Some new classes of upper embeddable graphs *

Duanfeng Liu [1,2] , Xinru Liu [1]

[1]Department of Mathematics Science and Computer Technology,Central South University,Changsha 410083,P.R.China

[2]Department of Applied Mathematics,Guangdong University of Technology, Guangzhou 510006,P.R.China

Abstract: Combining with specific degrees or edges of a graph, this paper provides some new classes of upper embeddable graphs and extends the results in [Y. Huang,Y. Liu, Some classes of upper embeddable graphs, Acta Mathematica Scientia,1997,17(Supp.): 154-161].

Keywords: Degree; maximum genus; upper embeddable

AMS Subject Classification: O5C

1 Introduction

Since the investigation of maximum genus was introduced by Nordhaus et al. [1] in 1971, the upper embeddability of graphs has received great emphasis. Nordhaus et al.[1], Nebesk[2], Ringeisen[3],and Skoviera[4] have shown that various classes of graphs are upper-imbeddable. In particular, every 4-edge connected graph is upper-imbeddable in Kundu[5]. However, there are examples of 3-edge connected graphs that are not upper imbeddable in Jungerment[6].

Combining with some invariants of a graph, many papers have provided distinct kinds of upper embeddable graphs with

[1]*Supported by youth funding the project of Guangdong University of Technology(062060)

[2]E-mail: Duanfeng Liu (ma.frank@163.com)

edge-connectivity ≤ 3 or have given the lower bounds of the maximum genus of graphs.

Recently, combining with the specific degrees or edges of a graph, Huang [7] gives some special upper embeddable graphs.

In this paper, we improve Huang's results as follows:

(1) Let G be a bipartite graph. If $d_G(v) = a(mod2a)$ holds for every vertex $v \in V(G)$, then G is upper embeddable, where $a = 0(mod2), a \geq 4$.

(2) Let G be a $k(k \geq 4)$-regular and 3-connected bipartite graph, then G is upper embeddable.

(3) Let G be a graph(loops and multiple edges are permitted). If G has at most two cut-vertices and every edge e(not a loop)in G is triangular edge, then G is upper embeddable.

(4) Let G be a graph(loops and multiple edges are permitted) and every edge e(not a loop)in G is a triangular edge. If $k(G) \geq 2$, then the inequality $\xi(G) \leq k(G)-1$ holds, where $k(G)$ denotes the number of cut-vertex in graph G.

2 Some definitions and notations

A graph, which may have multiple adjacencies or loops, is always assumed to be connected unless the context requires. The general background of this paper can be seen in White [8] or Liu [9], Huang [7].

Embedding a graph G in S means that the vertices and the edges of the graph are placed in the surface, and the edges may meet only at mutually incident vertices. A 2-cell embedding, or in other words, cellular embedding, of a graph G is the one in which each of the components of the complement of G in the surface is homeomorphic to an open disk. The components of the complement of G are called faces or regions.

The genus, denoted by $\gamma(G)$, of a connected graph G, is the smallest value of $g(S)$, where S is a surface in which G has a 2-cell embedding.

The maximum genus of a connected graph G, marked by $\gamma_M(G)$, is defined to be the maximum genus k of the orientable surface where a cellular embedding of G into the orientable surface of genus k exists.

From the Euler polyhedral formula, it can be seen obviously that $\gamma_M(G)$ has the upper bound

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ is the cycle rank (or Betti number) of the graph G. A connected graph G is upper embeddable if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$ holds exactly.

The deficiency $\xi(G,T)$ of a spanning tree T in a graph G is the number of components of $G \backslash E(T)$ which have an odd number of edges. The Betti deficiency $\xi(G)$ of the graph G is defined as the minimum of $\xi(G,T)$ over all spanning tree T of G. Note that $\xi(G) = \beta(G)(mod2)$.

For a subset $A \subseteq E(G)$,$c(G \backslash A)$ denotes the number of all components of $G \backslash A$, and $b(G \backslash A)$ denotes the number of components of $G \backslash A$ with odd circle ranks.

