NOTES ON ENDOMORPHISMS OF HENSON GRAPHS AND THEIR COMPLEMENTS

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ABSTRACT. We start by proving that the Henson graphs H_n, $n > 3$ (the homogeneous countable graphs universal for the class of all finite graphs omitting the clique of size n), are retract rigid. On the other hand, we provide a full characterization of retracts of the complement of H_3. Further, we prove that each countable partial order embeds in the natural order of retractions of the complements of Henson graphs. Finally, we show that graphs omitting sufficiently large null subgraphs omit certain configurations in their endomorphism monoids.

1. INTRODUCTION

Throughout the paper, all graphs (unless stated otherwise) are simple, undirected and countable. We recall that a *graph homomorphism* $f : G \to H$ is a mapping of vertex sets of G and H which preserves the edges: if $x, y \in V(G)$ are joined by an edge, so are $f(x), f(y) \in V(H)$. Since we are working with simple graphs only, this means in particular that if $f(x) = f(y)$, then x, y must be non-adjacent. Of course, if $H = G$, we obtain the definition of an *endomorphism* of G. A bijective endomorphism is an *automorphism* of G. If $f : G \to H$ is a graph homomorphism, the induced subgraph of H on $f(V(G))$ we simply denote by $f(G)$.

A graph G is *homogeneous* if every isomorphism $\varphi_0 : F_1 \to F_2$ of its finite induced subgraphs F_1, F_2 extends to an automorphism φ of G. Homogeneity is a very important property in mathematical logic and permutation group theory, see, for instance, [4, 13]. Classes of homogeneous graph-like structures which have been determined include countably infinite tournaments (Lachlan [14]), digraphs (Cherlin [7]), finite (Gardiner [11]) and countably infinite (simple undirected) graphs (Lachlan and Woodrow [15]). It is the main result of the latter paper that lies in the focus of our interest here. For $1 \leqslant n \leq \aleph_0$, let K_n denote the complete graph (clique) with n vertices. A countably infinite graph G is homogeneous if and only if G is isomorphic to one of the following:

(a) mK_n, the disjoint union of m copies of K_n, where either $m = \aleph_0$, or $n = \aleph_0$;
(b) the complements of the graphs from (a);
(c) the infinite random graph R (see [5] for a survey);
(d) the *Henson graphs* H_n, $n \geqslant 3$ (constructed first in [12]);
(e) the complements of Henson graphs.

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By a fundamental result of Fraïssé [9] (see also [10, 13]), a homogeneous graph G is uniquely determined by its *age*, the class of all finite subgraphs which embed in G. It is then usually said that G is the *Fraïssé limit* of its age. In the terminology just described, the infinite random graph R is the Fraïssé limit of the class of all finite graphs, H_n is the limit of all finite graphs omitting K_n, and consequently, the complement $\overline{H}_n$ is the limit of all finite graphs omitting the null graph $\overline{K}_n$.

Clearly, the graphs from (a) and (b) are sporadic and very easy to study. The infinite random graph R (also called the Erdős-Rado graph) was already a subject of thorough investigations: we direct to [5] for an overview of results, or to [6] for a number of more recent ones. In addition, let us mention a recent result [3] that $T_{\aleph_0}$, the full transformation monoid on an infinitely countable set, embeds in $End(R)$, and so does every countable semigroup.

It is the aim of this paper to investigate the analogous representation problem for Henson graphs and their complements. We provide some useful information on countable semigroups that embed in $End(H_n)$ and $End(\overline{H}_n)$, $n \geqslant 3$.

In understanding the structure of a semigroup, it is often an important step to consider its idempotents. If we are concerned with the endomorphism monoid $End(G)$ of a graph G, this means that we have to study its idempotent endomorphisms, called the *retractions*, and to describe the *retracts* of G — the images of retractions. For example, it was shown in [2] that a graph G is isomorphic to a retract of R if and only if it is *algebraically closed*, that is, if every finite set of vertices of G has a common neighbor.

Furthermore, the set of idempotents of a semigroup (and so the set of retractions of G) can be endowed with the so-called *natural order* defined by

$$e \leqslant f \text{ if and only if } ef = fe = e.$$

The minimal elements with respect to this order are called the *primitive idempotents*. It turns out that the natural order of all retractions of R is so complex that it embeds every countable order [1], and even more, the whole power set of the natural numbers, ordered by inclusion [8].

