CLOSED FORMULAE FOR GAUSS HYPERGEOMETRIC SERIES

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ABSTRACT. Combining integration method with series rearrangement, we establish several closed formulae for Gauss hypergeometric series with four free parameters, which extend essentially the related results found recently by Elsner (2005).

1. INTRODUCTION

For a complex number a and a nonnegative integer n, the shifted factorial is defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)\cdots(a+n-1) \quad \text{for} \quad n > 0.$$

Then we have the Gauss hypergeometric series

$$F[a,b;c;z] := {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| z\right] := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n, \tag{1.1}$$

where it is assumed that c is not a negative integer. When $z = 1$, the series is convergent provided that $\mathrm{Re}(c-a-b) > 0$ and evaluated by the well known Gauss summation formula:

$${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.2}$$

If $|z| < 1$, the Gauss hypergeometric series converges for arbitrary complex numbers a, b and c, but there does not exist closed formula in general. By applying Zeilberger's algorithm and Maple programs, Elsner [6] found recently the recurrence formulae of several classes of infinity sums. For

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example, with m and n being positive integers subject to $m \geq 2n$, the Gauss hypergeometric series of the following types

$$ {}_2F_1\left[\begin{matrix}1, & 1 \\ & m+1/2\end{matrix}\,\Big|\, \pm\frac{1}{4}\right] \quad \text{and} \quad {}_2F_1\left[\begin{matrix}n, & n+1 \\ & m+1/2\end{matrix}\,\Big|\, \pm\frac{1}{4}\right] $$

has been considered by Elsner [6]. By means of series rearrangement and integration method, this note will establish several closed formulae for Gauss hypergeometric series with four free parameters, which extend essentially the related results due to Elsner [6].

2. THE MAIN THEOREMS

Theorem 2.1. *Let n, ℓ and m be nonnegative integers with $m > n+\ell$. For any real number $a > 1$, we then have*

$$ {}_2F_1\left[\begin{matrix}n+1, & \ell+1 \\ & m+1/2\end{matrix}\,\Big|\, \frac{1}{a}\right] = p_1(a) + q_1(a)\sqrt{a-1}\,\operatorname{arccot}\sqrt{a-1}, \tag{2.1} $$

where

$$ p_1(a) = 2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}\Bigg\{\sum_{k=0}^{\ell}\sum_{i=0}^{n-1}(-1)^{\ell-k-1}\frac{\binom{\ell}{k}\binom{m-k-\frac{3}{2}}{i}}{2na^{n-i}\binom{n-1}{i}} $$

$$ +\sum_{k=0}^{\ell}\sum_{j=0}^{m-n-k-2}\frac{(-1)^{\ell-k}(1-a)^j}{2m-2n-2k-2j-3}\binom{\ell}{k}\binom{m-k-\frac{3}{2}}{n}\Bigg\}, $$

$$ q_1(a) = (-1)^{m-n-\ell-1}2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}\sum_{k=0}^{\ell}\binom{\ell}{k}\binom{m-k-\frac{3}{2}}{n}(a-1)^{m-n-k-2}. $$

Proof. By means of Euler's integral representation [1, §1.5, Eq. (1)]

$$ {}_2F_1\left[\begin{matrix}a, & b \\ & c\end{matrix}\,\Big|\, z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \tag{2.2} $$

we get the integral representation:

$$ {}_2F_1\left[\begin{matrix}n+1, \ell+1 \\ m+1/2\end{matrix}\,\Big|\, \frac{1}{a}\right] = \frac{a^{n+1}\Gamma(m+\frac{1}{2})}{\Gamma(\ell+1)\Gamma(m-\ell-\frac{1}{2})}\int_0^1 \frac{t^\ell(1-t)^{m-\ell-\frac{3}{2}}}{(a-t)^{n+1}}\,\mathrm{d}t. $$

