New Classes of Integral Trees of Diameter 4 *

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Abstract

A graph is called integral if all eigenvalues of its adjacency matrix are integers. In this paper, we investigate integral trees $S(r; m_i) = S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 with $s = 2, 3$. We give a better sufficient and necessary condition for the tree $S(a_1+a_2; m_1, m_2)$ of diameter 4 to be integral, from which we construct infinitely many new classes of such integral trees by solving some certain Diophantine equations. These results are different from those in the existing literature. We also construct new integral trees $S(a_1+a_2+a_3; m_1, m_2, m_3) = S(a_1+1+1; m_1, m_2, m_3)$ of diameter 4 with non-square numbers m_2 and m_3. These results generalize some well-known results of P.Z. Yuan, D.L. Zhang *et al.*

Key Words: Integral tree, Characteristic polynomial, Diophantine equation, Graph spectrum.

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1 Introduction

Throughout this paper, we consider only simple undirected graphs (i.e. undirected graphs without loops or multiple edges). Let $G = (V, E)$ be a simple graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Its adjacency matrix $A(G)$ is defined as $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and $a_{ij} = 0$ if they are non-adjacent. The characteristic polynomial of G is the polynomial $P(G, x) =$

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$det(xI_n - A(G))$, where I_n denotes the $n \times n$ identity matrix. The spectrum of $A(G)$ is also called the spectrum of G and denoted by $Spec(G)$ (see [3]).

We know that trees of diameter 4 can be formed by joining the centers of r stars $K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_r}$ to a new vertex v. The tree is denoted by $S(r; m_1, m_2, \ldots, m_r)$ or simply $S(r; m_i)$. Assume that the number of distinct integers of $m_1, m_2, \ldots, m_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $0 \leq m_1 < m_2 < \ldots < m_s$. Suppose that a_i is the multiplicity of m_i for each $i = 1, 2, \ldots, s$. The tree $S(r; m_i)$ is also denoted by $S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = 1 + \sum_{i=1}^{s} a_i(m_i + 1)$. For all other facts on graph spectra (or terminology), see [3].

A graph G is called integral if all eigenvalues of its characteristic polynomial $P(G, x)$ are integers. First observations on integral graphs were made by Harary and Schwenk in 1974 [4]. So far, there are many results for some particular classes of integral graphs [1]. In particular, some results on integral trees of diameter 4 were investigated in [1-9, 11-17]. In this paper, we investigate integral trees $S(r; m_i) = S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 with $s = 2, 3$. We give a better sufficient and necessary condition for the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 to be integral, from which we construct infinitely many new classes of such integral trees by solving some certain Diophantine equations. These results are different from those in the existing literature. We also construct new integral trees $S(a_1 + a_2 + a_3; m_1, m_2, m_3) = S(a_1 + 1 + 1; m_1, m_2, m_3)$ of diameter 4 with non-square numbers m_2 and m_3. These results generalize some well-known results of [9, 16, 17] *etc.*

2 Preliminaries

In this section, we state some known results on integral trees of diameter 4 and also obtain some new results on integral trees of diameter 4.

Lemma 1. *([7, 12]) For the tree $S(r; m_i) = S(a_1 + \ldots + a_s; m_1, \ldots, m_s)$ of diameter 4, then we have*

$$P[S(r; m_i), x] = P[S(a_1 + \ldots + a_s; m_1, \ldots, m_s), x] = x^{1+\sum_{i=1}^{s} a_i(m_i - 1)}$$
$$\cdot \prod_{i=1}^{s} (x^2 - m_i)^{a_i - 1} [\prod_{i=1}^{s} (x^2 - m_i) - \sum_{i=1}^{s} a_i \prod_{j=1, j \neq i}^{s} (x^2 - m_j)].$$

Lemma 2. *([17, 12]) The tree $S(r; m_i) = S(a_1 + \ldots + a_s; m_1, \ldots, m_s)$ of diameter 4 is integral if and only if (i) $a_i = 1$ must hold if m_i is not a* *perfect square, (ii) all solutions of the following equation are integers.*

$$\prod_{i=1}^{s}(x^2 - m_i) - \sum_{i=1}^{s} a_i \prod_{j=1, j\neq i}^{s} (x^2 - m_j) = 0 \tag{1}$$

We now discuss Eq.(1) to get more information. First, we divide both sides of Eq.(1) by $\prod_{i=1}^{s}(x^2 - m_i)$, and obtain

$$F(x) = \sum_{i=1}^{s} \frac{a_i}{x^2 - m_i} - 1 = 0. \tag{2}$$

Clearly, $\pm\sqrt{m_i}$ are not roots of Eq.(1) for $1 \leq i \leq s$. Hence, all solutions of Eq.(1) are the same as those of Eq.(2). By Lemma 2, and discussions of [6, 16, 17] , we can deduce the following theorem holds.

Theorem 3. *The tree $S(r; m_i) = S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 is integral if and only if (i) $a_i = 1$ must hold if m_i is not a perfect square, (ii) all solutions of the following equation are integers.*

$$\sum_{i=1}^{s} \frac{a_i}{x^2 - m_i} = 1 \tag{3}$$

Moreover, there exist positive integers $u_1, u_2, \ldots, u_s$ satisfying

$$0 \leq \sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < \ldots < u_{s-1} < \sqrt{m_s} < u_s < +\infty \tag{4}$$

such that the following linear equation system in $a_1, a_2, \ldots, a_s$ has positive integral solutions $(a_1, a_2, \ldots, a_s)$, and such that $a_i = 1$ must hold if m_i is not a perfect square.

$$\begin{cases} \frac{a_1}{u_1^2 - m_1} + \frac{a_2}{u_1^2 - m_2} + \ldots + \frac{a_s}{u_1^2 - m_s} = 1, \\ \cdots\cdots\cdots \quad \cdots\cdots\cdots \\ \frac{a_1}{u_s^2 - m_1} + \frac{a_2}{u_s^2 - m_2} + \ldots + \frac{a_s}{u_s^2 - m_s} = 1. \end{cases} \tag{5}$$

Theorem 4. *([17, 6]) The tree $S(r; m_i) = S(a_1 + \ldots + a_s; m_1, \ldots, m_s)$ of diameter 4 is integral if and only if there exist positive integers u_i and nonnegative integers m_i $(i = 1, 2, \ldots, s)$ such that $0 \leq \sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < \ldots < u_{s-1} < \sqrt{m_s} < u_s < +\infty$, and such that*

$$a_k = \frac{\prod_{i=1}^{s}(u_i^2 - m_k)}{\prod_{i=1, i\neq k}^{s}(m_i - m_k)}, \quad (k = 1, 2, \ldots, s) \tag{6}$$

are positive integers, and such that $a_i = 1$ must hold if m_i is not a perfect square.

Corollary 5. *If the tree $S(r; m_i) = S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 is integral, then we have the following results.*

(1) ([16, 17]) $a_1 > 1$. Moreover m_1 is a perfect square.

(2) ([16, 17]) $r = \sum_{i=1}^{s} a_i = \sum_{i=1}^{s} u_i^2 - \sum_{i=1}^{s} m_i$.

(3) $\prod_{i=1}^{s} u_i^2 = (\prod_{i=1}^{s} m_i)(1 + \sum_{i=1}^{s} \frac{a_i}{m_i})$.