In a sharp contrast to R, all Henson graphs turn out to be are *retract rigid* (see Proposition 1 below), which means that they have only one retraction — the identity mapping. As a consequence, a finite semigroup embeds into $End(H_n)$ if and only if it is a group (Corollary 2). On the other hand, the structure of retractions of complements of Henson graphs seems to be more involved than that of R. We are going to give a complete characterization of the retracts of $\overline{H}_3$ (Theorem 6). Following this, we will show that omission of the null graph $\overline{K}_n$ in a graph G implies the omission of a certain configuration in $End(G)$ (Proposition 11). Therefore, the situation with (countable) semigroups that embed in $End(\overline{H}_n)$ is something of an intermediate between that above two cases (R and H_n): not every finite semigroup can be embedded into $End(\overline{H}_n)$, but there are non-group finite semigroups that do. As an example, we prove that each countable semilattice can be represented by retractions of $\overline{H}_n$ for each $n \geqslant 3$ (Corollary 14).

2. Henson Graphs and Their Retracts

The infinite random graph R has many characterizations, but certainly the most used one is that R is the unique (up to an isomorphism) countable graph with the *existentially closed* (or *e.c.*) *property*: for any finite $A, B \subseteq V(R)$ with $A \cap B = \emptyset$ there exists a vertex $v \in V(R) \setminus (A \cup B)$ which is adjacent to all vertices from A and to none from B. There is a similar property, called K_{n-1}-*e.c.*, that characterizes Henson graphs:

> For any two disjoint, finite sets of vertices A, B such that A is K_{n-1}-free (does not contain a clique of size $n-1$ as an induced subgraph) there exists a vertex z adjacent to all vertices from A and to none from B.

By [12, Theorem 2.3], for each $n \geqslant 3$, any K_n-free graph Γ which satisfies K_{n-1}-e.c. must be isomorphic to H_n.

The following simple observation will easily settle the question of the representability of finite semigroups by endomorphisms of H_n.

Proposition 1. *Every endomorphism f of H_n ($n \geqslant 3$) is an isomorphism between H_n and $f(H_n)$. Consequently, each Henson graph H_n is retract rigid, that is, the identity mapping is its only retraction.*

Proof. First we prove that f must preserve non-edges. Assume to the contrary, that for two non-adjacent vertices x, y there is an edge between $f(x)$ and $f(y)$. Now let u_1 be a vertex adjacent to both x, y, and let $u_2, \ldots, u_{n-2}$ be further vertices of H_n chosen so that u_i is adjacent to all vertices from the set $\{x, y, u_1, \ldots, u_{i-1}\}$, $2 \leqslant i \leqslant n-2$, the existence of which is guaranteed by K_{n-1}-e.c. But then $v_i = f(u_i)$, $1 \leqslant i \leqslant n-2$, along with $f(x)$ and $f(y)$, form an clique of size n. A contradiction.

Furthermore, f must be injective. For suppose that $f(x) = f(y)$ for two distinct vertices x, y. By K_{n-1}-e.c., there is a vertex z adjacent to x and non-adjacent to y so that $f(x)$ and $f(z)$ must be adjacent. On the other hand, $f(x) = f(y)$ and $f(z)$ are non-adjacent by the previous paragraph. A contradiction, thus the proposition follows. □

Corollary 2. *Let S be a finite semigroup and $n \geqslant 3$. Then S embeds in $End(H_n)$ if and only if S is a group.*

Proof. If S is a finite semigroup which embeds in $End(H_n)$ then, by the above proposition, it has a unique idempotent e. Moreover, e must be an identity element of S. Since each element of S has an idempotent power, it follows that for each $a \in S$ we have $a^{m_a} = e$ for a suitable $m_a \geqslant 1$, showing that S is a group.

Conversely, Corollary 3.4 of [12] implies that $S_{\aleph_0}$, the symmetric group on an infinitely countable set, embeds in $Aut(H_n)$. Hence, every finite (and countable) group embeds in $End(H_n)$. □

3. Retractions of Complements of Henson Graphs

According to the above considerations, $\overline{H}_n$ is the unique (up to isomorphism) countable graph satisfying the property $\overline{K}_{n-1}$*-e.c.*, which is dual to K_{n-1}-e.c.:

> For any two disjoint, finite sets of vertices A, B such that B is $\overline{K}_{n-1}$-free, there exists a vertex z adjacent to all vertices from A and to none from B.

