Enumeration of Highly Irregular Trees by Automorphism Group

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Abstract

A connected graph is *highly irregular* if the neighbors of each vertex have distinct degrees. We will show that every highly irregular tree has at most one nontrivial automorphism. The question that motivated this work concerns the proportion of highly irregular trees that are asymmetric, i.e., have no nontrivial automorphisms. A *d-tree* is a tree in which every vertex has degree at most d. A technique for enumerating unlabeled highly irregular d-trees by automorphism group will be described for $d \geq 4$ and results will be given for d = 4. It will be shown that, for fixed d, $d \geq 4$, almost all highly irregular d-trees are asymmetric.

1. Introduction

Beginning with Cayley's work in 1857, enumeration problems have been solved for trees in general and for trees of many different types. In fact, the counting technique used by Pólya and Otter was generalized as a twenty step algorithm for counting various types of trees by Harary, Robinson and Schwenk [6]. In 1987, Alavi, Chartrand, Chung, Erdös, Graham and Oellermann [2] introduced a new class of graphs, highly irregular graphs. A connected graph is defined to be *highly irregular* if for each vertex v, the neighbors of v all have distinct degrees. This paper addresses the problem of enumerating unlabeled highly irregular trees by automorphism group. This work was motivated by the conjecture by P. R. Christopher [4] that almost all highly irregular trees have no nontrivial automorphisms. Alavi and Ruiz [3] showed that for any finite group Γ and positive integer n, there exists a highly irregular graph of order n with automorphism group Γ. However, the type of symmetries that can occur in a tree together with the highly irregular restriction result in the fact that all highly irregular trees have at most one nontrivial automorphism. We will prove this result and illustrate a method for enumerating highly irregular trees by automorphism group. A *d-tree* is a tree in which every vertex has degree at most d. We will show that, unlike ordinary trees, almost all of which have nontrivial automorphisms [5], almost all highly irregular d-trees have no nontrivial automorphisms.

2. Preliminaries

Let $\Gamma(T)$ denote the automorphism group of the tree T. If $|T| = 1$, then we say T is asymmetric; otherwise, T is symmetric.

Theorem 1 *The order of the automorphism group of a highly irregular tree is at most 2.*

PROOF. First we show that a rooted tree with a nontrivial automorphism has a vertex with at least two neighbors of the same degree. Let T be a rooted tree with root v and suppose φ is a nontrivial automorphism of T. Since the root must be a fixed point of any permutation, there must be a vertex that is fixed by φ and that has at least two neighbors, say a and b that are permuted by φ. Then these two vertices must have the same degree.

Now let T be a highly irregular tree. The center of any tree consists of either 1 or 2 vertices. Suppose the center of T consists of one vertex, v. Since v must be a fixed point of any permutation of the vertices of T, we can view T as a rooted tree with root v. Thus, by the above argument, if T is highly irregular and has one central vertex, then T has no nontrivial automorphisms.

Suppose the center of T consists of two vertices, u and v. Then T is formed by adding an edge between the roots of two rooted trees, T_1 and T_2, rooted at u and v, respectively. If T_1 had a nontrivial automorphism, then, as a rooted tree, some vertex of T_1 would have two neighbors, neither of which is the root of T_1, that have the same degree. This contradicts the highly irregular property of T. Similarly, T_2 cannot have any nontrivial automorphisms. Therefore, T has a nontrivial automorphism if and only if T_1 and T_2 are isomorphic. Then the edge uv is a symmetry edge and the only nontrivial automorphism of T permutes T_1 and T_2. □

Now we will enumerate highly irregular d-trees by automorphism group order. The highly irregular 3-trees must be dealt with as a special case. The only highly irregular trees with maximum degree at most 2 are K_2 and P_4, both of which have a nontrivial automorphism. Therefore, every highly irregular 3-tree of order at least 4 has maximum degree equal to 3. In [1] Alavi, Buckley, Shamula and Ruiz showed that there is exactly one highly irregular tree of maximum degree 3 and order n if and only if $n \geq 8$ and n is congruent to 2, 3, or 4 modulo 6 and that there are none if n is congruent to 0, 1, or 5 modulo 6. They also described a process for obtaining all highly irregular trees of maximum degree 3. The following theorem follows from the process for obtaining all highly irregular trees of maximum degree 3.