(4) ([16, 17]) $Spec(S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)) =$

$$\begin{pmatrix} 0 & \pm\sqrt{m_1} & \pm u_1 & \pm\sqrt{m_2} & \ldots & \pm\sqrt{m_s} & \pm u_s \\ 1 + \sum_{i=1}^{s} a_i(m_i - 1) & a_1 - 1 & 1 & a_2 - 1 & \ldots & a_s - 1 & 1 \end{pmatrix}.$$

Proof. We only prove (2)-(4). By Lemma 1, we have

$$\begin{aligned} &P[S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s), x] \\ &= x^{1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s} (x^2 - m_i)^{a_i - 1} [\prod_{i=1}^{s} (x^2 - m_i) \\ &- \sum_{i=1}^{s} a_i \prod_{j=1, j\neq i}^{s} (x^2 - m_j)] \\ &= x^{1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s} (x^2 - m_i)^{a_i - 1} [x^{2s} - (\sum_{i=1}^{s} a_i + \sum_{i=1}^{s} m_i) x^{2s-2} \\ &+ \ldots + (-1)^s (\prod_{i=1}^{s} m_i)(1 + \sum_{i=1}^{s} \tfrac{a_i}{m_i})] \\ &= x^{1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s} (x^2 - m_i)^{a_i - 1} \prod_{i=1}^{s} (x^2 - u_i^2). \end{aligned}$$

By using the relation between roots and coefficients of polynomials, and the inequality (4), we obtain the results in (2)-(4). □

Lemma 6. *Denote* $$\Psi_{\vec{a},\vec{m}}(x) = \sum_{i=1}^{s} \frac{a_i}{x^2 - m_i},$$

$$\Phi_{\vec{a},\vec{m}}(x) = [\prod_{i=1}^{s} (x^2 - m_i)][1 - \Psi_{\vec{a},\vec{m}}(x)],$$

where vectors $\vec{a} = (a_1, a_2, \ldots, a_s) \in (\mathbb{N} \setminus \{0\})^s, \vec{m} = (m_1, m_2, \ldots, m_s) \in \mathbb{N}^s, \mathbb{N} = \{0, 1, 2, \ldots\}$.

Let n be a positive integer. Then u is an integral root of $\Phi_{\vec{a},\vec{m}}(x)$ if and only if u/n is an integral root of $\Phi_{\vec{a}n^2,\vec{m}n^2}(x)$ too.

Proof. It is easy to see that v is a root of $\Phi_{\vec{a},\vec{m}}(x)$ if and only if vn is a root of $\Phi_{\vec{a}n^2,\vec{m}n^2}(x)$. Therefore if all roots of $\Phi_{\vec{a},\vec{m}}(x)$ are integers, then the roots of $\Phi_{\vec{a}n^2,\vec{m}n^2}(x)$ are integers as well.

Assume now that all roots of $\Phi_{\vec{a}n^2,\vec{m}n^2}(x)$ are integers and let v be one of them. Then v/n is a rational root of $\Phi_{\vec{a},\vec{m}}(x)$. Since $\Phi_{\vec{a},\vec{m}}(x)$ is a monic polynomial with integral coefficients, its rational roots should be integers. Therefore $v/n \in \mathbb{Z}$. □

From the above lemma we can obtain the following result.

Theorem 7. *If $m_1(\geq 0), m_2, \ldots, m_s$ are perfect squares, then the tree $S(a_1+a_2+\ldots+a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 is integral if and only if the tree $S(a_1n^2+a_2n^2+\ldots+a_sn^2; m_1n^2, m_2n^2, \ldots, m_sn^2)$ is integral for any positive integer n.*

Remark 8. *For $m_1(\geq 0), m_2, \ldots, m_s$ are perfect squares, the above Theorem 7 shows that we have to study Eq.(3) only for the case $(a_1, a_2, \ldots, a_s, m_1, m_2, \ldots, m_s) = 1$. Let us call such a vector $(\vec{a}, \vec{m})$ primitive.*

Next we shall give some facts on number theory.

Lemma 9. *([18]) Let K be an associate class of solutions of the Diophantine equation*

$$x^2 - dy^2 = m, \tag{7}$$

and let $u_0+v_0\sqrt{d}$ be the fundamental solution of the associate class K. Then all solutions of the class K are given by $x + y\sqrt{d} = \pm(u_0 + v_0\sqrt{d})(x_0 + y_0\sqrt{d})^n$, where n is an integer, and $x_0 + y_0\sqrt{d}$ is the fundamental solution of the Pell equation

$$x^2 - dy^2 = 1. \tag{8}$$

Lemma 10. *([10]) Let a, b and c be integers with $d = (a, b)$, we have*

(1) If $d \nmid c$, then the linear Diophantine equation in two variables

$$ax + by = c \tag{9}$$

does not have integral solutions.

(2) If $d|c$, then there are infinitely many integral solutions for Eq.(9). Moreover, if $x = x_0, y = y_0$ is a particular solution of Eq.(9), then all its solutions are given by

$$x = x_0 + (b/d)t, \quad y = y_0 - (a/d)t, \quad \textit{where } t \textit{ is an integer.}$$

3 Integral trees of diameter 4

In this section, we shall construct infinitely many new classes of integral trees $S(a_1 + \ldots + a_s; m_1, \ldots, m_s)$ of diameter 4, different from those of [1-9,11-17].

The idea of constructing such integral tree is as follows: First, we properly choose integers m_1 (≥ 0), m_2 (> 0), $\ldots, m_s$ (> 0). Then, we try to find proper positive integers u_i ($i = 1, 2, \ldots, s-1$) satisfying (4) such that there are positive integral solutions $(a_1, a_2, \ldots, a_s)$ for the linear equation system (5) (or such that all a_k's of (6) are positive integers). Finally, we obtain positive integers $a_1, a_2, \ldots, a_s$ such that all the solutions a_k's of Eq.(5) or Eq.(6) are integers. Thus, we have constructed many new classes of integral trees $S(a_1 + a_2 + \ldots + a_s; m_1, m_2, \ldots, m_s)$ of diameter 4.

Theorem 11. *(1) ([9, 13]) For $s = 1$, then the tree $S(r; m_i) = S(a_1; m_1)$ of diameter 4 is integral if and only if m_1 $(\neq 0)$ and $m_1 + r (= m_1 + a_1)$ are perfect squares.*

(2) ([9]) For $s = 2$, if m_1 is not a perfect square, then the tree $S(r; m_i) = S(a_1 + a_2;\ m_1,\ m_2)$ of diameter 4 is not an integral tree.

Theorem 12. *For $s = 2$, let $m_1 < m_2$. Then the tree $S(r; m_i) = S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral if and only if one of the following two cases holds:*

(1) When $m_2 (> m^2 + 2m + 1)$ is not a perfect square, $m_1 = m^2$, $m_2 = (m + q)^2 + u$, $a_1 = v(u + 1)$, $a_2 = 1$, and integers m (≥ 0), $q(\geq 1)$, $u(\geq 1)$, $v(\geq 1)$, $u_2(\geq 1)$ satisfy the following equations.

$$\begin{cases} uv = q(2m + q), \\ u_2^2 = 1 + u + v + (m + q)^2. \end{cases} \tag{10}$$

(2) When m_2 is a perfect square, $m_1 = m^2$, $m_2 = (m + k)^2$, $m \geq 0$, $k \geq 2$, let $(q(2m + q), (k - q)(2m + q + k)) = d$, $q(2m + q) = d\alpha$, $(k - q)(2m + q + k) = d\beta$, $(\alpha, \beta) = 1$, $a_1 = \alpha t + q(2m + q)$, $a_2 = \beta t$, where $t \geq 1$, $1 \leq q < k$, and integers $m(\geq 0)$, $k(\geq 2)$, $t(\geq 1)$, $u_2(\geq 1)$, $\alpha(\geq 1)$, $\beta(\geq 1)$ satisfy the following equation.

$$u_2^2 = (m + k)^2 + (\alpha + \beta)t. \tag{11}$$

Proof. Because $s = 2$, and $m_1 < m_2$, from Theorem 4, we know the tree $S(r; m_i) = S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral if and only if there exist positive integers u_i and nonnegative integers m_i $(i = 1, 2)$ such that $0 \leq \sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < +\infty$, and such that

$$a_k = \frac{\prod_{i=1}^{s}(u_i^2 - m_k)}{\prod_{i=1, i\neq k}^{s}(m_i - m_k)}, \quad (k = 1, 2) \tag{12}$$

are positive integers, and such that $a_i = 1$ must hold if m_i is not a perfect square.

We will discuss the following three cases:

Case 1. For $s = 2$, if m_1 is not a perfect square, by Theorem 11 (2), then the tree $S(r; m_i)$=$S(a_1 + a_2; m_1, m_2)$ of diameter 4 is not an integral tree.