G is a k-regular graph if and only if the degree of every vertex in the graph is k.

A cut-vertex is such a vertex that it will disconnect the graph when it is removed from the graph. A graph G is $k-$connected if the removal of any $k-1$ vertices in G does not disconnect the graph.

3 Some basic theorems

Firstly, two characterizations of the upper embeddability of graphs were stated. Their proofs can be seen in Liu [9] and Nebesky [10] respectively. Here, they were expressed in the following theorems:

Theorem A(Liu [9]) Given a graph G, then
(1) $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$;
(2) G is upper embeddable if and only if $\xi(G) \leq 1$ holds.
From the Theorem A(1), it shows clearly that the maximum

genus of G is mainly determined by the Betti deficiency $\xi(G)$, for which Nebesky [10] has given another combinatorial expression.

Theorem B(Nebesky [10]) Given a graph G, then we have

$$\xi(G) = max_{A\subseteq E(G)}\{c(G\backslash A) + b(G\backslash A) - |A| - 1\}$$

. Let F and H be two disjoint subgraphs of the graph G. Let $E(F,H)$ denote such edges that one endpoint is in $V(F)$ and the other in $V(H)$. Let $E(F,G)$ denote such edges that one endpoint is in $V(F)$ and the other not in $V(F)$.

The following theorem in [11] provides a structural characterization for a non-upper embeddable graph, i.e., graph G with $\xi(G) \geq 2$, and plays a fundamental role throughout this paper.

Theorem C(Huang [11]) If the graph G is not upper embeddable, i.e., $\xi(G) \geq 2$, then there exists an edge subset A of G satisfying the following properties:

(1) $c(G\backslash A) \geq 2$, and furthermore for any component F of $G\backslash A$, $\beta(F) \equiv 1(mod2)$;

(2) For any component F of $G\backslash A$, F is a vertex-induced subgraph of G;

(3) For any $k(\geq 2)$ different connected components $F_1, F_2, \cdots$, F_k, then $|E_G(F_1, F_2, \cdots, F_k)| \leq 2k - 3$;

(4) $\xi(G) = 2c(G\backslash A) - |A| - 1$.

Supposed A is such a chosen edge subset of G as in theorem C above, the following result can be obtained, as a continuation of Theorem C:

Theorem D(Huang [11]) Under the conditions and the conclusions of Theorem C, then we have

(1)For any connected component F of $G\backslash A$, let G be a graph with k-connectivity($k \geq 1$), then $|E(G,F)| \geq k$;

(2)$|A| = \frac{1}{2}\sum_F |E(G\backslash F, F)|$, where the sum is taken over all connected components F of $G\backslash A$.

4 Upper embeddable bipartite graphs with specific degrees

In this section, we investigate the upper embeddability of bipartite graphs. Firstly, the following lemma can be proved:

Lemma 1 Let G be a bipartite graph, and $d_G(v) = a(mod2a)$ holds for every vertex v in $V(G)$, where $a = 0(mod2), a \geq 4$. If there exist such two edges e_1,$e_2 \in E(G)$ that making $G\backslash\{e_1, e_2\}$ disconnected, then $|V(F)| = 0(mod2)$ and $\beta(F) = 0(mod2)$ hold for any connected components F of $G\backslash\{e_1, e_2\}$.

Proof G has no cut-edge, for it is Eulerian. If $G\backslash\{e_1, e_2\}$ is disconnected, then it has two connected components exactly. Let one be F and the other be H. Then, without loss of generality, let $e_1 = x_1y_1$ and $e_2 = x_2y_2$, where $x_1, x_2 \in V(F)$ and $y_1, y_2 \in V(H)$. Because G is a bipartite graph, F is too. Let $V(F) = V_1 \cup V_2$ be the bipartite partition of $V(F)$.

Claim 1 For the two vertices x_1, x_2, one of them is in V_1 and the other in V_2.