As a direct consequence of the above property, the graph $\overline{H}_n$ is algebraically closed for all $n \geqslant 3$.

We start by collecting some remarks on retractions of $\overline{K}_n$-free graphs.

Lemma 3. *Let Γ be a graph that omits $\overline{K}_n$ as a subgraph for some $n \geqslant 3$, let e be a retraction of Γ, $G = e(\Gamma)$ the corresponding retract, and*

$$A_k = \{x \in V(G) : |e^{-1}(x)| = k\}$$

for $k \geqslant 1$. Then:

(1) *if Γ is algebraically closed, so is G,*
(2) *$|e^{-1}(x)| \leqslant n-1$ for any $x \in V(G)$ (that is $A_k = \emptyset$ for all $k \geqslant n$),*
(3) *if the vertices $x_1, \ldots, x_r$ form a null graph and $x_i \in A_{k_i}$, $1 \leqslant i \leqslant r$, then*

$$\sum_{i=1}^{r} k_i \leqslant n-1.$$

Proof. (1) is immediate, as the property of being algebraically closed is obviously preserved by graph homomorphisms.

(2) follows from the fact that if $y, y' \in e^{-1}(x)$, then $e(y) = x = e(y')$, and so y, y' must be non-adjacent. Hence, $|e^{-1}(x)| \geqslant n$ for a vertex x would imply the existence of a null subgraph of Γ with at least n vertices.

(3) Note that if $u \in e^{-1}(x_i)$ and $v \in e^{-1}(x_j)$, $1 \leqslant i, j \leqslant r$, then u and v are non-adjacent, for otherwise $e(u) = x_i$ and $e(v) = x_j$ would be adjacent. Therefore, the vertices from $\bigcup_{i=1}^{r} e^{-1}(x_i)$ form a null graph. By the given conditions, this union can have at most $n-1$ elements, thus the required inequality follows. □

However, in order to work with $\overline{H}_n$ efficiently (for example, to obtain some of its retractions), we need a more constructive approach to these graphs. Recall (for example, from [5]) that there is an inductive construction of R which is in fact a special case of the canonical 'recipe' of constructing existentially closed structures (see [13]). Namely, take Γ_0 to be an arbitrary finite or countable graph. Assuming that the graph Γ_k has been constructed, for each finite subset $S \subseteq V(\Gamma_k)$ define a new vertex $z_S^{(k+1)} \notin V(\Gamma_k)$ whose neighbor set is precisely S. By adding all such vertices to Γ_k, we obtain Γ_{k+1}. As is easily verified, the union $\bigcup_{k<\omega} \Gamma_k$ of this chain of graphs has the existential property, so it is isomorphic to R.

Now modify the above construction such that the 'initial' graph Γ_0 is K_n-free ($n \geqslant 3$ is a fixed number), and in the process of obtaining Γ_{k+1} from Γ_k, expand Γ_k by vertices of the form $z_S^{(k+1)}$ *only* for those finite subsets $S \subseteq V(\Gamma_k)$ for which the induced subgraph of Γ_k on S is K_{n-1}-free.

Lemma 4. *Let $n \geqslant 3$, let Γ_0 be a K_n-free graph, and assume that the chain of graphs $\{\Gamma_k : k < \omega\}$ is constructed as above. Then*

$$\Gamma = \bigcup_{k<\omega} \Gamma_k$$

is isomorphic to H_n.

Proof. First of all, Γ is K_n-free, for otherwise there is a minimal $m \geqslant 1$ such that Γ_m contains K_n. By minimality of m, and since every two vertices from $V(\Gamma_m) \setminus V(\Gamma_{m-1})$ are non-adjacent, there is a unique $S \subseteq V(\Gamma_{m-1})$ such that $z_S^{(m)}$ is a vertex from the considered n-clique. The other vertices of that clique must belong to S, contradicting the fact that the graph formed by S must be K_{n-1}-free.