Case 2. For $s = 2$, if m_2 is not a perfect square, then $a_2 = 1$ must hold, we choose $m_1 = m^2$, $u_1 = m+q$, $m_2(> m^2 + 2m + 1)$ is not a perfect square, where $m \geq 0$, $q \geq 1$, $0 \leq m_1(= m^2) < u_1^2(= (m+q)^2) < m_2 < u_2^2$, then a_1 must be a positive integer, and $a_2 = 1$. By Theorem 4, we have

$$a_1 = \frac{(u_1^2 - m_1)(u_2^2 - m_1)}{m_2 - m_1} = \frac{q(2m+q)(u_2^2 - m^2)}{m_2 - m^2}, \tag{13}$$

$$a_2 = \frac{(u_1^2 - m_2)(u_2^2 - m_2)}{m_1 - m_2} = \frac{(m_2 - (m+q)^2)(u_2^2 - m_2)}{m_2 - m^2} = 1. \tag{14}$$

From (13) and (14) or Theorem 3, we can get the following line equation system (15) in a_1, a_2 must have positive integral solutions $a_1, a_2(= 1)$.

$$\begin{cases} \frac{a_1}{q(2m+q)} + \frac{a_2}{(m+q)^2 - m_2} = 1, \\ \frac{a_1}{u_2^2 - m^2} + \frac{a_2}{u_2^2 - m_2} = 1. \end{cases} \tag{15}$$

By the first equation of Eqs.(15), we deduce $a_1 = q(2m+q)[a_2 + m_2 - (m+q)^2]/[m_2 - (m+q)^2]$. Since $([a_2 + m_2 - (m+q)^2], [m_2 - (m+q)^2]) = 1$, we obtain $(m_2 - (m+q)^2)|(q(2m+q))$ must hold. So, let $m_2 - (m+q)^2 = u$, $q(2m+q) = uv$, where u, v are positive integers. Then all positive integral solutions of the first equation of Eqs.(15) are $a_1 = v(u+1)$, $a_2 = 1$. By Corollary 5 (2), we know $r = \sum_{i=1}^{2} a_i = \sum_{i=1}^{2} u_i^2 - \sum_{i=1}^{2} m_i$. Then we deduce $u_2^2 - m_1 = u_2^2 - m^2 = a_1 + a_2 + m_2 - u_1^2 = v(u+1) + 1 + m_2 - (m+q)^2 = (u+1)(v+1)$. By Eq.(14), we get

$$u_2^2 - m_2 = \frac{m_2 - m^2}{m_2 - (m+q)^2} = \frac{u + q(2m+q)}{u} = v + 1.$$

Therefore, $a_1 = v(u+1)$, $a_2 = 1$ are also all positive integral solutions of the second equation of Eqs.(15).

Hence, when $m_2(> m^2 + 2m + 1)$ is not a perfect square, $m_1 = m^2$, $m_2 = (m+q)^2 + u$, $a_1 = v(u+1)$, $a_2 = 1$, $u_1 = m + q$, where integers $q(\geq 1)$, $m(\geq 0)$, $u(\geq 1)$, $v(\geq 1)$, $u_2(\geq 1)$ satisfy Eqs.(10), and such that $0 \leq m_1(= m^2) < u_1^2(= (m+q)^2) < m_2 < u_2^2$,

Case 3. For $s = 2$, when m_2 is a perfect square, then we choose $m_1 = m^2$, $m_2 = (m+k)^2$, $u_1 = m + q$, where $m \geq 0$, $k \geq 2$, $1 \leq q < k$, $0 \leq m_1(= m^2) < u_1^2(= (m+q)^2) < m_2(= (m+k)^2 < u_2^2$. From Theorem 4, we have

$$a_1 = \frac{(u_1^2 - m_1)(u_2^2 - m_1)}{m_2 - m_1} = \frac{q(2m+q)(u_2^2 - m^2)}{k(2m+k)}, \tag{16}$$

and

$$a_2 = \frac{(u_1^2 - m_2)(u_2^2 - m_2)}{m_1 - m_2} = \frac{(k-q)(2m+k+q)(u_2^2 - (m+k)^2)}{k(2m+k)}. \quad (17)$$

From (16), we get

$$u_2^2 = \frac{k(2m+k)a_1}{q(2m+q)} + m^2. \quad (18)$$

So, when m_1, m_2 are perfect squares, by Theorem 4, then the tree $S(r; m_i) = S(a_1+a_2; m_1, m_2)$ of diameter 4 is integral if and only if a_1, a_2 are positive integers and $\frac{k(2m+k)a_1}{q(2m+q)} + m^2 (= u_2^2)$ must be a perfect square. From (16) and (17), we can get the following Diophantine equation (19) in a_1, a_2.

$$\frac{a_1}{q(2m+q)} - \frac{a_2}{(k-q)(2m+q+k)} = 1. \quad (19)$$

Assume that $(q(2m+q), (k-q)(2m+q+k)) = d$, and let $q(2m+q) = d\alpha$, $(k-q)(2m+q+k) = d\beta$, $(\alpha, \beta) = 1$. Thus, Eq.(19) can be changed into

$$\beta a_1 - \alpha a_2 = d\alpha\beta. \quad (20)$$

From elementary number theory knowledge, all positive integral solutions of Eq.(20) are given by $a_1 = \alpha t + d\alpha = \alpha t + q(2m+q)$, $a_2 = \beta t$, where t is a positive integer. Hence, by Corollary 5(2), we get $u_2^2 = a_1 + a_2 + m_1 + m_2 - u_1^2 = (m+k)^2 + (\alpha+\beta)t$. Then $(m+k)^2 + (\alpha+\beta)t$ must be a perfect square.

Hence, when m_2 is a perfect square, $m_1 = m^2$, $m_2 = (m+k)^2$, $m \geq 0$, $k \geq 2$, let $(q(2m+q), (k-q)(2m+q+k)) = d$, $q(2m+q) = d\alpha$, $(k-q)(2m+q+k) = d\beta$, $(\alpha, \beta) = 1$, $a_1 = \alpha t + q(2m+q)$, $a_2 = \beta t$, $u_1 = m+q$, and integers $m(\geq 0)$, $k(\geq 2)$, $t(\geq 1)$, $u_2(\geq 1)$, $\alpha(\geq 1)$, $\beta(\geq 1)$ satisfy Eq.(11), and such that $0 \leq m_1 (= m^2) < u_1^2 (= (m+q)^2) < m_2 (= (m+k)^2) < u_2^2$, where $m \geq 0$, $k \geq 2$, $1 \leq q < k$.

Thus the theorem is proved. □

Corollary 13. *(i) For* $s = 2$, *let* $m_1 = m^2$, $m_2 = (m+q)^2 + u$, $a_1 = v(u+1)$, $a_2 = 1$, $u_1 = m+q$, *then the tree* $S(a_1+a_2; m_1, m_2)$ *of diameter 4 is integral with nonsquare number* m_2 *if one of the following cases holds where* $m(\geq 0)$, $q(\geq 1)$, $u(\geq 1)$, $v(\geq 1)$ *are integers:*

(1) ([16]) $u = q$, $v = 2m+q$, *where* $u_2 = m+q+1$.

(2) ([16]) $u = 2m+q$, $v = q$, *where* $u_2 = m+q+1$.

(3) ([11]) $u = k$, $v = q(2m+q)/k$, *where integers* $k(\geq 1)$, $q(\geq 1)$, $m(\geq 0)$, $u_2(\geq 1)$ *satisfy the following Diophantine equation*

$$ku_2^2 - (k+1)(m+q)^2 + m^2 = k(k+1). \quad (21)$$

(4) $u = q(2m+q)/k$, $v = k$, *where integers* $k(\geq 1)$, $q(\geq 1)$, $m(\geq 0)$, $u_2(\geq 1)$ *satisfy Eq.*(21).

