The sum numbers and the integral sum numbers of $\overline{P_n}$ and $\overline{F_n}$

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Abstract Let $G = (V, E)$ be a simple graph with the vertex set V and the edge set E. G is a *sum graph* if there exists a labelling f of the vertices of G into distinct positive integers such that $uv \in E$ if and only if $f(w) = f(u) + f(v)$ for some vertex $w \in V$. Such a labelling f is called a sum labelling of G. The *sum number* $\sigma(G)$ of G is the smallest number of isolated vertices which result in a sum graph when added to G. Similarly, the *integral sum graph* and the *integral sum number* $\zeta(G)$ are also defined. The difference is that the labels may be any distinct integers. In this paper, we will determine that

$$\begin{cases} 0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1; \\ 1 = \zeta(\overline{P_5}) < \sigma(\overline{P_5}) = 2; \\ 3 = \zeta(\overline{P_6}) < \sigma(\overline{P_6}) = 4; \\ \zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 0, \quad n = 1, 2, 3; \\ \zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 2n - 7, \quad n \geq 7. \end{cases}$$

and

$$\begin{cases} 0 = \zeta(\overline{F_5}) < \sigma(\overline{F_5}) = 1; \\ 2 = \zeta(\overline{F_6}) < \sigma(\overline{F_6}) = 3; \\ \zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 0, \ n = 3, 4; \\ \zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 2n - 8, n \geq 7. \end{cases}$$

Keywords The sum graph; The integral sum graph; The sum number; The integral sum number; Path; Fan.

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1. Introduction

Let $G = (V, E)$ be a simple graph with the vertex set V and the edge set E. The complement $\overline{G}$ of G with order n is the graph with the vertex set V and the edge set $E(K_n) - E$. A path P_n is a graph with the vertex set $\{a_1, a_2, ..., a_n\}$ and the edge set $\{a_1a_2, a_2a_3, ..., a_{n-1}\ a_n\}$, and a_1 and a_n are called the end vertices of P_n. A fan F_n is a graph with the vertex set $\{c, a_1, a_2, ..., a_n\}$ and the edge set $\{ca_1, ca_2, ..., ca_n\} \cup \{a_1a_2,\ a_2a_3, ..., a_{n-1}a_n\}$. It is obvious that $\overline{F_n} = \overline{P_n} \cup K_1$.

A *sum graph* and an *integral sum graph* were introduced by Frank Harary in [2] and [3]. G is a *sum graph* if there exists a labelling f of the vertices of G into distinct positive integers such that $uv \in E$ if and only if $f(w) = f(u) + f(v)$ for some vertex $w \in V$. Such a labelling f is called a sum labelling of G. A sum graph cannot be connected. There must always be at least one isolated vertex. The *sum number* $\sigma(G)$ of G is the smallest number of isolated vertices which result in a sum graph when added to G. Similarly, an *integral sum graph* and an *integral sum number* $\zeta(G)$ are also defined. The difference is that the labels may be any distinct integers. Obviously $\zeta(G) \leq \sigma(G)$.

A vertex w of G is *working* if its label corresponds to an edge uv of G. G is *exclusive* if none of the vertices in V is working. For example, K_n and W_{2n-1} are exclusive in [9].

To simplify the notations, we may assume that the vertices of G are identified with their labels throughout this paper. And let $\overline{V_i}$ and E_i denote the set of the vertices independent of a_i and the set of the edges adjacent to a_i in $\overline{P_n}$ respectively. Besides, some results have been obtained as follow.

Lemma 1([2]) $\sigma(P_n) = 1$ and $\zeta(P_n) = 0$ for $n \geq 2$.
Lemma 2([1][8]) $\zeta(K_n) = \sigma(K_n) = 2n - 3$ for $n \geq 4$.
Lemma 3([8]) $\zeta(C_n) = \zeta(W_n) = 0$ for $n \neq 5$.
Lemma 4([2]) For $n \geq 3$, $\quad \sigma(C_n) = \begin{cases} 2, & n \neq 4, \\ 3, & n = 4. \end{cases}$
Lemma 5([10][7]) For $n \geq 3$, $\sigma(W_n) = \begin{cases} \frac{n}{2} + 2, & n \text{ even}, \\ n, & n \text{ odd}. \end{cases}$

In this paper, we will determine the sum numbers and the integral sum numbers of $\overline{P_n}$ and $\overline{F_n}$ for $n \geq 1$.