Theorem 2 *The highly irregular trees of maximum degree 3 and order congruent to 2 or 4 modulo 6 are symmetric. The highly irregular trees of maximum degree 3 and order congruent to 3 modulo 6 are asymmetric.*

PROOF. If n is congruent to 3 modulo 6, then n is odd. Therefore, by Theorem 1, a highly irrregular 3-tree whose order is congruent to 3 modulo 6 is asymmetric.

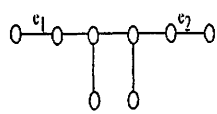

Figure 1: T_3, the unique highly irregular 3-tree of minimum order.

T_3, the tree in Figure 1, is the highly irregular tree with maximum degree 3 and minimum order 8 and is symmetric. Alavi, Buckley, Shamula and Ruiz [1] showed that each highly irregular tree T of maximum degree 3 and order congruent to 2 modulo 6 can be formed by linking $k \geq 1$ copies of T_3. It can easily be checked that the symmetry of T follows from the construction of T together with the symmetry of T_3.

Now let T be a highly irregular tree of maximum degree 3 and order n congruent to 4 modulo 6. Alavi, Buckley, Shamula and Ruiz [1] showed that T can be obtained from the highly irregular maximum degree 3 tree of order $n-2$, T'. The symmetry of T follows directly from the symmetry of T' and the construction of T. □

In the following section, we will describe a general technique for enumerating highly irregular d-trees with $d \geq 4$.

3. Enumeration Technique

The method for counting highly irregular d-trees will be described for $d = 4$ and results will be given for $d = 4$. Note that since 2^d is the minimum order of a highly irregular d-tree that actually has a vertex of degree d [1], applying this technique to count highly irregular d-trees of order at most $2^{d+1} - 1$ actually counts all highly irregular trees of those orders.

The standard enumeration technique for trees [6] relies on the fact that all planted, rooted and edge-rooted trees of many types may be formed using planted trees of the specified type as building blocks. For example, taking two planted 4-trees and identifying their roots results in a rooted 4-tree while then joining a new vertex adjacent to the root of this rooted tree results in a planted 4-tree. Because of the highly irregular degree restriction, two difficulties occur when this approach is attempted with highly irregular trees.

In the case of forming rooted highly irregular trees, only rooted highly irregular trees in which the vertices adjacent to the root have no neighbors of degree 1 can be built from planted highly irregular trees. Suppose T_1 and T_2 are two planted highly irregular trees with u adjacent to the root of T_1 and v adjacent to the root of T_2. Let T be the rooted tree resulting from identifying the roots of T_1 and T_2. Since both u and v each have a neighbor of degree 1 in T_1 and T_2 respectively, neither will have a neighbor of degree 1 in T. The rooted tree in Figure 2 is an example of a rooted highly irregular tree that cannot be built from planted highly irregular trees. To deal with this difficulty, we define *planted almost highly irregular trees* which will serve as the building blocks. In a *planted almost highly irregular tree*, every vertex except the vertex adjacent to the root must satisfy the highly irregular condition. The vertex adjacent to the root may have two neighbors of degree 1, the root and one other vertex; all its other neighbors must have distinct degrees.

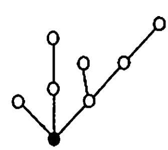

Figure 2: A rooted highly irregular tree.

However, not all planted almost highly irregular trees can be combined to form a tree in which the highly irregular property is maintained. If, in the above example, $\deg(u) = \deg(v)$, or either u or v has a neighbor of degree 2 in T_1 or T_2 respectively, then T is not highly irregular. To deal with this problem, we must keep track of the degree of the vertex adjacent to the root and the degrees of its neighbors other than the root. To each planted almost highly irregular d-tree, we assign a vector $\mathbf{X}$, a d-tuple in which the ith component X_i is 1 if the vertex adjacent to the root has a neighbor (other than the root) of degree i and is 0 otherwise (see Figure 3).

Then the degree of the vertex adjacent to the root is $1 + \sum_{i=1}^{d} \mathbf{X}_i$ and since each vector has at least one zero, there are $2^d - 1$ vectors which correspond to the planted almost highly irregular d-trees.