Proof By contradiction, without loss of generality, we assume that $x_1, x_2 \in V_1$. Because F is a bipartite graph, $\sum_{x\in V_1} d_G(x)$ $-2 = \sum_{y\in V_2} d_G(y)$ could be obtained easily. However, $d_G(v) = a(mod2a)$holds for every vertex $v \in V(G)$, so the above equality implies $-2 = 0(moda)$, whilst $a = 0(mod2), a \geq 4$. A contradiction appears.

Claim 2 $|V(F)| = 0(mod2)$

Proof Since F is a bipartite graph and Claim 1 holds, so $|E(F)| = \sum_{x\in V_1} d_G(x) - 1 = \sum_{y\in V_2} d_G(y) - 1$ could be derived, then $\sum_{x\in V_1} d_G(x) = \sum_{y\in V_2} d_G(y)$ holds. As $d_G(v) = a(mod2a)$ holding for every vertex $v \in V(G)$, where $a = 0(mod2), a \geq 4$, it follows that $|V_1| = |V_2|(mod2)$. Thus $|V(F)| = 0(mod2)$ holds.

Claim 3 $\beta(F) = 0(mod2)$

Proof Since the Claim 2 and $|E(F)| = \sum_{x\in V_1} d_G(x) - 1 = 1(mod2)$ hold. So we could obtain $\beta(F) = |E(F)| - |V(F)| + 1 =$

$0(mod2)$.

Since the arbitrariness in the choice of F, the claim is obtained.

Theorem 1 Let G be a bipartite graph. If $d_G(v) = a(mod2a)$ holds for every vertex $v \in V(G)$, then G is upper embeddable, where $a = 0(mod2), a \geq 4$.

Proof Assuming G is not upper embeddable. By Theorem C, there exists $A \subseteq E(G)$ making $G \backslash A$ satisfied all the properties (1)-(4) of Theorem C . Let $F_1, F_2, \cdots, F_l (l \geq 2)$ be all the connected components of $G \backslash A$, where $l = c(G \backslash A) \geq 2$. Because G is Eulerian, then $|E(F_i, G)|$ is even for any F_i. Since G is connected, obviously, $|E(F_i, G)| \neq 0$ for any F_i. And, we could claim that $|E(F_i, G)| \neq 2$ for any F_i. Otherwise, if there exists some $F_i (1 \leq i \leq l)$ makes $|E(F_i, G)| = 2$ hold, then, we have $\beta(F) = 0(mod2)$ by Lemma 1. This contradicts to the property(1)of Theorem C. Hence for any F_i, $|E(F_i, G)| \geq 4$ holds. Because every $F_i (1 \leq i \leq l)$ is an induced subgraph of G, we have $|A| = \frac{1}{2} \sum_{i=1}^{l} |E(F_i, G)| \geq 2l$. Finally we could obtain a contradictory that $\xi(G) = 2l - |A| - 1 \leq -1$ from Theorem C(4). So the theorem 1 is proved.

Remark 1 If $a = 2$ in Theorem 1, then $2a$ should be equal to $4n, n \in \mathbb{N}$, and the condition of "bipartite graph" can be removed, we obtain the same conclusion.

Proof Details of the proof can be seen in Huang [7].

Remark 2 If $a = 0$ in Theorem 1, then $2a$ should be equal to $2n, n \in \mathbb{N}$, and the condition of "bipartite graph" can be removed, but we should strengthen the connected condition of the graph G with 3-connected, we obtain the same conclusion.

Proof Details of the proof can be seen in Huang [12].

Corollary 1(Huang [7]) Let G be a bipartite graph. If $d_G(v) = 4(mod8)$ for every vertex $v \in V(G)$, then G is upper embeddable.

Proof It is a direct result of Theorem 1.

On the upper embeddability of regular bipartite graphs, we obtain the theorem 2 below.

Theorem 2 Let G be a k-regular and 3-connected bipartite graph, then G is upper embeddable, where $k \geq 4$.