2. Main results

Let $\overline{P_n} = (V, E)$ and $S = V \cup C$, where $V = \{a_1, a_2, ..., a_n\}$ and C is the isolated vertex set. It is clear that $\overline{P_2} = 2K_1$ and $\overline{P_3} = P_2 \cup K_1$ and $\overline{P_4} = P_4$. By Lemma 1 and Lemma 2, $\zeta(\overline{P_i}) = \sigma(\overline{P_i}) = 0$ for $i = 1, 2, 3$ and $0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1$. In this section, we only consider $n \geq 5$.

Lemma 2.1 $\overline{P_n}$ is not an integral sum graph for $n \geq 5$.

Proof: Let $|a_x| = \max\{|a| : a \in V\}$ and $a_x \in V$. Assume that $a_x > 0$ (A similar argument work for $a_x < 0$). By contradiction. If $\overline{P_n}$ is an integral sum graph for $n \geq 5$ then $0 \notin V$ and $a_x + a_i \in V$ for all $a_x a_i \in E$. Then $a_x + a_i > 0$ and $a_i < 0$ according to the choice of a_x. So we get at least $n-3$ distinct positive vertices $a_x + a_i$ in V. Meanwhile, we also get at least $n-3$ distinct negative vertices a_i. So $2(n-3)+1 \leq n$, that is, $n \leq 5$. Since $n \geq 5$, only $n = 5$.

Assume that $V(\overline{P_5}) = \{a_x, a_1, a_2, a_x + a_1, a_x + a_2\}$ with $a_i < 0$ $(i = 1, 2)$. By the choice of a_x, $(a_x + a_i)a_x \notin E$ with $i = 1, 2$ and $\overline{V_x} = \{a_x + a_1, a_x + a_2\}$ (see Figure 1). So $(a_x + a_1)(a_x + a_2) \in E$, that is, $(a_x + a_1) + (a_x + a_2) \in \{a_x, a_x + a_1, a_x + a_2\}$. Thus, $(a_x + a_1) + (a_x + a_2) = a_x$, that is, $a_x + a_1 = -a_2$ and $a_x + a_2 = -a_1$. So $(a_x + a_1)a_2 \notin E$ and $(a_x + a_2)a_1 \notin E$ and $a_1 a_2 \notin E$ (see Figure 1), contracting the structure of $\overline{P_5}$.

Thus, Lemma 2.1 holds. □

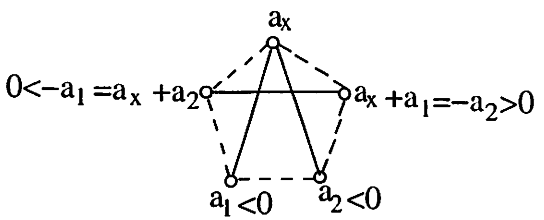


Figure 1

Lemma 2.2 $\zeta(\overline{P_5}) = 1$.

Proof: By Lemma 2.1, $\zeta(\overline{P_5}) \geq 1$. Below we will give an integral sum labelling of $\overline{P_5} \cup K_1$ (see Figure 2). So $\zeta(\overline{P_5}) \leq 1$. Thus, $\zeta(\overline{P_5}) = 1$. □

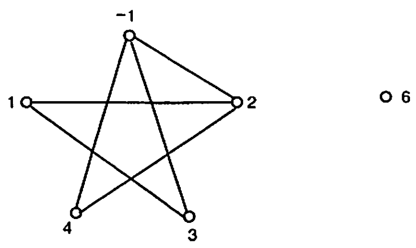


Figure 2

Lemma 2.3 $\sigma(\overline{P_5}) = 2$.