Performing the replacement $t = 1 - u^2$ in the above integral, we have

$$ {}_2F_1\left[\begin{matrix} n+1, \ell+1 \\ m+1/2 \end{matrix} \,\middle|\, \frac{1}{a}\right] = 2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}\int_0^1 \frac{(1-u^2)^\ell u^{2m-2\ell-2}}{(a-1+u^2)^{n+1}}\,du $$

$$ = 2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}\sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}\int_0^1 \frac{u^{2(m-k-1)}}{(a-1+u^2)^{n+1}}\,du. \qquad (2.3) $$

For positive integers r and n, let

$$ I(r,n) = \int_0^1 \frac{u^{2r}}{(a-1+u^2)^n}\,du. $$

Through integration by parts, we get the following recurrence relation:

$$ \begin{aligned} I(r,n) &= \left.\frac{-u^{2r-1}}{2(n-1)(a-1+u^2)^{n-1}}\right|_0^1 + \frac{2r-1}{2(n-1)}\int_0^1 \frac{u^{2r-2}}{(a-1+u^2)^{n-1}}\,du \\ &= -\frac{1}{2(n-1)a^{n-1}} + \frac{2r-1}{2(n-1)}\,I(r-1,n-1). \end{aligned} $$

If $r \geq n-1$, iterating the last relation $n-1$ times and then simplifying the result, we obtain

$$ I(r,n) = \frac{-1}{2(n-1)}\sum_{i=0}^{n-2}\frac{\binom{r-\frac{1}{2}}{i}}{a^{n-i-1}\binom{n-2}{i}} + \binom{r-\frac{1}{2}}{n-1} I(r-n+1,1). \qquad (2.4) $$

With s being positive integer, we can derive another recurrence relation:

$$ \begin{aligned} I(s,1) &= \int_0^1 u^{2s-2}du - (a-1)\int_0^1 \frac{u^{2s-2}}{a-1+u^2}\,du \\ &= \frac{1}{2s-1} - (a-1)\,I(s-1,1). \end{aligned} $$

Iterating this relation s times, we find

$$ I(s,1) = \sum_{j=0}^{s-1}\frac{(1-a)^j}{2s-2j-1} + (1-a)^s\,I(0,1), \qquad (2.5) $$

where

$$ I(0,1) = \int_0^1 \frac{du}{a-1+u^2} = \frac{\operatorname{arccot}\sqrt{a-1}}{\sqrt{a-1}}. $$

When $m > n+\ell$, the combination of (2.4), (2.5) and $I(0,1)$ leads us to the following identity

$$I(m-k-1,n+1) = -\frac{1}{2n}\sum_{i=0}^{n-1}\frac{\binom{m-k-\frac{3}{2}}{i}}{a^{n-i}\binom{n-1}{i}} + \binom{m-k-\frac{3}{2}}{n} I(m-n-k-1,1)$$

$$= -\frac{1}{2n}\sum_{i=0}^{n-1}\frac{\binom{m-k-\frac{3}{2}}{i}}{a^{n-i}\binom{n-1}{i}} + \binom{m-k-\frac{3}{2}}{n}\sum_{j=0}^{m-n-k-2}\frac{(1-a)^j}{2m-2n-2k-2j-3}$$

$$+\binom{m-k-\frac{3}{2}}{n}\frac{(1-a)^{m-n-k-1}}{\sqrt{a-1}}\operatorname{arccot}\sqrt{a-1}.$$

Substituting the last identity into (2.3) and simplifying the result, we obtain the identity displayed in (2.1). □

Remark 2.2. By rewriting ${}_2F_1[n+1,\ell+1;m+1/2;1/a]$ as

$$\frac{(2m-1)!!}{n!\,\ell!}\sum_{k=0}^{\infty}\frac{(k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}}\left(\frac{4}{a}\right)^k$$

we may state formula (2.1) in terms of infinite series identity

$$\sum_{k=0}^{\infty}\frac{(k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}}\left(\frac{4}{a}\right)^k$$
$$= \frac{n!\,\ell!}{(2m-1)!!}\left\{p_1(a)+q_1(a)\sqrt{a-1}\operatorname{arccot}\sqrt{a-1}\right\}. \qquad (2.6)$$