Further, let A, B be disjoint, finite subsets of $V(\Gamma)$ such that the induced subgraph on A is K_{n-1}-free. Then there is an integer m such that $A \cup B \subseteq V(\Gamma_m)$, and $z_A^{(m+1)} \in V(\Gamma_{m+1}) \subseteq V(\Gamma)$ is the vertex required by K_{n-1}-e.c. □

By considering the dual construction to the above one, it follows that $\overline{H}_n$ can be obtained by starting from an arbitrary at most countable $\overline{K}_n$-free graph Γ_0, and then by constructing a sequence of graphs Γ_k, $k < \omega$, such that Γ_k is enlarged to Γ_{k+1} by a countable complete graph consisting of vertices $z_S^{(k+1)}$ for all finite $\overline{K}_{n-1}$-free subsets $S \subseteq V(\Gamma_k)$, where $z_S^{(k+1)}$ is joined to all vertices of Γ_k, except to those from S. By the above lemma, the union of the chain $\{\Gamma_k : k < \omega\}$ must be isomorphic to $\overline{H}_n$.

We recall that the *join* $G_1 \vee G_2$ of graphs G_1, G_2 is the graph having $V(G_1) \cup V(G_2)$ as the set of vertices (while assuming that $V(G_1) \cap V(G_2) = \emptyset$) and

$$E(G_1) \cup E(G_2) \cup (V(G_1) \times V(G_2)) \cup (V(G_2) \times V(G_1))$$

as the set of edges.

Now we have the necessary prerequisites to exhibit a class of graphs whose copies occur as retracts of $\overline{H}_n$.

Proposition 5. *For $n \geqslant 3$, let H be a $\overline{K}_n$-free graph, and let $G \cong H \vee K_{\aleph_0}$. Then there exists a retraction e of $\overline{H}_n$ such that $e(\overline{H}_n) \cong G$.*

Proof. Let Γ_k, $k < \omega$, be a sequence of graphs as above, such that its union Γ is isomorphic to $\overline{H}_n$. First of all, enumerate the vertices of H:

$$V(H) = \{u_0, u_1, \ldots\}.$$

Now choose $v_0 \in V(\Gamma_0)$ arbitrarily, and then for each $i \geqslant 1$ a vertex $v_i \in V(\Gamma_i) \setminus V(\Gamma_{i-1})$ such that

(i) the induced subgraph of Γ on $\{v_0, v_1, \ldots, v_i\}$ is isomorphic to the induced subgraph of H on $\{u_0, u_1, \ldots, u_i\}$, and

(ii) v_i is adjacent to every vertex of Γ_{i-1} which is not one of v_j, $j < i$.

Of course, if H is finite, the process of taking vertices $v_0, v_1, \ldots$ terminates in a finite number of steps. Clearly, the required choice is $v_i = z_{S_i}^{(i)}$ where

$$S_i = \{v_j : j < i, \ u_j \text{ is non-adjacent to } u_i\}.$$

Further, let $B_0 = \emptyset$ and $C_0 = V(\Gamma_0) \setminus \{v_0\}$. For each $k \geqslant 0$ define

$$\begin{aligned} B_{k+1} &= \{z^{(k+1)}_{\{x\}} : x \in B_k\}, \\ C_{k+1} &= V(\Gamma_k) \setminus (B_{k+1} \cup \{v_{k+1}\}). \end{aligned}$$

Finally, let

$$B = \{v_0, v_1, \dots\} \cup \bigcup_{k>0} B_k,$$

and consider $e : V(\Gamma) \to B$ given by

$$e(x) = \begin{cases} x & x \in B, \\ z^{(k+1)}_{\{x\}} & \text{if } x \in C_k,\ k \geqslant 0. \end{cases}$$

We claim that the induced subgraph of Γ on B is isomorphic to G, and that e is a retraction of Γ.

For the first part, since the induced subgraph of Γ on $\{v_0, v_1, \dots\}$ is isomorphic to H, it suffices to show that for each $k \geqslant 1$ and $x \in B_k$, x is adjacent to all other vertices from B. Firstly, x is obviously adjacent to all other vertices from B_k and to v_k. Furthermore, $x \in B_k$ means that $x = z^{(k)}_{\{y\}}$ for some $y \in C_{k-1}$, so that x is adjacent to all vertices from $\bigcup_{0 \leqslant i<k} \Gamma_i$ except y, and so, in particular, to all vertices from $B_1, \dots, B_{k-1}$ and $\{v_0, \dots, v_{k-1}\}$. Also, by the condition (ii) above, x and v_j are adjacent for all $j > k$. Finally, if $x' \in B_j$ for some $j > k$, then $x' = z^{(j)}_{\{u\}}$ for some $u \in C_{j-1}$, and thus x' and x are adjacent.

It is obvious that we have $e^2 = e$, so it remains to prove that e is a homomorphism. Let u and u' form an edge in Γ.