Observe that planted almost highly irregular d-trees and rooted highly irregular d-trees can be formed from planted almost highly irregular d-trees. A rooted tree in which the root has degree m is formed by taking an appropriate collection of m planted almost highly irregular d-trees and identifying their roots to form the root of the new tree. Adding a new vertex adjacent to the root of this rooted tree results in a planted almost highly irregular d-tree in which the degree of the vertex adjacent to the root is $m + 1$. To illustrate the role the vectors play in determining when a collection of planted almost highly irregular d-trees can be combined to form planted almost highly irregular or rooted highly irregular d-trees, consider the three 4-trees in Figure 3. If we identify their roots to form the rooted tree T_1 in Figure 4, the root u has degree 3. However, since $\mathbf{X}_3 = 1$, one of the vertices adjacent to u already has a neighbor of degree 3. Consequently, the rooted tree in Figure 4 is not highly irregular. If we then add a new vertex adjacent to u to form the planted tree T_2 in Figure 4, the result is highly irregular. This can be confirmed with the vectors **X**, **Y** and **Z**. Since $1 + \sum_{i=1}^{d} \mathbf{X}_i$, $1 + \sum_{i=1}^{d} \mathbf{Y}_i$, and $1 + \sum_{i=1}^{d} \mathbf{Z}_i$ are all distinct, the degrees of the neighbors of the vertex adjacent to the vertex u are all distinct. $\mathbf{X}_4 = \mathbf{Y}_4 = \mathbf{Z}_4 = 0$ indicates that none of the vertices adjacent to u have another neighbor of degree 4.

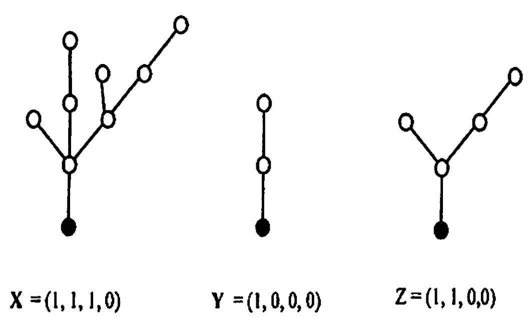

Figure 3: Planted almost highly irregular trees and their vectors.

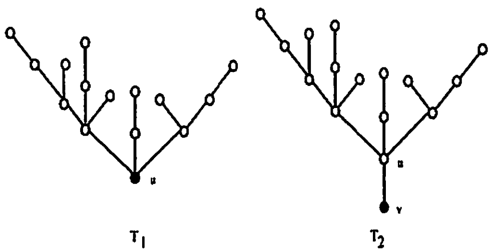

Figure 4: Trees formed from planted almost highly irregular trees.

Using this method, recurrence relations for the number of planted almost highly irregular d-trees with a given vector and rooted highly irregular d-trees are now derived for $d = 4$.

Let $P_{\mathbf{V}}(n)$ be the number of planted almost highly irregular d-trees with vector $\mathbf{V}$ and order n.

Theorem 3 *For d=4,*

(i) if $n = 2$*, then* $P_{\mathbf{V}}(n) = 1$*,*

(ii) if $n \geq 3$ *and* $\mathbf{V}$ *has exactly one entry equal to 1, then* $P_{\mathbf{V}}(n) = \sum_{\mathbf{X}} P_{\mathbf{X}}(n-1)$*, where* $\mathbf{X}$ *ranges over all vectors such that* $\mathbf{X}_2 = 0$ *and the position of the 1 in* $\mathbf{V}$ *is* $d_1 = 1 + \sum_{i=1}^{4} \mathbf{X}_i$

(iii) if $n \geq 5$ *and* $\mathbf{V}$ *has exactly two entries equal to 1, then* $P_{\mathbf{V}}(n) = \sum_{\mathbf{X},\mathbf{Y}} \sum_{r=2}^{n-2} P_{\mathbf{X}}(r) P_{\mathbf{Y}}(n-r)$ *, where* $\mathbf{X}$ *and* $\mathbf{Y}$ *range over all sets of two vectors such that* $\mathbf{X}_3 = \mathbf{Y}_3 = 0$ *and the positions of the 1's in* $\mathbf{V}$ *are* $d_1 = 1 + \sum_{i=1}^{4} \mathbf{X}_i$ *and* $d_2 = 1 + \sum_{i=1}^{4} \mathbf{Y}_i$