Proof: It is clear that the sum number of a graph must be at least as large as the minimum degree of the graph, so $\sigma(\overline{P_5}) \geq 2$. Figure 3 below shows that $\sigma(\overline{P_5}) \leq 2$. Thus, $\sigma(\overline{P_5}) = 2$. □

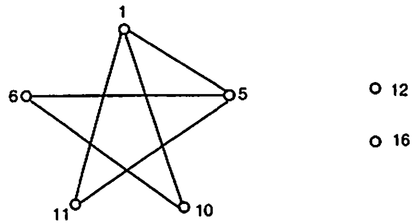


Figure 3

Lemma 2.4 If $a_x \in V$ with $|a_x| = \max\{|a| : a \in V\}$, then there exists one edge $a_x a_{j_0} \in E$ such that $a_x + a_{j_0} \in C$ for $n \geq 6$.

Proof: Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $a_x > 0$ (A similar argument works for $a_x < 0$.). By contradiction. Suppose to the contrary that $a_x + a_j \in V$ for all $a_x a_j \in E$. According to the choice of a_x, $a_x + a_j > 0$ and $a_j < 0$. Then there are at least $n-3$ distinct positive vertices $a_x + a_j$ and $n-3$ distinct negative vertices adjacent to a_x. So $(n-3)+(n-3)+1 \leq n$, i.e., $n \leq 5$, contradicting the fact $n \geq 6$.

Thus, Lemma 2.4 holds. □

Lemma 2.5 $\zeta(\overline{P_6}) = 3$.

Proof: Below we give an integral sum labelling of $\overline{P_6}$ (see Figure 4). So $\zeta(\overline{P_6}) \leq 3$. What we need to do is only to prove $\zeta(\overline{P_6}) \geq 3$.

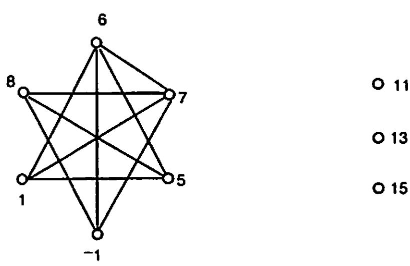


Figure 4

Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $a_x > 0$ (A similar argument works for $a_x < 0$). By Lemma 2.2, there exists one edge $a_x a_{j_0} \in E_x$ such that $a_x + a_{j_0} \in C$. Firstly, we will show Claim 1 and Claim 2 and Claim 3.

Claim 1: There exists another edge $a_x a_l \in E_x - a_x a_{j_0}$ such that $a_x + a_l \in C$.

By contradiction. Suppose to the contrary that $a_x + a_l \in V$ for all $a_x a_l \in E_x - a_x a_{j_0}$. By the choice of a_x, $a_x + a_l > 0$ and $a_l < 0$. Since $a_x + a_{j_0} \in C$, $(a_x + a_{j_0}) + a_l = (a_x + a_l) + a_{j_0} \notin S$. Then $a_x + a_l \in \{a_{j_0}\} \cup \overline{V_{j_0}}$, denoted (1).

If a_x is an end vertex of P_6 then $|E_x| = 4$. So there are four distinct positive vertices and at least three distinct negative vertices in V. So $n > 6$, a contraction. So a_x is not an end vertex of P_6.

Let $\overline{V_x} = \{a_{x+1}, a_{x-1}\}$ and $V = \{a_x, a_{x+1}, a_{x-1}, a_{j_0}, a_{l_1}, a_{l_2}\}$ (see Figure 5). Assume a_i, a_j are two end vertices of P_6. Then $a_i a_j \in E$ and $a_i + a_j \in S$.

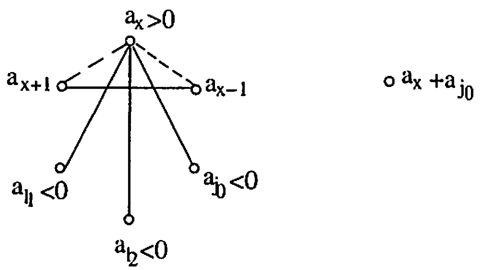


Figure 5

If $\overline{V_{j_0}} \subseteq \{a_{l_1}, a_{l_2}\}$ and $a_i a_j \in \{a_{x+1} a_{l_1}, a_{j_0} a_{l_2}, a_{l_2} a_{x-1}\}$ then we may assume that $a_{l_1} a_{j_0} \notin E$ (see Figure 6). By (1), $a_x + a_{l_1} = a_{j_0}$ and $a_x + a_{l_2} = a_{j_0}$, a contraction. So $\overline{V_{j_0}} \subseteq \{a_{l_1}, a_{x+1}\}$.