Its special cases for $a = 4,\ 2,\ 4/3$ may be displayed respectively as

$$\sum_{k=0}^{\infty}\frac{(k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}} = \frac{n!\,\ell!}{(2m-1)!!}\left\{p_1(4)+q_1(4)\frac{\sqrt{3}\pi}{6}\right\},$$

$$\sum_{k=0}^{\infty}\frac{2^k(k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}} = \frac{n!\,\ell!}{(2m-1)!!}\left\{p_1(2)+q_1(2)\frac{\pi}{4}\right\},$$

$$\sum_{k=0}^{\infty}\frac{3^k(k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}} = \frac{n!\,\ell!}{(2m-1)!!}\left\{p_1(\tfrac{4}{3})+q_1(\tfrac{4}{3})\frac{\sqrt{3}\pi}{9}\right\}.$$

We point out that Elsner's Theorem 4 (i) can be recovered from (2.6) by letting $n = \ell = 0$, $a = 4$ and $m \to m+1$:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)\cdots(2k+2m+1)\binom{2k}{k}}$$

$$= \frac{4}{(2m-1)!!}\left\{ (-1)^m 3^{m-\frac{3}{2}} \frac{\pi}{2} + \sum_{\nu=0}^{m-1} \frac{(-3)^\nu}{2m-2\nu-1} \right\}.$$

Similarly, setting $\ell = n-1$, $a = 4$ in (2.6), we find Elsner's Theorem 6 which does not give explicit formula.

Theorem 2.3. *Let n, ℓ and m be nonnegative integers and $m > n + \ell$. For any real number $a > 1$, we then have*

$$ {}_2F_1\left[\begin{matrix} n+1, & \ell+1 \\ & m+1/2 \end{matrix} \,\middle|\, -\frac{1}{a}\right] = p_2(a) + q_2(a)\sqrt{1+a}\,\ln\frac{1+\sqrt{1+a}}{\sqrt{a}}, \qquad (2.7)$$

where

$$p_2(a) = 2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}\left\{\sum_{k=0}^{\ell}\sum_{i=0}^{n-1}(-1)^{\ell-k-i}\frac{\binom{\ell}{k}\binom{m-k-\frac{3}{2}}{i}}{2na^{n-i}\binom{n-1}{i}}\right.$$

$$\left.+\sum_{k=0}^{\ell}\sum_{j=0}^{m-n-k-2}\frac{(-1)^{n+\ell-k+1}(1+a)^j}{2m-2n-2k-2j-3}\binom{\ell}{k}\binom{m-k-\frac{3}{2}}{n}\right\},$$

$$q_2(a) = 2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}\sum_{k=0}^{\ell}(-1)^{n+\ell-k}\binom{\ell}{k}\binom{m-k-\frac{3}{2}}{n}(1+a)^{m-n-k-2}.$$

Proof. Similarly to the proof of Theorem 2.1, we have the following identity

$$ {}_2F_1\left[\begin{matrix} n+1, \ell+1 \\ m+1/2 \end{matrix} \,\middle|\, -\frac{1}{a}\right] = 2(\ell+1)a^{n+1}\binom{m-\frac{1}{2}}{\ell+1}$$

$$\times\sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}\int_0^1\frac{u^{2(m-k-1)}}{(1+a-u^2)^{n+1}}\,du. \qquad (2.8)$$

For positive integers r and n, let

$$J(r,n) = \int_0^1 \frac{u^{2r}}{(1+a-u^2)^n}\,du.$$

By applying integration by parts, we get the recurrence relation:

$$J(r,n) = \frac{u^{2r-1}}{2(n-1)(1+a-u^2)^{n-1}}\bigg|_0^1 - \frac{2r-1}{2(n-1)}\int_0^1 \frac{u^{2r-2}}{(1+a-u^2)^{n-1}}\,du$$

$$= \frac{1}{2(n-1)a^{n-1}} - \frac{2r-1}{2(n-1)}J(r-1,n-1).$$

Supposing $r \geq n-1$, iterating this relation $n-1$ times and then simplifying the result, we have

$$J(r,n) = \sum_{i=0}^{n-2} \frac{(-1)^i\binom{r-\frac{1}{2}}{i}}{2(n-1)a^{n-i-1}\binom{n-2}{i}} + (-1)^n\binom{r-\frac{1}{2}}{n-1}J(r-n+1,1). \quad (2.9)$$