(5) *Let* m_i, a_i $(i = 1, 2)$ *be positive integers as in Table 1, and let* q, m, u, v, u_1, u_2 *be as in Theorem 12 (1), they are different from those of (1) and (2) in Corollary 13 (i).(Table 1 is obtained by computer search, where* $1 \leq u_2 \leq 25$ *for Eqs.*(10).*)*

(ii) For $s = 2$, *when* $d = (q(2m+q), (k-q)(2m+q+k)) = 1$, *let* $m_1 = m^2$, $m_2 = (m+k)^2$, $m \geq 0$, $k \geq 2$, $a_1 = q(2m+q)(t+1)$, $a_2 = (k-q)(2m+q+k)t$, $t \geq 1$, $1 \leq q < k$, *and integers* $m(\geq 0)$, $k(\geq 2)$, $t(\geq 1)$, $u_2(\geq 1)$ *satisfy the following equation.*

$$u_2^2 = (m+k)^2 + k(2m+k)t. \tag{22}$$

Then the tree $S(a_1 + a_2; m_1, m_2)$ *of diameter 4 is integral.*

(iii) For $s = 2$, *when* $d = (q(2m+q), (k-q)(2m+q+k)) = 1$, *let* $m_1 = m^2$, $m_2 = (m+k)^2$, $a_1 = q(2m+q)(t+1)$, $a_2 = (k-q)(2m+q+k)t$, *where* $u_1 = m+q$, $m \geq 0$, $k \geq 2$, $t \geq 1$, $1 \leq q < k$. *Then the tree* $S(a_1 + a_2; m_1, m_2)$ *of diameter 4 is integral if one of the following cases holds where* $l(\geq 1)$, $m(\geq 0)$, $k(\geq 2)$, $p(\geq 1)$, $q(\geq 1)$, $v(\geq 1)$ *are integers:*

(1) $m = 0$, $t = p^2 - 1 > 0$, *where* $u_2 = kp$.

(2) *Let* m_i, a_i $(i = 1, 2)$ *be positive integers as in Table 2, and let* a_1, a_2, m_1, m_2, u_2, t, m, k *be as in Corollary 13 (ii), they are different from those of (1) in Corollary 13 (iii).(Table 2 is obtained by computer search, where* $1 \leq u_2 \leq 12$ *for Eq.*(22).*)*

(3) ([16]) $t = [k(2m+k)v + 2(m+k)]v$, *where* $u_2 = k(2m+k)v + (m+k)$.

(4) $t = [k(2m+k)v - 2(m+k)]v > 0$, *where* $u_2 = k(2m+k)v - (m+k) > 0$.

(5) $t = k(2m+k)v^2 + 2mv - 1 > 0$, *where* $u_2 = k(2m+k)v + m$.

(6) $t = k(2m+k)v^2 - 2mv - 1 > 0$, *where* $u_2 = k(2m+k)v - m > 0$.

(7) $t = 1$, $m = 2(p-1)^2 - 1 > 0$, $k = 2p$, *where* $u_2 = 2p^2 - 1 > 0$.

(8) $t = 2p$, $m = pl$, $k = 2l$, *where* $u_2 = 3pl + 2l$.

(9) $t = p$, $m = pl$, $k = 4l$, *where* $u_2 = 3pl + 4l$.

(10) $t = 4p$, $m = pk$, *where* $u_2 = 3pk + k$.

(11) $t = 3p$, $m = 4p + 1$, $k = 2$, *where* $u_2 = 8p + 3$.

(12) $t = v(l+v) - l - 1 > 0$, $m = pl$, $k = 2pv$, *where* $u_2 = p(2v^2 + 2vl - l) > 0$.

(13) $t = 2lp+p^2-2l-1 > 0$, $m = lv$, $k = pv$, where $u_2 = v(2lp+p^2-l) > 0$.

(iv) For $s = 2$, *when* $d = (q(2m+q),(k-q)(2m+q+k)) > 1$, *let* $q(2m+q) = d\alpha$, $(k-q)(2m+q+k) = d\beta$, $(\alpha,\beta) = 1$, $m_1 = m^2$, $m_2 = (m+k)^2$, $a_1 = \alpha t + q(2m+q)$, $a_2 = \beta t$, *where* $u_1 = m+q$, $m \geq 0$, $k \geq 2$, $t \geq 1$, $1 \leq q < k$. *Then the tree* $S(a_1+a_2; m_1, m_2)$ *of diameter 4 is integral if one of the following cases holds where* $\alpha(\geq 1)$, $\beta(\geq 1)$, $m(\geq 0)$, $k(\geq 2)$, $q(\geq 1)$, $t(\geq 1)$, $v(\geq 1)$ *are integers:*

(1) *Let* a_1, a_2, m_1, m_2, u_1, u_2, m, k, q, $e(= q(2m+q))$, $f(= (k-q)(2m+q+k))$, d *be positive integers as in Table 3, and let* a_1, a_2, m_1, m_2, u_2, m, k, q, d *be as in Theorem 12 (2), where* $u_1 = m+q$, $m \geq 0$, $k \geq 2$, $1 \leq q < k$. *(Table 3 is obtained by computer search, where* $1 \leq u_2 \leq 9$, $d > 1$ *for Eq.(11).)*

(2) $t = (\alpha+\beta)v^2 + 2(m+k)v$, *where* $u_2 = (\alpha+\beta)v + (m+k)$.

(3) $t = (\alpha+\beta)v^2 - 2(m+k)v > 0$, *where* $u_2 = (\alpha+\beta)v - (m+k) > 0$.

(4) $m = 1$, $k = 3$, $q = 1$, $t = 5v^2 \pm 8v > 0$, *where* $u_2 = 5v \pm 4 > 0$.

(5) $m = 0$, $k = 4$, $q = 2$, $t = v^2 - 4 > 0$, *where* $u_2 = 2v$.

(6) $m = 1$, $k = 4$, $q = 1$, $t = 2v^2 \pm 5v > 0$, *where* $u_2 = 4v \pm 5 > 0$.

(7) $m = 1$, $k = 4$, $q = 2$, $t = 3v^2 \pm 10v > 0$, *where* $u_2 = 3v \pm 5 > 0$.

(8) $m = 1$, $k = 4$, $q = 3$, $t = 2v^2 \pm 5v > 0$, *where* $u_2 = 4v \pm 5 > 0$.

(9) $m = 3$, $k = 4$, $q = 2$, $t = 5v^2 \pm 14v > 0$, *where* $u_2 = 5v \pm 7 > 0$.

(10) $m = 2$, $k = 5$, $q = 2$, $t = 15v^2 \pm 14v > 0$ *or* $t = 15v^2 \pm 4v - 3 > 0$, *where* $u_2 = 15v \pm 7 > 0$ *or* $u_2 = 15v \pm 2 > 0$.

(11) $m = 1$, $k = 5$, $q = 3$, $t = 7v^2 \pm 12v > 0$, *where* $u_2 = 7v \pm 6 > 0$.

(12) $m = 2$, $k = 5$, $q = 3$, $t = 15v^2 \pm 14v > 0$ *or* $t = 15v^2 \pm 4v - 3 > 0$, *where* $u_2 = 15v \pm 7 > 0$ *or* $u_2 = 15v \pm 2 > 0$.

(13) $m = 0$, $k = 6$, $q = 3$, $t = v^2 - 9 > 0$, *where* $u_2 = 2v$.

(14) $m = 2$, $k = 3$, $q = 2$, $t = 7v^2 \pm 10v > 0$, *where* $u_2 = 7v \pm 5 > 0$.

(15) $m = 1$, $k = 6$, $q = 1$, $t = 4v^2 \pm v - 3 > 0$, *where* $u_2 = 8v \pm 1 > 0$.

(16) $m = 0$, $k = 6$, $q = 2$, $t = v^2 - 4 > 0$, *where* $u_2 = 3v$.

(17) $m = 1$, $k = 6$, $q = 3$, $t = 4v^2 \pm v - 3 > 0$, *where* $u_2 = 8v \pm 1 > 0$.

(18) $m = 0$, $k = 6$, $q = 4$, $t = v^2 - 4 > 0$, *where* $u_2 = 3v$.

m_1	m_2	a_1	a_2	u_1	u_2	u	v	m	q
9	65	110	1	8	11	1	55	3	5
9	119	56	1	8	11	55	1	3	5
1	171	252	1	13	16	2	84	1	12
1	253	170	1	13	16	84	2	1	12
1	145	286	1	12	17	1	143	1	11
16	198	270	1	14	17	2	90	4	10
16	286	182	1	14	17	90	2	4	10
1	287	144	1	12	17	143	1	1	11
16	170	306	1	13	18	1	153	4	9
64	259	256	1	16	18	3	64	8	8
4	260	315	1	16	18	4	63	2	14
4	319	256	1	16	18	63	4	2	14
64	320	195	1	16	18	64	3	8	8
16	322	154	1	13	18	153	1	4	9
121	363	360	1	19	22	2	120	11	8
4	364	476	1	19	22	3	119	2	17
4	480	360	1	19	22	119	3	2	17
121	481	242	1	19	22	120	2	11	8
121	325	406	1	18	23	1	203	11	7
25	403	500	1	20	23	3	125	5	15
25	525	378	1	20	23	125	3	5	15
121	527	204	1	18	23	203	1	11	7
4	290	570	1	17	24	1	285	2	15
4	574	286	1	17	24	285	1	2	15
25	325	598	1	18	25	1	299	5	13
25	623	300	1	18	25	299	1	5	13

Table 1: Integral trees $S(a_1 + a_2; m_1, m_2)$ of diameter 4.