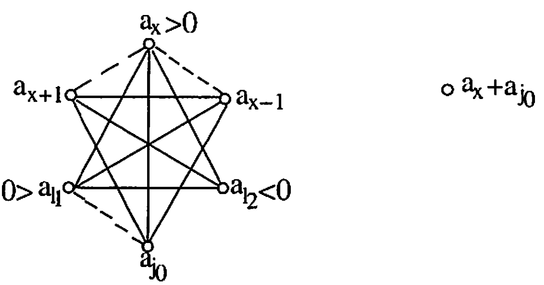


Figure 6

By (1) and $a_{l_1} < 0$, $\{a_x + a_{l_1}, a_x + a_{l_2}\} = \{a_{j_0}, a_{x+1}\}$ and $a_{x+1}a_{j_0} \notin E$ (see Figure 7).

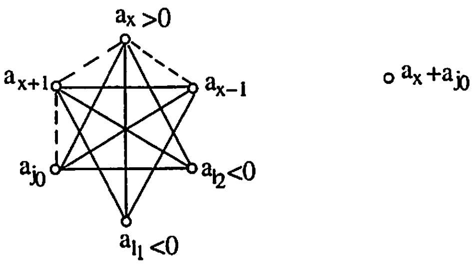


Figure 7

If $a_x + a_{l_1} = a_{x+1}$ and $a_x + a_{l_2} = a_{j_0}$ then $a_{l_1} + a_{j_0} = a_{x+1} + a_{l_2} \in S$. So $a_{l_1}a_{j_0} \in E$. Since $a_{x-1} + a_{j_0} = a_{x-1} + (a_x + a_{l_2}) = a_x + (a_{x-1} + a_{l_2}) \in S$, $a_{x-1} + a_{l_2} = a_{l_1}$, contracting $a_{x-1}a_{l_2} \notin E$.

If $a_x + a_{l_1} = a_{j_0}$ and $a_x + a_{l_2} = a_{x+1}$ then $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in S$. Uniting $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in S$ and $a_{x-1} + a_{l_1} = a_{l_2}$, we have $a_{j_0} + a_{x-1} = a_{x+1}$. If $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in V$ then $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} = a_{x-1}$. Since $a_{x-1} = a_{j_0} + a_{l_2} = a_{j_0} + (a_{x-1} + a_{l_1})$, $a_{j_0} + a_{l_1} = 0$. Thus, $(a_x + a_{j_0}) + a_{l_1} = a_x + (a_{j_0} + a_{l_1}) = a_x \in S$, contracting $a_x + a_{j_0} \in C$. If $a_{l_1} + a_{x+1} = a_{j_0} + a_{l_2} \in C$ then $a_{x+1} + a_{l_2} = (a_{j_0} + a_{x-1}) + a_{l_2} = (a_{j_0} + a_{l_2}) + a_{x-1} \in S$, contracting $a_{j_0} + a_{l_2} \in C$.

Thus, Claim 1 holds.

Up to now, we may assume that $a_x + a_{j_0} \in C$ and $a_x + a_{l_1} \in C$ with $\{a_x a_{j_0}, a_x a_{l_1}\} \subseteq E_x$.

Claim 2: If a_x is one end vertex of P_6 then $\zeta(\overline{P_6}) \geq 3$.

In fact, if a_x is an end vertex of P_6 then $|\overline{V_x}| = 1$. Let $\overline{V_x} = \{a_{x+1}\}$. Assume that $a_{j_0}a_{x-1} \in E$ (see Figure 8) (if not, we can consider $a_{l_1}a_{x-1} \in E$).