Again for a positive integer s, we have

$$\begin{aligned} J(s,1) &= -\frac{1}{2s-1} + (1+a)J(s-1,1) \\ &= -\sum_{j=0}^{s-1}\frac{(1+a)^j}{2s-2j-1} + (1+a)^s J(0,1) \qquad (2.10) \end{aligned}$$

where

$$J(0,1) = \int_0^1 \frac{du}{1+a-u^2} = \frac{1}{\sqrt{1+a}}\ln\frac{1+\sqrt{1+a}}{\sqrt{1+a}}.$$

If $m > n+\ell$, we derive from (2.9), (2.10) and $J(0,1)$ the following formula

$$J(m-k-1,n+1) = \sum_{i=0}^{n-1}\frac{(-1)^i\binom{m-k-\frac{3}{2}}{i}}{2na^{n-i}\binom{n-1}{i}} + (-1)^n\binom{m-k-\frac{3}{2}}{n}J(m-n-k-1,1)$$

$$= \sum_{i=0}^{n-1}\frac{(-1)^i\binom{m-k-\frac{3}{2}}{i}}{2na^{n-i}\binom{n-1}{i}} + (-1)^{n+1}\binom{m-k-\frac{3}{2}}{n}\sum_{j=0}^{m-n-k-2}\frac{(1+a)^j}{2m-2n-2k-2j-3}$$

$$+ (-1)^n\binom{m-k-\frac{3}{2}}{n}\frac{(1+a)^{m-n-k-1}}{\sqrt{1+a}}\ln\frac{1+\sqrt{1+a}}{\sqrt{a}}.$$

Replacing the integrals in (2.8) by the last expression and then simplifying the result, we get identity (2.7). □

Remark 2.4. Similarly, rewriting (2.7) in terms of infinite series identity

$$\sum_{k=0}^{\infty}\frac{(k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}}\left(-\frac{4}{a}\right)^k$$

$$= \frac{n!\,\ell!}{(2m-1)!!}\left\{p_2(a) + q_2(a)\sqrt{1+a}\ln\frac{1+\sqrt{1+a}}{\sqrt{a}}\right\}. \qquad (2.11)$$

we may state its special cases $a = 4,\ 2,\ 4/3,\ 8$ as follows:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}} = \frac{n!\,\ell!}{(2m-1)!!}\Big\{p_2(4) + q_2(4)\sqrt{5}\,\ln\tfrac{1+\sqrt{5}}{2}\Big\},$$

$$\sum_{k=0}^{\infty} \frac{(-2)^k (k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}} = \frac{n!\,\ell!}{(2m-1)!!}\Big\{p_2(2) + q_2(2)\sqrt{3}\,\ln\tfrac{1+\sqrt{3}}{\sqrt{2}}\Big\},$$

$$\sum_{k=0}^{\infty} \frac{(-3)^k (k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}} = \frac{n!\,\ell!}{(2m-1)!!}\Big\{p_2(\tfrac{4}{3}) + q_2(\tfrac{4}{3})\tfrac{\sqrt{21}}{3}\,\ln\tfrac{\sqrt{3}+\sqrt{7}}{2}\Big\},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (k+1)_n\,(k+1)_\ell}{(2k+1)(2k+3)\cdots(2k+2m-1)\binom{2k}{k}2^k} = \frac{n!\,\ell!}{(2m-1)!!}\Big\{p_2(8) + \frac{3}{2}\,q_2(8)\,\ln 2\Big\}.$$

We note also that Elsner's Theorem 4 (iii) is in fact the case $n = \ell = 0$, $a = 4$, and $m \to m+1$ of (2.11):

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)\cdots(2k+2m+1)\binom{2k}{k}} = \frac{4}{(2m-1)!!}\Big\{5^{m-\frac{1}{2}}\ln\tfrac{1+\sqrt{5}}{2} - \sum_{\nu=0}^{m-1}\frac{5^\nu}{2m-2\nu-1}\Big\}.$$

In addition, putting $\ell = n-1$ and $a = 4$ in the formula (2.6), we prove Elsner's Theorem 6 which does not give explicit formula.