(19) $m = 1$, $k = 6$, $q = 4$, $t = 2v^2 \pm 14v > 0$, *where* $u_2 = 2v \pm 7 > 0$.

Proof. (i) We only prove (4). The results in (1)-(3) and (5) can be proved similarly by Theorem 12 (1).

When $s = 2$, $m_1 < m_2$, $m_2(> m^2 + 2m + 1)$ is not a perfect square, by Theorem 12 (1), we know the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral if and only if $m_1 = m^2$, $m_2 = (m+q)^2 + u$, $a_1 = v(u+1)$, $a_2 = 1$, and integers m (≥ 0), $q(\geq 1)$, $u(\geq 1)$, $v(\geq 1)$, $u_2(\geq 1)$ satisfy Eqs.(10).

Because $u = q(2m+q)/k$, $v = k$, we have

$$uv = q(2m+q)/k \cdot k = q(2m+q),$$

m_1	m_2	u_2	t	m	k	m_1	m_2	u_2	t	m	k
1	9	5	2	1	2	1	25	7	1	1	4
1	9	7	5	1	2	4	16	8	4	2	2
1	9	9	9	1	2	4	36	10	2	2	4
4	16	10	7	2	2	25	49	11	3	5	2
1	25	11	4	1	4	9	25	11	6	3	2
1	16	11	7	1	3	1	9	11	14	1	2
9	36	12	4	3	3	-	-	-	-	-	-

Table 2: Integral trees $S(a_1+a_2;m_1,m_2)$ of diameter 4, where $a_1 = q(2m+q)(t+1)$, $a_2 = (k-q)(2m+q+k)t$, $k \geq 2$, $1 \leq q < k$.

m_1	m_2	a_1	a_2	u_1	u_2	m	k	q	e	f	d
1	16	7	16	2	6	1	3	1	3	12	3
0	16	9	15	2	6	0	4	2	4	12	4
1	25	6	21	2	7	1	4	1	3	21	3
1	25	16	16	3	7	1	4	2	8	16	8
1	25	30	9	4	7	1	4	3	15	9	3
0	16	16	36	2	8	0	4	2	4	12	4
1	25	21	26	3	8	1	4	2	8	16	8
9	49	22	9	5	8	3	4	2	16	24	8
4	49	16	11	4	8	2	5	2	12	33	3
1	36	27	16	4	8	1	5	3	15	20	5
4	49	28	8	5	8	2	5	3	21	24	3
0	36	16	21	3	8	0	6	3	9	27	9
1	16	16	52	2	9	1	3	1	3	12	3
4	25	44	24	4	9	2	3	2	12	9	3
1	25	10	49	2	9	1	4	1	3	21	3
1	25	50	21	4	9	1	4	3	15	9	3
1	49	5	30	2	9	1	6	1	3	45	3
0	36	9	40	2	9	0	6	2	4	32	4
1	49	25	22	4	9	1	6	3	15	33	3
0	36	36	25	4	9	0	6	4	16	20	4
1	49	40	16	5	9	1	6	4	24	24	24

Table 3: Integral trees $S(a_1+a_2;m_1,m_2)$, where $e = q(2m+q)$, $f = (k-q)(2m+q+k)$, $d = (e,f)$.

and $\quad u_2^2 = 1+u+v+(m+q)^2 = 1+q(2m+q)/k+k+(m+q)^2.$

$\Rightarrow ku_2^2-(k+1)(m+q)^2+m^2 = k(k+1).$

Hence, by Theorem 12 (1), when $m_1 = m^2$, $m_2 = (m+q)^2+u$, $a_1 = v(u+1)$, $a_2 = 1$, $u = q(2m+q)/k$, $v = k$, and integers k (≥ 1), $q(\geq 1)$, $m(\geq 0)$, $u_2(\geq 1)$ satisfy Eq.(21). Then the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral.

(ii) For $s = 2$, when $d = (q(2m+q), (k-q)(2m+q+k)) = 1$, $m_1 = m^2$, $m_2 = (m+k)^2$, $m \geq 0$, $k \geq 2$, $1 \leq q < k$, by Theorem12 (2), we obtain $\alpha = q(2m+q)$, $\beta = (k-q)(2m+q+k)$, $a_1 = \alpha t+q(2m+q) = q(2m+q)(t+1)$, $a_2 = \beta t = (k-q)(2m+q+k)t$, where $m \geq 0$, $k \geq 2$, $t \geq 1$, $1 \leq q < k$, and integers $m(\geq 0)$, $k(\geq 2)$, $t(\geq 1)$, $u_2(\geq 1)$, $\alpha(\geq 1)$, $\beta(\geq 1)$ satisfy $u_2^2 = (m+k)^2 + (\alpha+\beta)t = (m+k)^2 + k(2m+k)t$. Thus, by Theorem 12 (2), the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral.

(iii) We only prove (1). The results in (2)-(13) can be proved similarly by Corollary 13 (ii) or Theorem 12 (2).

When $s = 2$, $m_1 < m_2$, m_2 is a perfect square, $d = (q(2m+q), (k-q)(2m+q+k)) = 1$, by Corollary 13 (ii) or Theorem 12 (2), then the tree $S(r; m_i)$=$S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral if $m_1 = m^2$, $m_2 = (m+k)^2$, $a_1 = q(2m+q)(t+1)$, $a_2 = (k-q)(2m+q+k)t$, where $m \geq 0$, $k \geq 2$, $1 \leq q < k$, $t \geq 1$, and integers $m(\geq 0)$, $k(\geq 2)$, $t(\geq 1)$, $u_2(\geq 1)$ satisfy Eq.(22).

Because $m = 0$, $t = p^2 - 1 > 0$, we have

$u_2^2 = (m+k)^2 + k(2m+k)t = (0+k)^2 + k(0+k)(p^2-1) = k^2p^2.$

Hence, when $m_1 = m^2 = 0$, $m_2 = (m+k)^2 = k^2$, $k \geq 2$, $t = p^2 - 1 > 0$, $a_1 = q(2m+q)(t+1) = p^2q^2$, $a_2 = (k-q)(2m+q+k)t = (k-q)(q+k)(p^2-1)$, $1 \leq q < k$, by Corollary 13 (ii) or Theorem 12 (2), the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral.

(iv) We only prove (10). The results in (1)-(9) and (11)-(19) can be proved similarly by Theorem 12 (2). Here integral trees $S(a_1 + a_2; m_1, m_2)$ of diameter 4 in (3)-(19) are constructed from Table 3 by Theorem 12 (2).

When m_2 is a perfect square, $m_1 < m_2$, m_2 is a perfect square, $d = (q(2m+q), (k-q)(2m+q+k)) > 1$, by Theorem 12 (2), then the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral if $m_1 = m^2$, $m_2 = (m+k)^2$, $m \geq 0$, $k \geq 2$, let $q(2m+q) = d\alpha$, $(k-q)(2m+q+k) = d\beta$, $(\alpha, \beta) = 1$, $a_1 = \alpha t + q(2m+q)$, $a_2 = \beta t$, where $t \geq 1$, $1 \leq q < k$, and integers $m(\geq 0)$, $k(\geq 2)$, $t(\geq 1)$, $u_2(\geq 1)$, $\alpha(\geq 1)$, $\beta(\geq 1)$ satisfy Eq.(11).

Since $m = 2$, $k = 5$, $q = 2$, $d = (q(2m+q), (k-q)(2m+q+k)) = (12, 33) = 3$, Thus, $\alpha = 4$, $\beta = 11$. Hence, Eq.(11) can be changed into

$$u_2^2 = (m+k)^2 + (\alpha+\beta)t = 15t + 49. \tag{23}$$

From elementary number theory knowledge, all positive integral solutions of Eq.(23) are given by $(t, u_2) = (15v^2 \pm 14v, 15v \pm 7)$ or $(t, u_2) =$

$(15v^2 \pm 4v - 3, 15v \pm 2)$, where v, t, u_2 are positive integers.