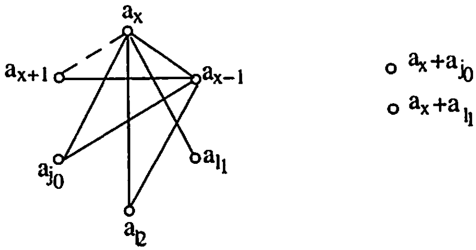


Figure 8

By contradiction. Suppose to the contrary that $\zeta(\overline{P_6}) \le 2$. By Claim 1, $\zeta(\overline{P_6}) \ge 2$. So $\zeta(\overline{P_6}) = 2$. Let $C = \{a_x + a_{j_0}, a_x + a_{l_1}\}$. Then $\{a_x + a_{x-1}, a_x + a_{l_2}\} \subset (\{a_{j_0}\} \cup \overline{V_x}) \cap (\{a_{l_1}\} \cup \overline{V_{l_1}})$.

According to the choice of a_x, $a_{x-1} < 0$ and $a_{l_2} < 0$. So there is only one case of $\{a_x+a_{x-1}, a_x+a_{l_2}\} = \{a_{j_0}, a_{l_1}\}$ and $P_6 = a_x a_{x+1} a_{j_0} a_{l_2} a_{l_1} a_{x-1}$ (see Figure 9).

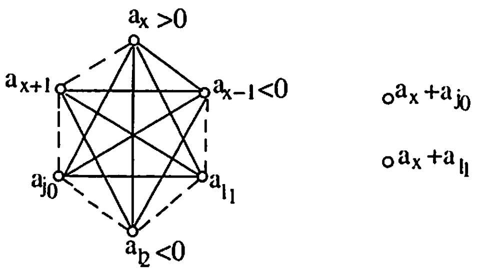


Figure 9

If $a_x + a_{x-1} = a_{j_0}$ and $a_x + a_{l_2} = a_{l_1}$ then $a_{j_0} + a_{l_1} = (a_x + a_{x-1}) + a_{l_1} = (a_x + a_{l_1}) + a_{x-1} \in S$, contracting $a_x + a_{l_1} \in C$.

If $a_x + a_{x-1} = a_{l_1}$ and $a_x + a_{l_2} = a_{j_0}$ then $a_{j_0} + a_{l_1} = (a_x + a_{l_2}) + a_{j_0} = (a_x + a_{j_0}) + a_{l_2} \in S$, contracting $a_x + a_{j_0} \in C$.

Thus, Claim 2 holds.

Claim 3: If a_x is not an end vertex of P_6 then $\zeta(\overline{P_6}) \ge 3$.

In fact, if a_x is not an end vertex of P_6 then $|\overline{V_x}| = 2$. Let $\overline{V_x} = \{a_{x+1}, a_{x-1}\}$. Then $a_x a_{x+1} \notin E$ and $a_x a_{x-1} \notin E$. Let $a_x a_{l_2} \in E_x - \{a_x a_{j_0}, a_x a_{l_1}\}$. Since $\{a_x + a_{j_0}, a_x + a_{l_1}\} \subseteq C$, $a_x + a_{l_2} \in [(\{a_{j_0}\} \cup \overline{V_{j_0}}) \cap (\{a_{l_1}\} \cup \overline{V_{l_1}})] \cup C$.

By contradiction. Suppose to the contrary that $\zeta(\overline{P_6}) \le 2$. By Claim 1, $\zeta(\overline{P_6}) \ge 2$. So $\zeta(\overline{P_6}) = 2$. Let $C = \{a_x + a_{j_0}, a_x + a_{l_1}\}$. Then $a_x + a_{l_2} \in V$. So $a_{l_2} < 0$ and it is impossible that both of a_{j_0} and a_{l_1} are adjacent to a_{l_2}. Assume $a_i a_j \in \{a_{x+1} a_{j_0}, a_{j_0} a_{l_1}, a_{l_1} a_{l_2}, a_{l_2} a_{x-1}\}$. Then $a_x + a_{l_2} \in \{a_{j_0}, a_{l_1}\}$ with $a_{j_0} a_{l_1} \notin E$ (if $a_{j_0} a_{l_1} \in E$ then $a_x + a_{l_2} \in (\{a_{j_0}\} \cup \overline{V_{j_0}}) \cap (\{a_{l_1}\} \cup \overline{V_{l_1}}) = \emptyset$. It is impossible.)(see Figure 10).