Theorem 2.5. *Let n, ℓ and m be nonnegative integers and $m > \ell \geq n$, For any real number $a > 1$, we then have*

$$ {}_2F_1\left[\begin{matrix} n+1, & \ell+1/2 \\ & m+1/2 \end{matrix}\,\Big|\,\frac{1}{a}\right] = p_3(a) + q_3(a)\sqrt{a}\,\ln\frac{1+\sqrt{a}}{\sqrt{a-1}}, \tag{2.12}$$

where

$$p_3(a) = (-1)^{m-\ell}\,2(m-\ell)a^{n+1}\binom{m-\frac{1}{2}}{m-\ell}$$

$$\times\Bigg\{\sum_{k=0}^{m-\ell-1}\sum_{i=0}^{n-1}(-1)^{k+i+1}\frac{\binom{m-\ell-1}{k}\binom{m-k-\frac{3}{2}}{i}}{2n\,(a-1)^{n-i}\binom{n-1}{i}}$$

$$+\sum_{k=0}^{m-\ell-1}\sum_{j=0}^{m-n-k-2}\frac{(-1)^{n-k}\,a^j}{2m-2n-2k-2j-3}\binom{m-\ell-1}{k}\binom{m-k-\frac{3}{2}}{n}\Bigg\},$$

$$q_3(a) = (-1)^{m+n-\ell-1}\,2(m-\ell)\binom{m-\frac{1}{2}}{m-\ell}\sum_{k=0}^{m-\ell-1}(-1)^k\binom{m-\ell-1}{k}\binom{m-k-\frac{3}{2}}{n}a^{m-k-1}.$$

Proof. According to (2.2), we have

$$ {}_2F_1\left[\begin{matrix} n+1, \ell+1/2 \\ m+1/2 \end{matrix} \,\middle|\, \frac{1}{a}\right] = \frac{a^{n+1}\Gamma(m+\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})\Gamma(m-\ell)} \int_0^1 \frac{t^{\ell-\frac{1}{2}}(1-t)^{m-\ell-1}}{(a-t)^{n+1}}\,\mathrm{d}t. $$

Performing the replacement $t = u^2$ in the above integral and then simplify the result, we see that the above equation becomes

$$ {}_2F_1\left[\begin{matrix} n+1, \ell+1/2 \\ m+1/2 \end{matrix} \,\middle|\, \frac{1}{a}\right] = 2(m-\ell)a^{n+1}\binom{m-\frac{1}{2}}{m-\ell}\int_0^1 \frac{u^{2\ell}(1-u^2)^{m-\ell-1}}{(a-u^2)^{n+1}}\,\mathrm{d}u $$

$$ = 2(m-\ell)a^{n+1}\binom{m-\frac{1}{2}}{m-\ell}\sum_{k=0}^{m-\ell-1}(-1)^{m-\ell-k-1}\binom{m-\ell-1}{k}\int_0^1 \frac{u^{2(m-k-1)}}{(a-u^2)^{n+1}}\,\mathrm{d}u. \tag{2.13} $$

Following the computation of $J(m-k-1, n+1)$, we can establish, for $m > \ell \geq n$, the following expression

$$ \int_0^1 \frac{u^{2(m-k-1)}}{(a-u^2)^{n+1}}\,\mathrm{d}u = (-1)^n a^{m-n-k-2}\sqrt{a}\binom{m-k-\frac{3}{2}}{n}\ln\frac{1+\sqrt{a}}{\sqrt{a}-1} $$

$$ +\sum_{i=0}^{n-1}\frac{(-1)^i\binom{m-k-\frac{3}{2}}{i}}{2n(a-1)^{n-i}\binom{n-1}{i}} + (-1)^{n+1}\binom{m-k-\frac{3}{2}}{n}\sum_{j=0}^{m-n-k-2}\frac{a^j}{2m-2n-2k-2j-3}. $$