Hence, when $m = 2$, $k = 5$, $q = 2$, $m_1 = m^2 = 4$, $m_2 = (m+k)^2 = 49$, $a_1 = \alpha t + q(2m+q) = 4t + 12 > 0$, $a_2 = \beta t = 11t > 0$, where $t = 15v^2 \pm 14v > 0$ or $t = 15v^2 \pm 4v - 3 > 0$, $u_1 = m + q = 4$, $u_2 = 15v \pm 7 > 0$ or $u_2 = 15v \pm 2 > 0$, and $v(\geq 1)$ is an integer. Then the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral. Thus we can construct infinitely many integral trees $S(a_1 + a_2; m_1, m_2)$ of diameter 4 from the ninth row in Table 3. □

Next we shall give a method by the following cases for finding the solutions of the Diophantine equation (21). We discuss k and m. Choosing (1) $k = 1$, $m = m$, (2) $k = k$, $m = k$, (3) $k = k$, $m = k + 1$,..., where $k(\geq 1)$, $m(\geq 0)$ are integers, we get the following corollary.

Corollary 14. *For $s = 2$, let $m_1 = m^2$, $m_2 = (m+q)^2 + u$, $a_1 = v(u+1)$, $a_2 = 1$, $u_1 = m + q$, where (i) $u = k$, $v = q(2m+q)/k$, or (ii) $u = q(2m+q)/k$, $v = k$. Then the tree $S(a_1 + a_2; m_1, m_2)$ of diameter 4 is integral with nonsquare number m_2 if one of the following cases holds where $m(\geq 0)$, $k(\geq 1)$, $q(\geq 1)$, $u(\geq 1)$, $v(\geq 1)$, $u_2(\geq 1)$ are integers:*

(1) When $k = 1$, and integers $q(\geq 1)$, $m(\geq 0)$, $u_2(\geq 1)$ satisfy the following Diophantine equation

$$u_2^2 - 2(m+q)^2 = 2 - m^2. \tag{24}$$

(2) When $k = k$, $m = k$, and integers $q(\geq 1)$, $k(= m \geq 0)$, $u_2(\geq 1)$ satisfy the following Diophantine equation

$$u_2^2 - k(k+1)[(k+q)/k]^2 = 1. \tag{25}$$

(3) When $k = k$, $m = k + 1$, and positive integers q, k, u_2 satisfy the following Diophantine equation

$$[(k+1)+q]^2 - k(k+1)[u_2/(k+1)]^2 = 1. \tag{26}$$

Proof. We can check the validity by Theorem 12 (1) or (3) and (4) of Corollary 13 (i). □

Using Lemma 9 and elementary number theory knowledge, all positive integral solutions of the Diophantine equations (24), (25) and (26) are given respectively by

$$u_2 + (m+q)\sqrt{2} = (3 + 2\sqrt{2})^n[(m+2) + (m+1)\sqrt{2}], \quad n = 0, 1, 2, \ldots, \tag{27}$$

$$u_2 + [(k+q)/k]\sqrt{k(k+1)} = [(2k+1) + 2\sqrt{k(k+1)}]^n, \quad n = 1, 2, \ldots, \tag{28}$$

$$(k+1+q)+[u_2/(k+1)]\sqrt{k(k+1)} = [(2k+1)+2\sqrt{k(k+1)}]^n,\ n=1,2,\ldots. \tag{29}$$

The authors of [2,8,11,12,14-17] constructed integral trees $S(a_1+a_2+\ldots+a_s;\ m_1,\ m_2,\ \ldots,\ m_s)$ of diameter 4 which the number of nonsquares among $m_1, m_2, \ldots, m_s$ is 1 and 2 respectively. We shall give a kind of integral trees $S(a_1+a_2+\ldots+a_s; m_1, m_2, \ldots, m_s)$ of diameter 4 which the number of nonsquares among $m_1, m_2, \ldots, m_s$ is 2. This result generalizes a result of Zhang at el. [17].

Theorem 15. *For $s = 3$, let integers $m_i\ (\geq 0)$, $a_i\ (\geq 1)$, $u_i\ (\geq 1)$ $(i = 1,2,3)$ be those of Theorem 4, given in the following items (i) or (ii), where $\alpha_n = \sum_{j=0}^{[\frac{n}{2}]} 8^j 3^{n-2j} \binom{n}{2j}$, $\beta_n = 2\sum_{i=0}^{[\frac{n-1}{2}]} 8^i 3^{n-2i-1} \binom{n}{2i+1}$, and n is a positive integer. Then the tree $S(a_1+a_2+a_3;\ m_1,\ m_2,\ m_3)$ of diameter 4 is integral with nonsquares m_2 and m_3. (Note that the identity $\alpha_n^2 = 2\beta_n^2+1$ holds for any positive integer n.)*

(i) $m_1 = a^2$, $m_2 = [(a+1)\alpha_n + (2a+1)\beta_n]^2 - [(2a+1)\alpha_n + 2(a+1)\beta_n]$, $m_3 = [(a+1)\alpha_n + (2a+1)\beta_n]^2 + [(2a+1)\alpha_n + 2(a+1)\beta_n]$, $a_1 = [(a+1)\alpha_n + (2a+1)\beta_n]^2 - a^2$, $a_2 = a_3 = 1$, $u_2 = (a+1)\alpha_n + (2a+1)\beta_n$, $u_1 = u_2 - 1$, $u_3 = u_2 + 1$, *where a is a nonnegative integer.*

(ii) $m_1 = a^2$, $m_2 = [(a-1)\alpha_n + (2a-1)\beta_n]^2 - [(2a-1)\alpha_n + 2(a-1)\beta_n]$, $m_3 = [(a-1)\alpha_n + (2a-1)\beta_n]^2 + [(2a-1)\alpha_n + 2(a-1)\beta_n]$, $a_1 = [(a+1)\alpha_n + (2a+1)\beta_n]^2 - a^2$, $u_2 = (a-1)\alpha_n + (2a-1)\beta_n$, $u_1 = u_2 - 1$, $u_3 = u_2 + 1$, $a_2 = a_3 = 1$, *where $a\ (\geq 2)$ is an integer.*

Proof. For $s = 3$, assume that $m_1 = a^2$, $m_2 = d^2 - k$, $m_3 = d^2 + k$, $u_1 = d - 1$, $u_2 = d$, $u_3 = d + 1$, where $d \geq a + 2$, $1 \leq k \leq 2d - 2$. By Corollary 5, we get that $a_1 = \sum_{i=1}^3 u_i^2 - \sum_{i=1}^3 m_i - \sum_{i=2}^3 a_i$. From Theorem 4, we know that the tree $S(a_1+a_2+a_3; m_1, m_2, m_3)$ of diameter 4 is integral if and only if

$$a_2 = \frac{(2d-1-k)(2d+1+k)}{2(d^2-k-a^2)} = 1, \tag{30}$$

$$a_3 = \frac{(2d-1+k)(2d+1-k)}{2(d^2+k-a^2)} = 1. \tag{31}$$

From (30) and (31), we get the same Diophantine equation

$$k^2 - 2d^2 = 2a^2 - 1. \tag{32}$$

For any given nonnegative integer a, let K be any associate class of solutions of the Diophantine equation (32), and let k^*, d^* be the fundamental solution of the class K of Eq.(32), by Lemma 9, we get that all positive integral solutions of the class K of Eq.(32) are given by

$$k + d\sqrt{2} = (3 + 2\sqrt{2})^n (k^* + d^*\sqrt{2}). \tag{33}$$

From (33), it deduces that $k + d\sqrt{2} = (\alpha_n + \beta_n\sqrt{2})(k^* + d^*\sqrt{2})$, where $\alpha_n = \sum_{j=0}^{[\frac{n}{2}]} 8^j 3^{n-2j} \binom{n}{2j}$ and $\beta_n = 2\sum_{i=0}^{[\frac{n-1}{2}]} 8^i 3^{n-2i-1} \binom{n}{2i+1}$, and it is not difficult to prove that the identity $\alpha_n^2 = 2\beta_n^2 + 1$ holds for any positive integer n.

