. For $n = 0$ the result holds. Now assume that $n > 0$ and that our assumption holds for $n-1$. Then

$$x_{2(2k+1)n-4k-1} = \frac{Ax_{-2k}}{-A+p}, x_{2(2k+1)n-4k} = \frac{1}{A}x_{-(2k-1)}(-A+p),$$

$$x_{2(2k+1)n-4k+1} = \frac{Ax_{-(2k-2)}}{-A+p}, x_{2(2k+1)n-4k+2} = \frac{1}{A}x_{-(2k-3)}(-A+p),$$

$$\dots$$

$$x_{2(2k+1)n-2k-2} = \frac{1}{A}x_{-1}(-A+p), x_{2(2k+1)n-2k-1} = \frac{Ax_0}{-A+p},$$

$$x_{2(2k+1)n-2k} = x_{-2k}, x_{2(2k+1)n-2k+1} = x_{-(2k-1)},$$

$$\dots$$

$$x_{2(2k+1)n-1} = x_{-1}, x_{2(2k+1)n} = x_0.$$

It follows from Eq.(4.14) that

$$x_{2(2k+1)n+1} = \frac{Ax_{2(2k+1)n-2k}}{-A+\prod_{i=0}^{2k} x_{2(2k+1)n-i}} = \frac{Ax_{-2k}}{-A+x_0x_{-1}\dots x_{-2k}}.$$

Hence, we have

$$x_{2(2k+1)n+1} = \frac{Ax_{-2k}}{-A+p}.$$

Also, we get from Eq.(4.14) that

$$x_{2(2k+1)n+2} = \frac{Ax_{2(2k+1)n-(2k-1)}}{-A+\prod_{i=0}^{2k} x_{2(2k+1)n+1-i}} = \frac{Ax_{-(2k-1)}}{-A+\frac{Ax_{-2k}}{-A+p}x_0x_{-1}\dots x_{-(2k-1)}} = \frac{Ax_{-(2k-1)}}{-A+\frac{Ap}{-A+p}}.$$

Hence, we have

$$x_{2(2k+1)n+1} = \frac{1}{A}x_{-(2k-1)}(-A+p).$$

Similarly, one can obtain the other cases. Thus, the proof is complete. □

Theorem 8. *Eq.(4.14) has a periodic solutions of period $(2k+1)$ iff $p = 2A$ and will be take the form $\{x_{-2k}, x_{-(2k-1)}, \dots, x_{-1}, x_0, x_1, x_2, \dots, x_{2k+1}, \dots\}$. If $p \neq 2A$, then periodicity number is $(4k+2)$.*

Proof. Firstly, assume that there exists a prime period $(2k+1)$ solution $x_{-2k}, x_{-(2k-1)}, \dots, x_{-1}, x_0, x_1, x_2, \dots, x_{2k+1}, \dots$ of Eq.(4.14).
We have from the form of solution of Eq.(4.14) that

$$x_{-2k} = \frac{Ax_{-2k}}{-A+p}, x_{-(2k-1)} = \frac{1}{A}x_{-(2k-1)}(-A+p), x_{-(2k-2)} = \frac{Ax_{-(2k-2)}}{-A+p},$$

$$x_{-(2k-3)} = \frac{1}{A}x_{-(2k-3)}(-A+p), \dots, x_{-1} = \frac{1}{A}x_{-1}(-A+p), x_0 = \frac{Ax_0}{-A+p}.$$

Then $p = 2A$.

Secondly, suppose that $p = 2A$. Then we have

$x_{2(2k+1)n+1} = x_{-2k}, x_{2(2k+1)n+2} = x_{-(2k-1)}, ..., x_{2(2k+1)n+2k+1} = x_0,$

$x_{2(2k+1)n+2k+2} = x_{-2k}, x_{2(2k+1)n+2k+3} = x_{-(2k-1)}, ..., x_{2(2k+1)n+2(2k+1)} = x_0.$

Thus, we obtain a period $(2k+1)$ solution.

Lastly, assume that $p \neq 2A$. Then we see that periodicity number is $(4k+2)$ from the form of solution of Eq.(4.14).

The proof is complete. □

4.3. On the Difference Equation $x_{n+1} = \frac{Ax_{n-2k}}{-A-\prod_{i=0}^{2k} x_{n-i}}$. In this section we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{Ax_{n-2k}}{-A-\prod_{i=0}^{2k} x_{n-i}}, \quad n = 0, 1, ..., \tag{4.15}$$

where k is a positive integer and initial conditions are non zero real numbers with $\prod_{i=0}^{2k} x_{-i} \neq -A$.

Theorem 9. *Assume that* $\prod_{i=0}^{2k} x_{-i} = p$ *and* $p \neq -A$. *Let* $\{x_n\}_{n=-2k}^{\infty}$ *be a solution of Eq.*(4.15). *Then for* $n = 0, 1, ...$

$x_{2(2k+1)n+1} = \frac{Ax_{-2k}}{-A-p}, x_{2(2k+1)n+2} = \frac{1}{A}x_{-(2k-1)}\left(-A-p\right),$

$x_{2(2k+1)n+3} = \frac{Ax_{-(2k-2)}}{-A-p}, x_{2(2k+1)n+4} = \frac{1}{A}x_{-(2k-3)}\left(-A-p\right),$

$...$

$x_{2(2k+1)n+2k} = \frac{1}{A}x_{-1}\left(-A-p\right), x_{2(2k+1)n+2k+1} = \frac{Ax_0}{-A-p},$

$x_{2(2k+1)n+2k+2} = x_{-2k}, x_{2(2k+1)n+2k+3} = x_{-(2k-1)},$

$...$

$x_{2(2k+1)n+4k+1} = x_{-1}, x_{2(2k+1)n+4k+2} = x_0.$

Proof. The proof is similar to Theorem 7 and it will be omitted. □

Theorem 10. *Eq.*(4.15) *has a periodic solutions of period* $(2k+1)$ *iff* $p = -2A$ *and will be take the form* $\{x_{-2k}, x_{-(2k-1)}, ..., x_{-1}, x_0, x_1, x_2, ..., x_{2k+1}, ...\}$. *If* $p \neq -2A$, *then periodicity number is* $(4k+2)$.

Proof. The proof is similar to Theorem 8 and it will be omitted. □

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