Substituting it into (2.13) and then simplifying the result, we get the identity (2.12). □

Remark 2.6. Noting that formula (2.12) is equivalent to

$$ \sum_{k=0}^{\infty}\frac{\binom{n+k}{k}}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)a^k} = \frac{(2\ell-1)!!}{(2m-1)!!}\left\{p_3(a)+q_3(a)\sqrt{a}\ln\frac{1+\sqrt{a}}{\sqrt{a}-1}\right\}, \tag{2.14} $$

we may display its particular $a = 4,\ 9,\ 5$ respectively as follows:

$$ \sum_{k=0}^{\infty}\frac{\binom{n+k}{k}}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)2^{2k}} = \frac{(2\ell-1)!!}{(2m-1)!!}\left\{p_3(4)+q_3(4)\ln 3\right\}, $$

$$ \sum_{k=0}^{\infty}\frac{\binom{n+k}{k}}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)3^{2k}} = \frac{(2\ell-1)!!}{(2m-1)!!}\left\{p_3(9)+\frac{3}{2}q_3(9)\ln 2\right\}, $$

$$ \sum_{k=0}^{\infty}\frac{\binom{n+k}{k}}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)5^k} = \frac{(2\ell-1)!!}{(2m-1)!!}\left\{p_3(5)+q_3(5)\sqrt{5}\ln\frac{1+\sqrt{5}}{2}\right\}. $$

When $n=0$, identity (2.14) becomes

$$\sum_{k=0}^{\infty}\frac{1}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)a^k}$$

$$=\frac{(-1)^{m-\ell}}{2^{m-\ell-1}(m-\ell-1)!}\left\{\sum_{k=0}^{m-\ell-1}\sum_{j=0}^{m-k-2}\frac{(-1)^k\,a^{j+1}}{2m-2k-2j-3}\binom{m-\ell-1}{k}\right.$$

$$\left.+\sum_{k=0}^{m-\ell-1}(-1)^{k+1}\binom{m-\ell-1}{k}a^{m-k-1}\sqrt{a}\,\ln\frac{1+\sqrt{a}}{\sqrt{a}-1}\right\}.$$

Its further specialization to the case $\ell=0$ reads as follows:

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2k+3)\cdots(2k+2m-1)a^k}$$

$$=\frac{(-1)^{m}}{2^{m-1}(m-1)!}\left\{\sum_{k=0}^{m-2}\sum_{j=0}^{m-k-2}\frac{(-1)^k\,a^{j+1}}{2m-2k-2j-3}\binom{m-1}{k}\right.$$

$$\left.+\sum_{k=0}^{m-1}(-1)^{k+1}\binom{m-1}{k}a^{m-k-1}\sqrt{a}\,\ln\frac{1+\sqrt{a}}{\sqrt{a}-1}\right\}.$$

Theorem 2.7. *Let n, ℓ and m be nonnegative integers and $m>\ell\geq n$, For any real number $a>1$, we then have*

$${}_2F_1\left[\begin{matrix}n+1, & \ell+1/2\\ & m+1/2\end{matrix}\,\middle|\,-\frac{1}{a}\right]=p_4(a)+q_4(a)\sqrt{a}\,\mathrm{arccot}\sqrt{a},\qquad(2.15)$$

where

$$p_4(a)=(-1)^{m-\ell}\,2(m-\ell)a^{n+1}\binom{m-\frac{1}{2}}{m-\ell}\left\{\sum_{k=0}^{m-\ell-1}\sum_{i=0}^{n-1}(-1)^k\frac{\binom{m-\ell-1}{k}\binom{m-k-\frac{3}{2}}{i}}{2n\,(a+1)^{n-i}\binom{n-1}{i}}\right.$$