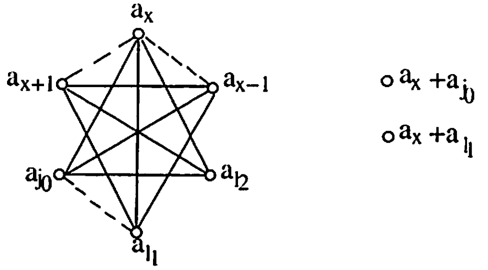


Figure 10

Similarly, $a_{j_0}+a_{x-1}\in\{a_x,a_{x+1}\}\cup C$; $a_{x+1}+a_{l_1}\in\{a_x,a_{x-1}\}\cup C$; $a_{x-1}+a_{l_1}\in\{a_x,a_{x+1}\}\cup C$; $a_{x+1}+a_{j_0}\in\{a_x,a_{x-1}\}\cup C$; $a_{j_0}+a_{l_2}\in\{a_{x-1},a_{x+1}\}\cup C$.

(1.1) If $a_x+a_{l_2}=a_{j_0}$ and $a_{j_0}+a_{x-1}=a_x$ then $a_x+a_{l_1}=(a_{j_0}+a_{x-1})+a_{l_1}=a_{j_0}+(a_{x-1}+a_{l_1})\in S$. So $a_{x-1}+a_{l_1}=a_{x+1}$, which implies $a_x+a_{l_1}=a_{j_0}+a_{x+1}$ and $a_{j_0}a_{x+1}\in E$. Since $(a_x+a_{j_0})+a_{x+1}=a_x+(a_{j_0}+a_{x+1})\notin S$, $a_{j_0}+a_{x+1}\in\{a_{x-1}\}\cup C$.

(1.1.1) If $a_{j_0}+a_{x+1}=a_{x-1}$ then $a_{l_1}+a_{j_0}=0$ (since $a_{x-1}+a_{l_1}=a_{x+1}$). So $a_x+a_{l_1}=(a_{j_0}+a_{x-1})-a_{j_0}=a_{x-1}\in V$, contracting $a_x+a_{l_1}\in C$.

(1.1.2) If $a_{x+1}+a_{j_0}\in C$, $(a_{x+1}+a_{j_0})+a_{l_1}=(a_{x+1}+a_{l_1})+a_{j_0}\notin S$. Then $a_{x+1}+a_{l_1}\in C$. So $\{a_{x+1}+a_{j_0},a_{x+1}+a_{l_1}\}\subseteq C=\{a_x+a_{j_0},a_x+a_{l_1}\}$, a contraction.

(1.2) If $a_x+a_{l_2}=a_{j_0}$ and $a_{j_0}+a_{x-1}=a_{x+1}$ then $a_{x+1}+a_{l_1}=(a_{j_0}+a_{x-1})+a_{l_1}=a_{j_0}+(a_{x-1}+a_{l_1})\in S$. So $a_{x-1}+a_{l_1}=a_x$. Since $a_{x+1}+a_{l_2}=(a_{j_0}+a_{x-1})+a_{l_2}=(a_{j_0}+a_{l_2})+a_{x-1}\in S$, $a_{j_0}+a_{l_2}=a_{x-1}$. Uniting $a_x+a_{l_2}=a_{j_0}$ and $a_{j_0}+a_{l_2}=a_{x-1}$, we have $a_x+2a_{l_2}=a_{x-1}$, contracting the choice of a_x.

(1.3) If $a_x+a_{l_2}=a_{j_0}$ and $a_{j_0}+a_{x-1}\in C$ then $a_{l_2}+(a_{j_0}+a_{x-1})=(a_{l_2}+a_{j_0})+a_{x-1}\notin S$. So $a_{l_2}+a_{j_0}=a_{x-1}$ (If not, then $a_{l_2}+a_{j_0}\in C$, but $a_{l_2}+a_{j_0}\notin C=\{a_x+a_{j_0},a_x+a_{l_1}\}$, a contraction.). Then $(a_{x+1}+a_{l_2})+a_{j_0}=a_{x+1}+(a_{l_2}+a_{j_0})=a_{x+1}+a_{x-1}\in S$. So $a_{x+1}+a_{l_2}=a_x$, contracting the choice of a_x.