Case 1. $u, u' \notin B$. Then $e(u), e(u') \in \bigcup_{k>0} B_k$, and therefore they are adjacent, both being universal vertices in the subgraph of Γ induced by B (that is, adjacent to all other vertices from B). Of course, they are different, since if $u \in C_k$ and $u' \in C_\ell$, $k, \ell \geqslant 0$, then $e(u) = z^{(k+1)}_{\{u\}} \neq z^{(\ell+1)}_{\{u'\}} = e(u')$.

Case 2. $u \notin B$ and $u' \in B$ (or the other way around). In this case, if $u \in C_k$, $k \geqslant 0$, then $e(u) = z^{(k+1)}_{\{u\}} \neq u' = e(u')$, for otherwise u and u' are non-adjacent, by construction. Hence, $e(u)$ and $e(u')$ must be adjacent.

Case 3. $u, u' \in B$. This case is trivial, as $e(u) = u$ and $e(u') = u'$.

Since all possibilities are exhausted, e is a graph homomorphism, and the proposition is proved. □

We have enough information to give a complete list of retracts of $\overline{H}_3$.

Theorem 6. *Let G be a graph. Then there exists a retraction e of $\overline{H}_3$ such that $e(\overline{H}_3) \cong G$ if and only if $G \cong H \vee K_{\aleph_0}$, where H is a $\overline{K}_3$-free graph.*

Proof. Sufficiency follows from the above proposition. For necessity, note that $e(\overline{H}_3)$ is an algebraically closed graph, so it cannot be finite. Also, it cannot be a cofinite subgraph of $\overline{H}_3$, for otherwise let V be the set of all vertices of $\overline{H}_3$ not belonging to $e(\overline{H}_3)$, and choose $v \in V$ arbitrarily. By $\overline{K}_2$-e.c., there is a vertex z adjacent to all vertices from V (thus $z \notin V$, so $e(z) = z$), and non-adjacent to

$e(v)$. This is, however, a contradiction, as v, z are adjacent, while $e(v), e(z)$ are not.

Hence, V is an infinite set of vertices, and, using the notation from Lemma 3, $A_2 = e(V)$. As a consequence of Lemma 3 (3), each vertex from A_2 is universal in $e(\overline{H}_3)$. Thus $e(\overline{H}_3) = A_1 \vee A_2$ and $A_2 \cong K_{\aleph_0}$, and so the theorem follows, since A_1 is finite or countable and, of course, $\overline{K}_3$-free. □

Problem 1. *Characterize the retracts of* $\overline{H}_n$ *for* $n \geqslant 4$.

We proceed by studying minimal retractions of the graphs $\overline{H}_n$. First we make some observations on minimal retractions in general.

Lemma 7. (a) *A retraction* e *of a graph* Γ *is minimal if and only if* $e(\Gamma)$ *is a retract-rigid graph.*

(b) *Let* C_n *be the cycle with* n *vertices,* X *the set of all odd numbers* $\geqslant 5$, *and* $I, J \subseteq X$. *If* $I \neq J$, *then* $\bigvee_{i \in I} \overline{C}_i \ncong \bigvee_{j \in J} \overline{C}_j$.

(c) *For each odd number* $n \geqslant 5$, *the graph* $\overline{C}_n$ *is retract-rigid.*

Proof. (a) A reference to Claim 1 in the proof of Theorem 4.1, p.142 of [2], suffices: although it is proved there for Γ being R, the same proof applies verbatim to the general case.

(b) This follows from the fact that two graphs are isomorphic if and only if their complements are isomorphic. Namely, the complement of $\bigvee_{i \in I} \overline{C}_i$ is the disjoint union of the cycles C_i, $i \in I$ (denote it by $\biguplus_{i \in I} C_i$). Since cycles of different length are mutually non-embeddable, $I \neq J$ obviously implies $\biguplus_{i \in I} C_i \ncong \biguplus_{j \in J} C_j$.