$$\left.+\sum_{k=0}^{m-\ell-1}\sum_{j=0}^{m-n-k-2}\frac{(-1)^{k+j+1}\,a^j}{2m-2n-2k-2j-3}\binom{m-\ell-1}{k}\binom{m-k-\frac{3}{2}}{n}\right\},$$

$$q_4(a)=(-1)^{n-\ell}\,2(m-\ell)\binom{m-\frac{1}{2}}{m-\ell}\sum_{k=0}^{m-\ell-1}\binom{m-\ell-1}{k}\binom{m-k-\frac{3}{2}}{n}a^{m-k-1}.$$

Proof. Similar to the proof of Theorem 2.5, we get the following identity by applying formula (2.2)

$$ {}_2F_1\left[\begin{matrix} n+1, \ell+1/2 \\ m+1/2 \end{matrix} \,\Big|\, -\frac{1}{a}\right] = 2(m-\ell)a^{n+1}\binom{m-\frac{1}{2}}{m-\ell} $$

$$ \times \sum_{k=0}^{m-\ell-1} (-1)^{m-\ell-k-1}\binom{m-\ell-1}{k}\int_0^1 \frac{u^{2(m-k-1)}}{(a+u^2)^{n+1}}\,du. \quad (2.16) $$

In the same manner as the the computation of $I(m-k-1, n+1)$, we can find the following formula with $m > \ell \geq n$

$$ \int_0^1 \frac{u^{2(m-k-1)}}{(a+u^2)^{n+1}}\,du = -(-a)^{m-n-k-2}\binom{m-k-\frac{3}{2}}{n}\sqrt{a}\,\mathrm{arccot}\sqrt{a} $$

$$ -\sum_{i=0}^{n-1} \frac{\binom{m-k-\frac{3}{2}}{i}}{2n(a+1)^{n-i}\binom{n-1}{i}} + \binom{m-k-\frac{3}{2}}{n}\sum_{j=0}^{m-n-k-2} \frac{(-a)^j}{2m-2n-2k-2j-3}. $$

Replacing the integrals in (2.16) by the last expression and then simplifying the result, we establish identity (2.15). □

Remark 2.8. The formula in (2.15) may also be written as

$$ \sum_{k=0}^{\infty} \frac{(-1)^k\binom{n+k}{k}}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)a^k} = \frac{(2\ell-1)!!}{(2m-1)!!}\left\{p_4(a)+q_4(a)\sqrt{a}\,\mathrm{arccot}\sqrt{a}\right\}. \quad (2.17) $$

When $a = 3$, the last identity reads as:

$$ \sum_{k=0}^{\infty} \frac{(-1)^k\binom{n+k}{k}}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)3^k} = \frac{(2\ell-1)!!}{(2m-1)!!}\left\{p_4(3)+q_4(3)\,\frac{\sqrt{3}\pi}{6}\right\}. $$

Putting $n = 0$ in identity (2.17), we have

$$ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2\ell+1)(2k+2\ell+3)\cdots(2k+2m-1)a^k} $$

$$ = \frac{(-1)^\ell}{2^{m-\ell-1}(m-\ell-1)!}\left\{\sum_{k=0}^{m-\ell-1}\binom{m-\ell-1}{k}a^{m-k-1}\sqrt{a}\,\mathrm{arccot}\sqrt{a}\right. $$

$$ \left. + \sum_{k=0}^{m-\ell-1}\sum_{j=0}^{m-k-2} \frac{(-1)^{m+k+j+1}a^{j+1}}{2m-2k-2j-3}\binom{m-\ell-1}{k}\right\}. $$

It may be further specialized by letting $\ell = 0$ to the following identity

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)\cdots(2k+2m-1)\,a^k}$$

$$= \frac{1}{2^{m-1}(m-1)!}\left\{ \sum_{k=0}^{m-1} \binom{m-1}{k} a^{m-k-1}\sqrt{a}\,\mathrm{arccot}\sqrt{a}\right.$$

$$\left. + \sum_{k=0}^{m-2}\sum_{j=0}^{m-k-2} \frac{(-1)^{m+k+j+1}\,a^{j+1}}{2m-2k-2j-3}\binom{m-1}{k}\right\}.$$