Proof We only need to prove that a k-regular 3-connected bipartite graph is 4-edge connected. Assuming $B = \{e_1, e_2, e_3\}$ is a set G, F is one of the connected components of $G \backslash B$. Clearly, F is a bipartite graph and $V(F_i) = V_1 \cup V_2$, where$(V_1 \cup V_2)$ is the bipartition. Supposed that the cut-edges set B has x_i end points in V_i, $i = 1, 2$. As F is a bipartite graph,then

$$\sum_{v \in V_1} d_G(v) - x_1 = \sum_{v \in V_2} d_G(v) - x_2$$

i.e.

$$k|V_1| - x_1 = k|V_2| - x_2$$

i.e.

$$k(|V_1| - |V_2|) = x_1 - x_2$$

But this is impossible, for $k \geq 4$ and $1 \leq |x_1 - x_2| \leq 3$, Hence, G is 4-edge connected and upper embeddable. So Theorem 2 is proved.

Remark 3 If k is odd, then the condition "G is 3-connected "can be weakened. An example of graph G_1 is shown in Fig.1. Obviously, G_1 is a 4-regular and 2-connected bipartite graph. Let T be a spanning tree of G_1, where $V(T) = V(G_1)$ and $E(T)$ is the set of red edges in G_1, then $\xi(G_1, T) = 1$ holds. Therefore, according to the definition of $\xi(G)$, $\xi(G_1) \leq 1$ can be obtained. So the graph G_1 is upper embeddable.

Remark 4 If k is even, then the condition "G is 3-connected "can not be weakened. An example of the graph G_2 is shown in Fig.2. Obviously, G_2 is a 5-regular and 2-connected bipartite graph. Let $A = \{e_1, e_2, e_3\}$, then we could obtain $c(G_2 \backslash A) = 3$, $b(G_2 \backslash A) = 3$ and $|A| = 3$. Thus $c(G_2 \backslash A) + b(G_2 \backslash A) - |A| - 1 = 3 + 3 - 3 - 1 = 2$. Therefore, according to Theorem B, we could obtain $\xi(G_2) \geq 2$, that is to say, the graph G_2 is not upper embeddable.

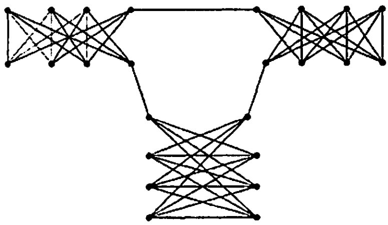

Figure 1: G_1

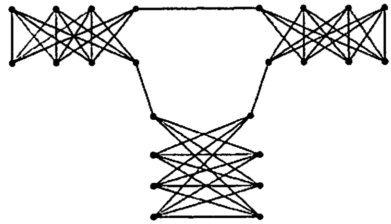

Figure 2: G_2

5 Graphs with specific edges

If the length of a circuit C in G is three (or four), C is called triangle (or quadrangle). Furthermore, if there exists a triangle (or quadrangle) C in G which the edge $e \in E(G)$(not a loop) belongs to, e is called triangular (or quadrangular) edge.

Theorem 3 Let G be a graph(loops and multiple edges are permitted). If G has at most two cut-vertices and every edge e(not a loop)in G is triangular edge, G is upper embeddable.

Proof Assuming G is not upper embeddable. Then, there exists $A \subseteq E(G)$ which makes $G\backslash A$ satisfy all the properties (1)-(4) of Theorem C. Let $F_1, F_2, \cdots, F_l (l \geq 2)$ be all the connected components of $G\backslash A$, where $l = c(G\backslash A) \geq 2$. Since every edge e(not a loop)in G is a triangular edge, G has no cut-edge. Thus, $|E(G, F_i)| \geq 2$ holds for any $F_i (1 \leq i \leq l)$.