(iv) if $n \geq 9$ *and* $\mathbf{V}$ *has exactly three entries equal to 1, then* $P_{\mathbf{V}}(n) = \sum_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \sum_{r=2}^{n-3} \sum_{s=2}^{n-1-r} P_{\mathbf{X}}(r) P_{\mathbf{Y}}(s) P_{\mathbf{Z}}(n+1-r-s)$*, where* $\mathbf{X}$*,*$\mathbf{Y}$ *and* $\mathbf{Z}$ *range over all sets of three vectors such that*

$X_4 = Y_4 = Z_4 = 0$ *and the positions of the 1's in* $\mathbf{V}$ *are*
$d_1 = 1 + \sum_{i=1}^{4} \mathbf{X}_i$, $d_2 = 1 + \sum_{i=1}^{4} \mathbf{Y}_i$ *and* $d_3 = 1 + \sum_{i=1}^{4} \mathbf{Z}_i$.

PROOF. Let T be a planted almost highly irregular 4-tree of order n, vector $\mathbf{V}$, and root w. The four formulas given above correspond to the cases that u, the vertex adjacent to the root of T has degree 1, 2, 3 or 4. The case of degree 1 is simply a planted K_2. If the degree of u in T is $m =$ 2, 3 or 4, then T is formed from a set of m- 1 planted almost highly irregular 4-trees, T_1, T_2, ..., T_{m-1} whose orders sum to $n - 1$. The restrictions on the T_i's are described by the restrictions on their vectors in the formulas given above. For each i = 1, 2, ..., m - 1, let v_i be the vertex adjacent to the root of T_i. In T, u is a neighbor of each v_i and has degree m. So for each i = 1, 2, ..., m, v_i cannot have a neighbor of degree m in T_i. Hence, for each T_i, the m^{th} coordinate of its vector must be zero.

Also the neighbors of u, excluding w, must have distinct degrees. These degrees are determined by the coordinates of $\mathbf{V}$ that are nonzero. If the vector of T_i is $\mathbf{X}$, then the degree of v_i (in T_i and in T) is $d_i = 1 + \sum_{j=1}^{d} \mathbf{X}_j$. Thus, $\{d_1, d_2, ..., d_{m-1}\}$ must be distinct and form the set of the positions of $\mathbf{V}$ that are nonzero.

□

Let $P(n)$ be the total number of planted almost highly irregular d-trees of order n. Clearly, $P(n) = \sum_{\mathbf{V}} P_{\mathbf{V}}(n)$, where $\mathbf{V}$ ranges over all $2^d - 1$ vectors which correspond to the planted almost highly irregular d-trees.

Relations expressing *Root*(n), the number of rooted highly irregular d-trees of order n, in terms of the $P_{\mathbf{V}}(i)$'s are derived in a similar manner.

Theorem 4 *For d = 4, Root(n)=*

$$\sum_{\mathbf{X}} P_{\mathbf{X}}(n) + \sum_{\mathbf{X},\mathbf{Y}} \sum_{r=2}^{n-1} P_{\mathbf{X}}(r) P_{\mathbf{Y}}(n+1-r)$$

$$+ \sum_{\mathbf{X},\mathbf{Y},\mathbf{Z}} \sum_{r=2}^{n-2} \sum_{s=2}^{n-r} P_{\mathbf{X}}(r) P_{\mathbf{Y}}(s) P_{\mathbf{Z}}(n+2-r-s)$$

$$+ \sum_{\mathbf{U},\mathbf{V},\mathbf{X},\mathbf{Y}} \sum_{r=2}^{n-3} \sum_{s=2}^{n-1-r} \sum_{t=2}^{n+1-r-s} P_{\mathbf{U}}(r) P_{\mathbf{V}}(s) P_{\mathbf{X}}(t) P_{\mathbf{Y}}(n+3-r-s-t)$$

Again, the four terms correspond to the four cases that the root has degree 1, 2, 3 or 4. The first sum in each term is taken over all appropriate sets of vectors that will result in a tree that is highly irregular. For example, in the third term, the three vectors $\mathbf{X}, \mathbf{Y}$, $\mathbf{Z}$ must satisfy $\mathbf{X}_3 = \mathbf{Y}_3 = \mathbf{Z}_3 = 0$, and $1 + \sum_{i=1}^{4} \mathbf{X}_i$, $1 + \sum_{i=1}^{4} \mathbf{Y}_i$, and $1 + \sum_{i=1}^{4} \mathbf{Z}_i$ are all distinct. In general, if the root has degree m, then the sum is taken over all sets of m vectors in which the m^{th} coordinate of each vector is zero and each of the m vectors has a different number of 1's.