Hence, we get all positive integral solutions of the class K of Eq.(32) are given by

$$\begin{cases} k = \alpha_n k^* + 2\beta_n d^*, \\ d = \beta_n k^* + \alpha_n d^*. \end{cases} \tag{34}$$

Next we discuss the following two cases.

Case 1. Because $k^* = 2a+1$, $d^* = a+1$ is the fundamental solution of one associate class K' of Eq.(32), all positive integral solutions of Eq.(32) are given by

$$\begin{cases} k = (2a+1)\alpha_n + 2(a+1)\beta_n, \\ d = (2a+1)\beta_n + (a+1)\alpha_n, \end{cases} \tag{35}$$

for any positive integer n. Clearly, it is easy to prove that k, d of (35) satisfy the conditions $d \geq a+2$ and $1 \leq k \leq 2d-2$.

Hence, when $m_1 = a^2$, $m_2 = d^2 - k = [(a+1)\alpha_n + (2a+1)\beta_n]^2 - [(2a+1)\alpha_n + 2(a+1)\beta_n]$, $m_3 = d^2 + k = [(a+1)\alpha_n + (2a+1)\beta_n]^2 + [(2a+1)\alpha_n + 2(a+1)\beta_n]$, $a_1 = \sum_{i=1}^3 u_i^2 - \sum_{i=1}^3 m_i - \sum_{i=2}^3 a_i = [(a+1)\alpha_n + (2a+1)\beta_n]^2 - a^2$, $a_2 = a_3 = 1$, $u_2 = d = (a+1)\alpha_n + (2a+1)\beta_n$, $u_1 = u_2 - 1$, $u_3 = u_2 + 1$, where n (≥ 1) and a (≥ 0) are integers. Then the tree $S(a_1+a_2+a_3; m_1, m_2, m_3)$ of diameter 4 is integral with nonsquares m_2 and m_3.

On the other hand, although $k^* = 2a+1$, $d^* = a+1$ is a solution of the associate class K' of Eq.(32), they do not satisfy the conditions $d^* \geq a+2$ and $1 \leq k^* \leq 2d^* - 2$. So we cannot construct such integral trees with nonsquares m_2 and m_3 from $k^* = 2a+1$, $d^* = a+1$, where a is a nonnegative integer.

Case 2. Similarly, because $k^* = 2a-1$, $d^* = a-1$ is the fundamental solution of the other associate class K'' of Eq.(32), where a (≥ 2) is an integer, all positive integral solutions of Eq.(32) are given by

$$\begin{cases} k = (2a-1)\alpha_n + 2(a-1)\beta_n, \\ d = (2a-1)\beta_n + (a-1)\alpha_n, \end{cases} \tag{36}$$

for any positive integer n.

Hence, when $m_1 = a^2$, $m_2 = d^2 - k = [(a-1)\alpha_n + (2a-1)\beta_n]^2 - [(2a-1)\alpha_n + 2(a-1)\beta_n]$, $m_3 = d^2 + k = [(a-1)\alpha_n + (2a-1)\beta_n]^2 + [(2a-1)\alpha_n + 2(a-1)\beta_n]$, $a_1 = \sum_{i=1}^{3} u_i^2 - \sum_{i=1}^{3} m_i - \sum_{i=2}^{3} = [(a+1)\alpha_n + (2a+1)\beta_n]^2 - a^2$, $u_2 = d = (a-1)\alpha_n + (2a-1)\beta_n$, $u_1 = u_2 - 1$, $u_3 = u_2 + 1$, $a_2 = a_3 = 1$, where n (≥ 1) and a (≥ 2) are integers. Then the tree $S(a_1 + a_2 + a_3; m_1, m_2, m_3)$ of diameter 4 is integral with nonsquares m_2 and m_3.

On the other hand, although $k^* = 2a - 1$, $d^* = a - 1$ is a solution of the associate class K'' of Eq.(32), they do not satisfy the conditions $d^* \geq a + 2$ and $1 \leq k^* \leq 2d^* - 2$. So we cannot construct such integral trees with nonsquare numbers m_2 and m_3 from $k^* = 2a - 1$, $d^* = a - 1$, where a (≥ 2) is an integer. □

Remark 16. *In fact, Zhang and Tan [17] constructed such integral trees* $S(a_1 + a_2 + a_3;\ m_1,\ m_2, m_3)$=$S(a_1 + 1 + 1; m_1, m_2, m_3)$ *of diameter 4 with nonsquares* m_2 *and* m_3 *from some positive integral solutions of Eq.*(32). *Here, Theorem 15 is obtained from all positive integral solutions of Eq.*(32).

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References

[1] K. T. Balińska, D. Cvetković, Z. Radosavljević, S. K. Simić, D. Stevanović, A survey on integral graphs, *Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat.* **13** (2002), 42-65.

[2] Z.F. Cao, On the integral trees of diameter R when $3 \leq R \leq 6$ (in Chinese), *J. Heilongjiang University* **2** (1988), 1-3, 95.

[3] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs–Theory and Application*, Academic Press, New York, Francisco, London 1980.

[4] F. Harary and A.J. Schwenk, Which graphs have integral spectra ?, In: R. Bari, F. Harary (eds.), *Graphs and Combinatorics*, Lecture Notes in Mathematics, Vol.406, Springer, Berlin 1974, pp.45-51.

[5] P. Híc and M. Pokorný, On integral balanced rooted trees of diameter 10, *Acta Univ. M. Belii Math* **10** (2003), 9-15.

[6] M.S. Li, W.S. Yang and J.B. Wang, Notes on the spectra of trees with small diameters, *J. Changsha Railway University* **18** (2) (2000), 84-87.

[7] X.L. Li and G.N. Lin, On integral trees problems(in English), *Chinese Science Bulletin* **33** (10) (1988), 802-806, or X.L. Li and G.N. Lin, On the problem of integral trees(in Chinese), *Chinese Science Bulletin* **32** (11) (1987), 813-816 .

[8] R.Y. Liu, Some new families of trees with integral eigenvalues (in Chinese), *J. Qinghai Normal University* **3** (1988), 1-5.

[9] H.Z. Ren, On integral trees with diameter four (in Chinese), *J. Qinghai Normal University* **1** (2000), 8-11.

[10] K.H. Rosen, *Elementary Number Theory and Its Applications*, Addison-Wesley Publishing Company, London, Amsterdam (1984), 87-90.

[11] L.G. Wang, X.L. Li and X.J. Yao, Integral trees with diameters 4, 6 and 8, *Australasian J. Combinatorics* **25** (2002), 29-44.

[12] L.G. Wang, X.L. Li and S.G. Zhang, Families of integral trees with diameters 4, 6 and 8, *Discrete Appl. Math.* **136** (2-3) (2004), 349-362.

[13] M. Watanabe and A.J. Schwenk, Integral startlike trees, *J. Austral. Math. Soc. Ser. A* **28** (1979), 120-128.

[14] D.S. Xu, A note on the integral trees with diameter 4 (in Chinese), *J. Qinghai University* **14** (2) (1996), 16-18.

[15] D.S. Xu, Some new classes of the integral trees with diameter 4 (in Chinese), *J. Hebei University* **17** (1) (1997), 12-16.

[16] P.Z. Yuan, Integral trees of diameter 4 (in Chinese), *J. Systems Sci. Math. Sci.* **18** (2) (1998), 177-181.

[17] D.L. Zhang and S.W. Tan, On integral trees of diameter 4 (in Chinese), *J. Systems Sci. Math. Sci.* **20** (3) (2000), 330-337.

[18] C.Z. Zhou, *Fibonacci and Lucas Sequence and Their Applications* (in Chinese), Hunan Science and Technology Press, Changsha (1993).

Some new classes of upper embeddable graphs *

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Abstract: Combining with specific degrees or edges of a graph, this paper provides some new classes of upper embeddable graphs and extends the results in [Y. Huang,Y. Liu, Some classes of upper embeddable graphs, Acta Mathematica Scientia,1997,17(Supp.): 154-161].