Theorem 2.9. *Let n, ℓ and m be nonnegative integers and $m > \ell$, For any real number $a > 1$, we then have*

$$\text{(i)}\quad {}_2F_1\left[\begin{matrix} n+1/2, & \ell+1 \\ & m+1 \end{matrix} \,\middle|\, \frac{1}{a}\right] = 2(m-\ell)\binom{m}{\ell}\sum_{k=0}^{m-1}\sum_{i=0}^{k}(-1)^i a^{\ell-i}$$

$$\times\ (1-a)^{m-\ell-k+i-1}\binom{\ell}{i}\binom{m-\ell-1}{k-i}\frac{a^{k+1}-a^n(a-1)^{k-n}\sqrt{a^2-a}}{2k-2n+1}. \tag{2.18}$$

$$\text{(ii)}\quad {}_2F_1\left[\begin{matrix} n+1/2, & \ell+1 \\ & m+1 \end{matrix} \,\middle|\, -\frac{1}{a}\right] = 2(m-\ell)\binom{m}{\ell}\sum_{k=0}^{m-1}\sum_{i=0}^{k}(-1)^{\ell-k} a^{\ell-i}$$

$$\times\ (1+a)^{m-\ell-k+i-1}\binom{\ell}{i}\binom{m-\ell-1}{k-i}\frac{a^n(1+a)^{k-n}\sqrt{a+a^2}-a^{k+1}}{2k-2n+1}. \tag{2.19}$$

Proof. We prove only part (i) for part (ii) can be done in the same method. First note the following integral representation:

$${}_2F_1\left[\begin{matrix} n+1/2, \ell+1 \\ m+1 \end{matrix} \,\middle|\, \frac{1}{a}\right] = \frac{a^{n+\frac{1}{2}}\Gamma(m+1)}{\Gamma(\ell+1)\Gamma(m-\ell)}\int_0^1 \frac{t^\ell(1-t)^{m-\ell-1}}{(a-t)^{n+1/2}}\,dt$$

$$= 2(m-\ell)a^{n+\frac{1}{2}}\binom{m}{\ell}\int_{\sqrt{a-1}}^{\sqrt{a}} u^{-2n}(a-u^2)^\ell(1-a+u^2)^{m-\ell-1}\,du \tag{2.20}$$

where the replacement $t = a - u^2$ has been made in the third representation.

Then for $m > \ell$, both $(a-u^2)^\ell$ and $(1-a+u^2)^{m-\ell-1}$ can be expanded through the binomial theorem:

$$(a-u^2)^\ell\,(1-a+u^2)^{m-\ell-1}$$

$$=\sum_{i=0}^{\ell}(-1)^i a^{\ell-i}\binom{\ell}{i}u^{2i}\sum_{j=0}^{m-\ell-1}(1-a)^{m-\ell-j-1}\binom{m-\ell-1}{j}u^{2j}$$

$$=\sum_{k=0}^{m-1}\sum_{i=0}^{k}(-1)^i a^{\ell-i}(1-a)^{m-\ell-k+i-1}\binom{\ell}{i}\binom{m-\ell-1}{k-i}u^{2k}$$

$$=\sum_{k=0}^{m-1}\sum_{i=k-m+\ell+1}^{\min\{k,\ell\}}(-1)^i a^{\ell-i}(1-a)^{m-\ell-k+i-1}\binom{\ell}{i}\binom{m-\ell-1}{k-i}u^{2k}.$$

Finally, substituting the last expression into (2.20) and then integrating the result, we get identity (2.18), where the limits of the inner sum with respect to i have been simplified for the presence of zero binomial coefficients. □

Remark 2.10. Other infinite series identities may be derived from setting $a = 2, 4, 5, 9$ in (2.18) and $a = 4, 8$ in formula (2.19). The details will not be produced.

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