(c) Assume that for some odd $n \geqslant 5$, $\overline{C}_n$ has a non-trivial retraction e. Let v be a vertex of $\overline{C}_n$ such that $e(v) \neq v$. Now enumerate the vertices of $\overline{C}_n$ by $v_0 = v, v_1, \ldots, v_{n-1}$ such that v_i and v_{i+1} are non-adjacent for all $0 \leqslant i \leqslant n-2$, as well as v_{n-1} and v_0. Clearly, v and $e(v)$ must be non-adjacent — so assume, without any loss of generality, that $e(v) = v_1$. We prove by induction that $e(v_{2k}) = e(v_{2k+1}) = v_{2k+1}$ for each $k \geqslant 0$. Indeed, $e(v_{2k+2})$ must be non-adjacent to v_{2k+2}, but on the other hand, it must be adjacent to v_{2k+1}, as v_{2k} and v_{2k+2} form an edge. This already implies $e(v_{2k+2}) = v_{2k+3}$, and so $e(v_{2k+3}) = v_{2k+3}$. In particular, $e(v_{n-3}) = e(v_{n-2}) = v_{n-2}$. Hence, $e(v_{n-1})$ must be non-adjacent to v_{n-1}, but adjacent to v_{n-2}, implying $e(v_{n-1}) = v_0$. However, this is impossible, since now we obtain $v_0 = e(v_{n-1}) = e(e(v_{n-1})) = e(v_0) = v_1$, a contradiction. □

Similarly to $End(R)$ (cf. Theorem 4.1 of [2]), there are uncountably many primitive idempotents in $End(\overline{H}_n)$ for each $n \geqslant 3$.

Theorem 8. *There exist* $2^{\aleph_0}$ *primitive idempotents in the endomorphism monoid of* $\overline{H}_n$, *for all* $n \geqslant 3$.

Proof. Let $\omega = \{0, 1, 2, \ldots\}$ be the set of natural numbers. Since for any $I \subseteq \omega$ the graph $\bigvee_{i \in I} \overline{C}_i$ is $\overline{K}_n$-free for all $n \geqslant 3$, by Proposition 5 there is a retraction e_I of $\overline{H}_n$ such that the image of e_I is isomorphic to $\bigvee_{i \in I} \overline{C}_i \vee K_{\aleph_0}$. By (b) of the above lemma, all these retractions are different, that is, $e_I = e_J$ if and only if $I = J$. Furthermore, all graphs of this form are retract rigid by (c) of the previous lemma, since $K_{\aleph_0}$ is retract rigid, and the join of retract rigid graphs remains retract rigid (cf. Claim 3 of the proof of Theorem 4.1 in [2]). Therefore, by (a) of the above lemma, all retractions e_I are minimal, forming a set of $2^{\aleph_0}$ primitive idempotents of $End(\overline{H}_n)$. □

Also, a result that parallels Theorem 3 of [8] is true for the graphs $\overline{H}_n$.

Theorem 9. *The power set of ω (ordered by inclusion) embeds into the natural order of retractions of $\overline{H}_n$, for each $n \geqslant 3$.*

Proof. Consider the graph

$$G \cong \left(\bigvee_{n \in \omega} \overline{C}_{2n+4} \right) \vee K_{\aleph_0}.$$

By Proposition 5, $\overline{H}_n$ contains an isomorphic copy of G' of G such that there is a retraction of $\overline{H}_n$ onto G'. Without any loss of generality, we identify G' and G, so that we have a retraction $e : \overline{H}_n \to G$.

Enumerate $V(\overline{C}_{2k}) = \{c_{0,2k}, c_{1,2k}, \dots, c_{2k-1,2k}\}$ (for $k \geqslant 2$) and define $f_k : V(\overline{C}_{2k}) \to V(\overline{C}_{2k})$ by

$$f_k(c_{i,2k}) = \begin{cases} c_{i,2k}, & \text{if } i \text{ is odd,} \\ c_{i+1,2k}, & \text{otherwise.} \end{cases}$$

It is easy to see that f_k is indeed an idempotent endomorphism of $\overline{C}_{2k}$, and that its image is a complete graph of k vertices. Now for each $I \subseteq \omega$ we define a mapping $e_I : V(G) \to V(G)$ such that

$$e_I(v) = \begin{cases} f_{k+2}(v), & \text{if } v \in V(\overline{C}_{2k+4}) \text{ and } k \notin I, \\ v, & \text{otherwise.} \end{cases}$$

It is straightforward to see that e_I is an idempotent endomorphism of G. It acts on G so that it turns the graphs $\overline{C}_{2k+4}$ for which $k \notin I$ into complete graphs of the form K_{k+2} — thus producing two new universal vertices in $e_I(G)$ — and leaves all the other cycle complements (as well as $K_{\aleph_0}$) fixed. Hence,

$$e_I(G) \cong \left(\bigvee_{k \in I} \overline{C}_{2k+4} \right) \vee K_{\aleph_0}.$$

Furthermore, $e_I e$ is a retraction of the whole $\overline{H}_n$ onto $G_I = e_I(G)$, since $e|_{V(G)}$ is an identity mapping.