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1 Introduction

Since the investigation of maximum genus was introduced by Nordhaus et al. [1] in 1971, the upper embeddability of graphs has received great emphasis. Nordhaus et al.[1], Nebesk[2], Ringeisen[3],and Skoviera[4] have shown that various classes of graphs are upper-imbeddable. In particular, every 4-edge connected graph is upper-imbeddable in Kundu[5]. However, there are examples of 3-edge connected graphs that are not upper imbeddable in Jungerment[6].

Combining with some invariants of a graph, many papers have provided distinct kinds of upper embeddable graphs with

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edge-connectivity ≤ 3 or have given the lower bounds of the maximum genus of graphs.

Recently, combining with the specific degrees or edges of a graph, Huang [7] gives some special upper embeddable graphs.

In this paper, we improve Huang's results as follows:

(1) Let G be a bipartite graph. If $d_G(v) = a(mod2a)$ holds for every vertex $v \in V(G)$, then G is upper embeddable, where $a = 0(mod2), a \geq 4$.

(2) Let G be a $k(k \geq 4)$-regular and 3-connected bipartite graph, then G is upper embeddable.

(3) Let G be a graph(loops and multiple edges are permitted). If G has at most two cut-vertices and every edge e(not a loop)in G is triangular edge, then G is upper embeddable.

(4) Let G be a graph(loops and multiple edges are permitted) and every edge e(not a loop)in G is a triangular edge. If $k(G) \geq 2$, then the inequality $\xi(G) \leq k(G)-1$ holds, where $k(G)$ denotes the number of cut-vertex in graph G.

2 Some definitions and notations

A graph, which may have multiple adjacencies or loops, is always assumed to be connected unless the context requires. The general background of this paper can be seen in White [8] or Liu [9], Huang [7].

Embedding a graph G in S means that the vertices and the edges of the graph are placed in the surface, and the edges may meet only at mutually incident vertices. A 2-cell embedding, or in other words, cellular embedding, of a graph G is the one in which each of the components of the complement of G in the surface is homeomorphic to an open disk. The components of the complement of G are called faces or regions.

The genus, denoted by $\gamma(G)$, of a connected graph G, is the smallest value of $g(S)$, where S is a surface in which G has a 2-cell embedding.

The maximum genus of a connected graph G, marked by $\gamma_M(G)$, is defined to be the maximum genus k of the orientable surface where a cellular embedding of G into the orientable surface of genus k exists.

From the Euler polyhedral formula, it can be seen obviously that $\gamma_M(G)$ has the upper bound

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ is the cycle rank (or Betti number) of the graph G. A connected graph G is upper embeddable if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$ holds exactly.

The deficiency $\xi(G,T)$ of a spanning tree T in a graph G is the number of components of $G \backslash E(T)$ which have an odd number of edges. The Betti deficiency $\xi(G)$ of the graph G is defined as the minimum of $\xi(G,T)$ over all spanning tree T of G. Note that $\xi(G) = \beta(G)(mod 2)$.

For a subset $A \subseteq E(G)$,$c(G\backslash A)$ denotes the number of all components of $G\backslash A$, and $b(G\backslash A)$ denotes the number of components of $G\backslash A$ with odd circle ranks.

G is a k-regular graph if and only if the degree of every vertex in the graph is k.

A cut-vertex is such a vertex that it will disconnect the graph when it is removed from the graph. A graph G is $k-$connected if the removal of any $k-1$ vertices in G does not disconnect the graph.

3 Some basic theorems

Firstly, two characterizations of the upper embeddability of graphs were stated. Their proofs can be seen in Liu [9] and Nebesky [10] respectively. Here, they were expressed in the following theorems:

Theorem A(Liu [9]) Given a graph G, then
(1) $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$;
(2) G is upper embeddable if and only if $\xi(G) \leq 1$ holds.
From the Theorem A(1), it shows clearly that the maximum

genus of G is mainly determined by the Betti deficiency $\xi(G)$, for which Nebesky [10] has given another combinatorial expression.

Theorem B(Nebesky [10]) Given a graph G, then we have

$$\xi(G) = max_{A \subseteq E(G)}\{c(G \backslash A) + b(G \backslash A) - |A| - 1\}$$

. Let F and H be two disjoint subgraphs of the graph G. Let $E(F, H)$ denote such edges that one endpoint is in $V(F)$ and the other in $V(H)$. Let $E(F, G)$ denote such edges that one endpoint is in $V(F)$ and the other not in $V(F)$.

The following theorem in [11] provides a structural characterization for a non-upper embeddable graph, i.e., graph G with $\xi(G) \geq 2$, and plays a fundamental role throughout this paper.

Theorem C(Huang [11]) If the graph G is not upper embeddable, i.e., $\xi(G) \geq 2$, then there exists an edge subset A of G satisfying the following properties:

(1) $c(G \backslash A) \geq 2$, and furthermore for any component F of $G \backslash A$, $\beta(F) \equiv 1(mod 2)$;

(2) For any component F of $G \backslash A$, F is a vertex-induced subgraph of G;

(3) For any $k (\geq 2)$ different connected components $F_1, F_2, \cdots,$ F_k, then $|E_G(F_1, F_2, \cdots, F_k)| \leq 2k - 3$;

(4) $\xi(G) = 2c(G \backslash A) - |A| - 1$.

Supposed A is such a chosen edge subset of G as in theorem C above, the following result can be obtained, as a continuation of Theorem C:

Theorem D(Huang [11]) Under the conditions and the conclusions of Theorem C, then we have

(1)For any connected component F of $G \backslash A$, let G be a graph with k-connectivity($k \geq 1$), then $|E(G, F)| \geq k$;

(2)$|A| = \frac{1}{2} \sum_F |E(G \backslash F, F)|$, where the sum is taken over all connected components F of $G \backslash A$.

4 Upper embeddable bipartite graphs with specific degrees

In this section, we investigate the upper embeddability of bipartite graphs. Firstly, the following lemma can be proved:

Lemma 1 Let G be a bipartite graph, and $d_G(v) = a(mod2a)$ holds for every vertex v in $V(G)$, where $a = 0(mod2), a \geq 4$. If there exist such two edges e_1,$e_2 \in E(G)$ that making $G\backslash\{e_1, e_2\}$ disconnected, then $|V(F)| = 0(mod2)$ and $\beta(F) = 0(mod2)$ hold for any connected components F of $G\backslash\{e_1, e_2\}$.

Proof G has no cut-edge, for it is Eulerian. If $G\backslash\{e_1, e_2\}$ is disconnected, then it has two connected components exactly. Let one be F and the other be H. Then, without loss of generality, let $e_1 = x_1y_1$ and $e_2 = x_2y_2$, where $x_1, x_2 \in V(F)$ and $y_1, y_2 \in V(H)$. Because G is a bipartite graph, F is too. Let $V(F) = V_1 \cup V_2$ be the bipartite partition of $V(F)$.

Claim 1 For the two vertices x_1, x_2, one of them is in V_1 and the other in V_2.

Proof By contradiction, without loss of generality, we assume that $x_1, x_2 \in V_1$. Because F is a bipartite graph, $\sum_{x\in V_1} d_G(x) -2 = \sum_{y\in V_2} d_G(y)$ could be obtained easily. However, $d_G(v) = a(mod2a)$holds for every vertex $v \in V(G)$, so the above equality implies $-2 = 0(moda)$, whilst $a = 0(mod2), a \geq 4$. A contradiction appears.

Claim 2 $|V(F)| = 0(mod2)$

Proof Since F is a bipartite graph and Claim 1 holds, so $|E(F)| = \sum_{x\in V_1} d_G(x) - 1 = \sum_{y\in V_2} d_G(y) - 1$ could be derived, then $\sum_{x\in V_1} d_G(x) = \sum_{y\in V_2} d_G(y)$ holds. As $d_G(v) = a(mod2a)$ holding for every vertex $v \in V(G)$, where $a = 0(mod2), a \geq 4$, it follows that $|V_1| = |V_2|(mod2)$. Thus $|V(F)| = 0(mod2)$ holds.

Claim 3 $\beta(F) = 0(mod2)$

Proof Since the Claim 2 and $|E(F)| = \sum_{x\in V_1} d_G(x) - 1 = 1(mod2)$ hold. So we could obtain $\beta(F) = |E(F)| - |V(F)| + 1 =$