Observe that these planted almost highly irregular d-trees can also be used to form unrooted or free highly irregular d-trees that have a nontrivial automorphism. Recall that these symmetric trees consist of two isomorphic subtrees that are joined by a symmetry edge. Thus, a symmetric highly irregular d-tree can be formed by taking two copies of a planted almost highly irregular d-tree and identifying the edges incident to the roots of the two planted trees. However, this identification does not always result in a highly irregular tree. This is illustrated in Figure 5. The tree T formed from two copies of T_1 is not highly irregular while the tree T' formed from two copies of T_2 is highly irregular.

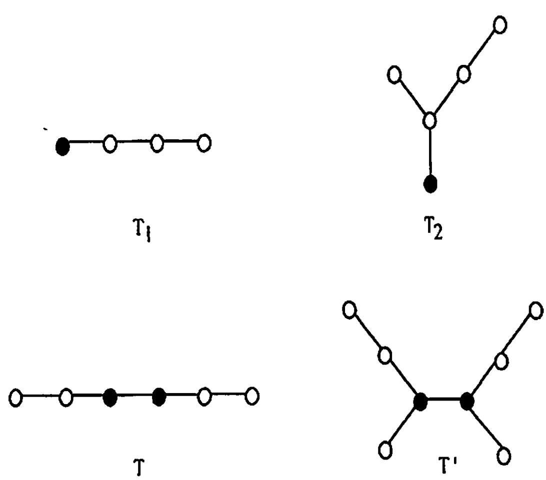

Figure 5: Symmetric trees formed from planted trees.

Let T be a planted almost highly irregular d-tree with vector $\mathbf{X}$. Let T' be the tree formed by identifying the edges incident to the roots of two copies of T. Then if u is a central vertex of T', the degree of u in T' is $1 + \sum_{i=1}^{d} \mathbf{X_i}$. Hence, T' is highly irregularly if and only if u has no neighbor of degree $1 + \sum_{i=1}^{d} \mathbf{X_i}$ in T. Let $j = 1 + \sum_{i=1}^{d} \mathbf{X_i}$. T' is highly irregular if and only if $\mathbf{X_j} = 0$.

This gives the following relation that expresses $Sym(n)$, the number of symmetric highly irregular d-trees in terms of the $P_{\mathbf{V}}(\mathrm{i})$'s. Note that n must be even.

Theorem 5 *For n even, $Sym(n)$, the number of highly irregular d-trees with a nontrivial automorphism and order n satisfies*

$$Sym(n) = \sum_{\mathbf{X}} P_{\mathbf{X}}(\frac{n}{2} + 1),$$

where X ranges over all vectors with 0 in position $j = 1 + \sum_{i=1}^{d} \mathbf{X_i}$.

Let $Asym(n)$ be the number of asymmetric highly irregular d-trees of order n. Since the number of different ways to root a graph at a vertex is the number of orbits of the vertices under the automorphism group of the graph, we have the relation

$$Root(n) = nAsym(n) + \frac{n}{2}Sym(n)$$

which can be solved to express $Asym(n)$ in terms of $Root(n)$ and $Sym(n)$. Then, clearly, $Free(n)$, the number of (unrooted) highly irregular d-trees of order n is just the sum of $Asym(n)$ and $Sym(n)$.

Theorem 6 *(i) $Asym(n)$, the number of highly irregular d-trees with no nontrivial automorphisms and order n, satisfies*

$$Asym(n) = \frac{Root(n)}{n} - \frac{1}{2}Sym(n)$$

(ii) $Free(n)$, the number of highly irregular d-trees of order n, satisfies

$$Free(n) = \frac{Root(n)}{n} + \frac{1}{2}Sym(n).$$

Table 1 contains the values for $P(n)$, $Root(n)$ and $Free(n)$ for 4-trees. Note that 16 is the smallest order for which there is a highly irregular 4-tree with a vertex of degree 4; however both such trees are symmetric. In Table 2, the numbers of (free) highly irregular 4-trees are broken down according to the automorphism group. Since $Sym(n) = 0$ if n is odd, the values are given only for even orders.