We claim that $e_I e \leqslant e_J e$ if and only if $I \subseteq J$, which would obviously finish the proof of the theorem.

($\Rightarrow$) First of all, note that we have $e_I e = e_I e e_J e = e_I e_J e = e_J e_I e$ by the assumptions, since $e e_I = e_I$ and $e e_J = e_J$. Let $k \in I$, let i be even, and $v = c_{i,2k+4}$. Then $v = e(v) = e_I(v)$, so $v = e_I e(v) = e_J e_I e(v) = e_J(v)$. The latter equality implies $k \in J$, for otherwise $e_J(v) = c_{i+1,2k+4} \neq v$. Hence, $I \subseteq J$.

($\Leftarrow$) Let $I \subseteq J$. We prove that $e_I = e_I e_J = e_J e_I$ holds. If $v \in V(K_{\aleph_0})$, or $v \in V(\overline{C}_{2k+4})$ such that $k \in J$, then $e_J(v) = v$ and $e_I(v) = e_I e_J(v)$. Also,

$e_I(v) = e_J e_I(v)$, since by definition we have respectively $e_I(v) = v$ in the former, and $e_I(v) \in V(\overline{C}_{2k+4})$ in the latter case (regardless whether $k \in I$ or not). So, it remains to consider the case when $k \notin I \Rightarrow k \notin J$. We distinguish two possibilities. First,if $v = c_{2i,2k+4}$ then $e_I(v) = e_J(v) = c_{2i+1,2k+4}$, so $e_I e_J(v) = v = e_J e_I(v)$. On the other hand, if $v = c_{2i+1,2k+4}$, then $e_I(v) = e_J(v) = v$, thus $e_I e_J(v) = v = e_J e_I(v)$ follows immediately.

Now by the above remarks, $e_I e = e_I e_J e = e_I e e_J e = (e_I e)(e_J e)$, and similarly $e_I e = (e_J e)(e_I e)$, that is, $e_I e \leqslant e_J e$. □

Corollary 10. *Every countable order embeds in the natural order of idempotents of $End(\overline{H}_n)$, $n \geqslant 3$.*

4. Partial Results on Embeddings into $End(\overline{H}_n)$

Proposition 11. *Let $n \geqslant 3$, and let e, f be endomorphisms of a $\overline{K}_n$-free graph G such that $e^2 = e$ and $ef = fe = e$. Then*

$$f^{n-2} = f^{n-2+(n-2)!}\,.$$

Proof. Let x be a vertex of G. Along with x, consider the vertices $e(x)$ and $f^i(x)$, $1 \leqslant i \leqslant n-2$. Since all these vertices have the same image $e(x)$ under e, by Lemma 3 (2) two of them must coincide. If $e(x) = x$, then

$$f(x) = fe(x) = e(x) = x.$$

Further, if $e(x) = f^i(x)$ for some $1 \leqslant i \leqslant n-2$, then

$$f^{i+1}(x) = fe(x) = e(x) = f^i(x).$$

The remaining possibility is that

$$f^i(x) = f^j(x)$$

holds for some i, j such that $0 \leqslant i < j \leqslant n-2$. Summing up, for all $x \in V(G)$ we have $f^i(x) = f^j(x)$ for some distinct $i, j \in \{0, 1, \ldots, n-1\}$ (which, of course, may depend on x) such that $|i - j| \leqslant n-2$.

By assuming $i < j$ and setting $d = j - i$, it follows that $f^i(x) = f^{i+kd}(x)$ for all $k \geqslant 0$. Since $i \leqslant n-2$, this implies

$$f^{n-2}(x) = f^{n-2+kd}(x)$$

for all $k \geqslant 0$. Finally, $d \leqslant n-2$, so $d \mid (n-2)!$. Thus,

$$f^{n-2}(x) = f^{n-2+(n-2)!}(x)$$

holds for all $x \in V(G)$, whence we are done. □

Remark 12. For $n = 3$, the conclusion of the above lemma is $f = f^2$. This means in particular that no nonidempotent semigroup with zero embeds in $End(\overline{H}_3)$.