The following conclusion would be proved: if $V(F_i)$ has no cut-vertex of G, then $|E(G, F_i)| \geq 4$ holds for any F_i $(1 \leq i \leq l)$. Otherwise, let us assume that $|E(G, F_i)| = 2$ or 3, the following cases are dealt with:

Case 1 $|E(G, F_i)| = 2$. Let $E(G, F_i) = \{e_1, e_2\}$ and $e_1 = x_1y_1$,$e_2 = x_2y_2$. Without loss of generality, let $x_1, x_2 \in V(F_i)$,y_1, y_2 not in $V(F_i)$. According to the property (4) of Theorem C, there exist two distinct connected components, donated them by F_j and F_k of $G\backslash A$, such that $y_1 \in V(F_j)$ and $y_2 \in V(F_k)(1 \leq i, j, k \leq l; i \neq j \neq k)$. On the condition of $|E(G, F_i)| = 2$, if $x_1 \neq x_2$, the edge $e_1 = x_1y_1$ is not a triangular edge of G. This is a contradiction to the given condition; however if $x_1 = x_2$, the vertex x_1 is a cut-vertex. It is a contradiction to the assume of $V(F_i)$ having no cut-vertex of G.

Case 2 $|E(G, F_i)| = 3$. Similar to the analysis of Case 1, the same contradictions will be obtained: either existing an edge $e \in E(G, F_i)$ is not a triangular edge of G; or $V(F_i)$ having cut-vertices of G.

Consequently, according to the discussions above, we obtain $|E(G, F_i)| \geq 4$ for any F_i without cut-vertices of G. Since the graph G has at most two cut-vertices, there are at most some two F_m and F_n satisfied the equality $|E(G, F_i)| = 2$ or 3. Therefore, we obtain $|A| = \frac{1}{2}\sum_i^l |E(G, F_i)| \geq 2l - 2$ via Theorem D(2). Finally, we obtain $\xi(G) = 2l - |A| - 1 \leq 1$ via Theorem C(4). A contradiction as well. So the graph G is upper embeddable.

Moreover, we further discuss about the general relationship between the Betti deficiency $\xi(G)$ and the number of cut-vertices of a graph. Before obtaining the main result, some related lemmas are proved first.

Lemma 2 Let G be a graph, the edge $e \in E(G)$(not a loop), then $\xi(G \cdot e) \leq \xi(G)$ holds, where $G \cdot e$ is a graph obtained by shrinking the edge e in $E(G)$.

Details of the proof can be shown in Liu [9].

Lemma 3 Let G be a graph and the edge e be a cut-edge of G, then $\xi(G) = \xi(G_1) + \xi(G_2)$ holds, where G_1 and G_2 are the two connected components of $G\backslash e$.

Details of the proof can be seen in Liu [9] also.

Lemma 4 Let G_1 and G_2 be two graph, then $\xi(G_1\{v_1, v_2\}G_2) \leq \xi(G_1)+\xi(G_2)$ holds, where $G_1\{v_1, v_2\}G_2$ denotes a graph formed through conglutinating the two vertices v_1 and v_2 together to one vertex, where $v_i \in V(G_i)(i = 1, 2)$.

Proof Let G' be a graph obtained by adding such an edge e' that one endpoint is v_1 and the other is v_2, where $v_i \in V(G_i)(i = 1, 2)$. Obviously, the edge e' is a cut-edge of G', meanwhile, G_1 and G_2 are the two connected components of $G'\backslash e'$. By lemma 3, we obtain $\xi(G') = \xi(G_1) + \xi(G_2)$. In addition, the graph $G_1\{v_1, v_2\}G_2$ can be obtained by shrinking the edge e' of G', then $\xi(G_1\{v_1, v_2\}G_2) \leq \xi(G') = \xi(G_1) + \xi(G_2)$ holds via lemma

2. So the lemma 4 is proved.

Under the conditions of the theorem 3, if the number of cut-vertices in graph G greater than or equal to 2, the following result can be obtained:

Theorem 4 Let G be a graph(loops and multiple edges are permitted) and every edge e(not a loop)in G is a triangular edge. If $k(G) \geq 2$, then $\xi(G) \leq k(G) - 1$ holds, where $k(G)$ denotes the number of cut-vertex in graph G.