Vertices	Planted	Rooted	Free
16	24	16	2
17	41	34	2
18	73	72	5
19	124	133	7
20	196	250	14
21	286	399	19
22	414	572	28
23	607	828	36
24	878	1176	51
25	1270	1700	68
26	1824	2509	99
27	2553	3456	128
28	3553	4494	164
29	5008	5887	203
30	7213	7980	272

Table 1: Numbers of planted almost highly irregular, rooted highly irregular and free highly irregular 4-trees.

Vertices	Asymmetric	Symmetric	Per Cent Asymmetric
18	3	2	60.00
20	11	3	78.57
22	24	4	85.71
24	47	4	92.16
26	94	5	94.95
28	157	7	95.73
30	260	12	95.59
32	553	22	96.17
34	1339	35	97.45
36	3393	54	98.43
38	8427	79	99.07
40	19833	109	99.45
42	43573	162	99.63
44	91372	251	99.73
46	189584	375	99.80
48	400687	554	99.86
50	888756	802	99.90

Table 2: Number of (free) highly irregular 4-trees by automorphism group order.

4. Conclusion

As is illustrated for $d = 4$ in Table 2, the proportion of highly irregular d-trees that are asymmetric is approaching 1 as the number of vertices increases. The enumeration technique provides the tools for verifying this result.

Theorem 7 *For fixed $d \geq 4$, almost all highly irregular d-trees have no nontrivial automorphisms.*

PROOF. We show that the proportion of all highly irregular d-trees of order n that are symmetric approaches 0 as n increases. Let H_k be the set of all highly irregular d-trees of order $2k$. Let S_k be the set of symmetric highly irregular d-trees of order $2k$ and, for each appropriate vector $\mathbf{V}$ with a 0 in position $j = 1 + \sum_{i=1}^{d} \mathbf{V}_i$, let $S_{k,\mathbf{V}}$ be the subset of trees in S_k that are formed by combining two copies of a planted almost highly irregular d-tree of order $k+1$ and vector $\mathbf{V}$. Then $S_k = \bigcup S_{k,\mathbf{V}}$ where the union is taken over all appropriate vectors $\mathbf{V}$ and the proportion of all highly irregular d-trees of order $2k$ that are symmetric is

$$\frac{|S_k|}{|H_k|} = \sum_{\mathbf{V}} \frac{|S_{k,\mathbf{V}}|}{|H_k|}. \tag{1}$$

Now let $A_{k,\mathbf{V}}$ be the set of asymmetric highly irregular d-trees of order $2k$ that are formed by identifying the edges incident to the roots of two non-isomorphic planted almost highly irregular d-trees with vector $\mathbf{V}$ and order $k+1$. For each vector $\mathbf{V}$ with $|S_{k,\mathbf{V}}| \neq 0$,

$$\frac{|S_k|}{|H_k|} < \frac{|S_{k,\mathbf{V}}|}{|A_{k,\mathbf{V}}|} = \frac{P_{\mathbf{V}}(k+1)}{\frac{1}{2}P_{\mathbf{V}}(k+1)[P_{\mathbf{V}}(k+1)-1]} = \frac{2}{[P_{\mathbf{V}}(k+1)-1]}$$

which approaches 0 as k goes to infinity since $P_{\mathbf{V}}(k+1)$ increases with k.

Thus, $\frac{|S_k|}{|H_k|}$ approaches 0 as k goes to infinity and almost all highly irregular d-trees are asymmetric. □

For any positive integer N, the enumeration technique presented in this paper can be applied to count all highly irregular trees of order at most N since such trees are highly irregular d-trees with $d = \lfloor \log_2 N \rfloor$. However, the proof of Theorem 7 does not hold if the degree restriction is removed. In order to show that almost all highly irregular trees are asymmetric, we must let d, the bound on the maximum degree, increase along with n, the number of vertices. Then the number of vectors increases also and the sum (1) in the proof of Theorem 7 becomes an infinite sum. Consequently, information about how each term of the sum is approaching 0 is required before we may conclude that the infinite sum is equal to 0.

There is a jump in the number of highly irregular trees when the order is a power of 2 and the maximum possible degree goes up by 1, i.e., at 32 vertices trees with maximum degree 5 are introduced, at 64 vertices trees with maximum degree 6 are introduced, etc. It appears that the behavior of the number of asymmetric trees relative to the total number of trees of these and slightly larger orders must be studied further to determine if the stronger conjecture that almost all highly irregular trees are asymmetric is true. Numerical data that has been produced for highly irregular trees on up to 54 vertices does support this stronger conjecture.

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