Furthermore, there are finite semigroups that cannot be represented by endomorphisms of $\overline{H}_n$ for other values of n. A *0-group* is a semigroup obtained from a group G by adjoining a zero element. We denote it by G^0.

Corollary 13. *Let $\mathbb{C}_m$ denote the cyclic group with m elements. Then $\mathbb{C}_m^0$ does not embed in $End(\overline{H}_n)$ if $m > (n-2)!$.*

However, it turns out that every countable semilattice embeds in $End(\overline{H}_n)$ for all $n \geqslant 3$. As known, a semilattice may be also viewed as an ordered set in which every two elements have a greatest lower bound (with respect to the ordering defined by $x \leqslant y$ if and only if $xy = x$).

Corollary 14. *Every countable semilattice embeds in $End(\overline{H}_n)$ for all $n \geqslant 3$.*

Proof. By the proof of Theorem 9, $I \mapsto e_I e$ $(I \subseteq \omega)$ is an embedding of the partial order $\Omega = (2^\omega, \subseteq)$ into the natural order of retractions of $\overline{H}_n$. However, Ω itself is an lower semilattice, and the considered embedding preserves the intersections, since one easily verifies that

$$(e_I e)(e_J e) = e_I e_J e = e_{I \cap J} e$$

holds for all $I, J \subseteq \omega$. Therefore, if we consider Ω as a semigroup, the above embedding is in fact a semigroup embedding of Ω into $End(\overline{H}_n)$. As each countable semilattice embeds in Ω, the result follows. □

Let us remind that once more that by Corollary 3.4 of [12], every countable group embeds in $End(\overline{H}_n)$, too (via $Aut(\overline{H}_n) = Aut(H_n)$). On the other hand, we have seen that there are finite semigroups whose copies are not contained in $End(\overline{H}_n)$, such as large 0-groups (which might be considered as 'combinations' of semilattices and groups). Hence, the following question emerges.

Problem 2. *Characterize the countable semigroups which embed in $End(\overline{H}_n)$ for $n \geqslant 3$.*

REFERENCES

[1] A.Bonato, On retracts of the random graph and their natural order, *Monatsh. Math.* **135** (2002), 1–9.

[2] A.Bonato and D.Delić, The monoid of the random graph, *Semigroup Forum* **61** (2000), 138–148.

[3] A.Bonato, D.Delić and I.Dolinka, All countable monoids embed in the monoid of the infinite random graph, *Discrete Math.* (to appear).

[4] P.J.Cameron, *Oligomorphic Permutation Groups*, London Math. Soc. Lecture Notes Ser., Vol. 152, Cambridge University Press, Cambridge, 1990.

[5] P.J.Cameron, The random graph, in: R.L.Graham, J.Nešetřil (eds.), *The Mathematics of Paul Erdős, II*, Algorithms and Combinatorics, Vol. 14, pp. 333–351, Springer-Verlag, New York, 1997.

[6] P.J.Cameron, The random graph revisited, in: *Proceedings of the European Congress of Mathematics, Vol. I* (Barcelona, 2000), Progr. Math., Vol. 201, pp. 267–274, Birkhäuser, Basel, 2001.

[7] G.L.Cherlin, The classification of homogeneous directed graphs and countable homogeneous n-tournaments, *Memoirs Amer. Math. Soc.* **621** (1998).

[8] D.Delić and I.Dolinka, The endomorphism monoid of the random graph has uncountably many ideals, *Semigroup Forum* **69** (2004), 75–79.

[9] R.Fraïssé, Sur certains relations qui généralisent l'ordre des nombres rationelles, *C. R. Acad. Sci. Paris* **237** (1953), 540–542.

[10] R.Fraïssé, *Theory of Relations*, North-Holland, Amsterdam, 1986.

[11] A.Gardiner, Homogeneous graphs, *J. Comb. Theory, Ser. B* **20** (1976), 94–102.

[12] C.W.Henson, A family of countable homogeneous graphs, *Pacific J. Math.* **38** (1971), 69–83.
[13] W.A.Hodges, *Model Theory*, Cambridge University Press, Cambridge, 1993.
[14] A.H.Lachlan, Countable homogeneous tournaments, *Trans. Amer. Math. Soc.* **284** (1984), 431–461.
[15] A.H.Lachlan and R.E.Woodrow, Countable ultrahomogeneous undirected graphs, *Trans. Amer. Math. Soc.* **262** (1980), 51–94.

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