Proof If $k(G) \leq 2$, it is just the result of Theorem 3. Here, we assume that $k(G) \geq 3$ and take a cut-vertex v from G randomly. Let G' and G'' be two subgraphs of G as follows: G' is a union composed of such all blocks having the vertex v, while G'' is a union composed of such all blocks without the vertex v. Obviously, the two graphs G' and G'' have the following characters: G' has one cut-vertex at most, whilst the number of cut-vertices in G'' is no more than $k(G) - 1$. Let $V = \{x|x \in V(G') \cap V(G''), x \in V(G)\}$ and $|V| = m(m \in \mathbb{N}, m \geq 1)$, then V can be expressed by set: $V = \{x_1, x_2, \cdots, x_m\}$,and G'' is a union composed of these blocks $G''_1, G''_2, \cdots, G''_m$ including the vertices $x_1, x_2, \cdots, x_m$ respectively. Evidently, the two graphs G' and G'' satisfy the condition that every edge of G' and G'' is a triangular edge. Because G' has one cut-vertex v at most, $\xi(G') \leq 1$ holds via Theorem 3. Meanwhile, we obtain $\xi(G'') \leq k(G'') - 1$ from the inductive assumption of $k(G'') \leq k(G) - 1 < k(G)$. The graph G can be seen as a graph formed by conglutinating the vertices $x_1, x_2, \cdots, x_m$ in $V(G')$ and the vertices $x_1, x_2, \cdots, x_m$ in $V(G'')$ correspondingly. The detailed process construction of graph G as follows: by conglutinating vertex x_1 in V, G' and G''_1 can be formed a new graph G^*_1 first, then conglutinating vertex x_2 in V, G^*_1 and G''_2 can be formed the graph G^*_2. Repeating the similar process until the graph G^*_m is formed, so the the graph G can be expressed as $G'\{x_1, x_2, \cdots, x_m\}G''$. Via Lemma 4, we obtain the following relations:

$\xi(G^*_2) \leq \xi(G^*_1) + \xi(G''_2) \leq k(G''_1) + k(G''_2) - 1;$

$\xi(G^*_3) \leq \xi(G^*_2) + \xi(G''_3) \leq k(G''_1) + k(G''_2) + k(G''_3) - 1;$

$\cdots\cdots\cdots$

$\xi(G) = \xi(G^*_m) \leq \xi(G^*_{m-1}) + \xi(G''_m) \leq k(G''_1) + k(G''_2) + \cdots + k(G''_m) - (m-1)$;

then

$\xi(G) \leq k(G) - 1 - (m-1) \leq k(G) - 1$

Therefore the theorem is proved.

References

[1] E.A. Nordhause, B.M. Stewart, A.T. White, On the maximum genus of a graph, *J. Combin. Theory* 1971,11:285-267

[2] L. Nebesky, Every connected locally connected graph is upper embeddable, *J. Graph Theory*, 1981,5:205-207

[3] R. Ringeisen, The maximum genus of a graph, Ph.D. Thesis, Department of Mathematics, Michigan State University, 1970

[4] M. Skoviera, The maximum genus of graphs of diameter two, *Discrete Mathematics*, 1991,87:175-180

[5] S. Kundu, Bounds on number of disjoint spanning trees, *J. Combinatiorial Theory B*, 1974,17;199-203

[6] M. Jungerment, A characterization of upper embeddable graphs, *Trans. Amer. Math. Soc.*, 1978,241:401-406

[7] Y. Huang, Y. Liu, Some classes of upper embeddable graphs, *Acta Mathematica Scientia*, 1997,17(Supp.):154-161

[8] A.T. White, Graphs, Groups, and Surface, North-Holland, Amsterdram, 1973

[9] Y. Liu, Embeddability in Graph, Kluwer & Science, 1995.

[10] L. Nebesky, A new characterization of the maximum of a graph. *J. Czechoslovak Math*, 1981,31(106):604- 613

[11] Y. Huang, Y. Liu, An improvement of theorem on the maximum genus for graphs. *Mathematica Applicata*, 1998,11(2):109-112

[12] Y. Huang, Y. Liu, The classes of upper embeddable graphs with the same value of degree of vertex under modulo 4, *Acta Mathematica Scientia*, 2000,20(2):251-255