(2) If $a_x+a_{l_2}=a_{l_1}$ then $a_{l_1}+a_{x+1}=(a_x+a_{l_2})+a_{x+1}=a_x+(a_{l_2}+a_{x+1})\in S$. So $a_{l_2}+a_{x+1}=a_{j_0}$ and $a_x+a_{j_0}=a_x+(a_{l_2}+a_{x+1})=(a_x+a_{l_2})+a_{x+1}=a_{l_1}+a_{x+1}\in C$. Since $a_{j_0}+a_{x-1}=(a_{l_2}+a_{x+1})+a_{x-1}=a_{l_2}+(a_{x+1}+a_{x-1})\in S$ and $(a_{l_1}+a_{x+1})+a_{x-1}=a_{l_1}+(a_{x+1}+a_{x-1})\notin S$, $a_{x+1}+a_{x-1}\in\{a_{l_1},a_{l_2}\}$.

(2.1) If $a_{x+1}+a_{x-1}=a_{l_1}$ then $a_{j_0}+a_{x-1}=a_{l_1}+a_{l_2}\in S$. So $a_{l_1}a_{l_2}\in E$ and $a_{j_0}+a_{x-1}=a_{l_2}+a_{l_1}=a_{x+1}$ (Since $a_{j_0}+a_{x-1}=a_{l_2}+a_{l_1}=a_{x+1}\notin C=\{a_x+a_{j_0},a_x+a_{l_1}\}$).

Since $a_{j_0}+a_{x-1}=(a_{l_2}+a_{x+1})+a_{x-1}=a_{x+1}$, $a_{l_2}+a_{x-1}=0$. Uniting $a_{l_1}=a_x+a_{l_2}$, we have $a_{l_1}+a_{x-1}=a_x$.

Since $C=\{a_x+a_{j_0},a_x+a_{l_1}\}$ and the choice of a_x, we have $a_{j_0}+a_{l_2}\notin C$. So $a_{j_0}+a_{l_2}=a_{x-1}$ (note that $a_{j_0}+a_{x-1}=a_{x+1}$).

Assume that $a_{x-1}=x$. By the above, $a_{j_0}=2x$ and $a_{x+1}=3x$ and $a_x=5x$. Since $a_{x+1}+a_{j_0}=5x=a_x$, contracting $a_{x+1}a_{j_0}\notin E$.

(2.2) If $a_{x+1}+a_{x-1}=a_{l_2}$ then $2a_{l_2}=a_{j_0}+a_{x-1}=a_{x+1}$ (If $a_{j_0}+a_{x-1}\in C$ then $(a_{j_0}+a_{x-1})+a_{x+1}=a_{j_0}+(a_{x-1}+a_{x+1})=a_{j_0}+a_{l_2}\in S$, a contraction). Since $a_{j_0}+a_{l_2}\neq a_{x+1}=a_{j_0}+a_{x-1}$, $a_{j_0}+a_{l_2}=a_{x-1}$ or $a_{j_0}+a_{l_2}\in C$.

(2.2.1) If $a_{j_0}+a_{l_2}=a_{x-1}$ then $-a_{l_2}=a_{x-1}=a_{j_0}+a_{l_2}=(a_{l_2}+a_{x+1})+a_{l_2}$. So $a_{x+1}=-3a_{l_2}$, contracting $a_{x+1}=2a_{l_2}$.

(2.2.2) If $a_{j_0}+a_{l_2}\in C=\{a_x+a_{j_0},a_x+a_{l_1}\}$ then $a_{j_0}+a_{l_2}=a_x+a_{l_1}=a_x+(a_x+a_{l_2})=2a_x+a_{l_2}$. So $a_{j_0}=2a_x$, contracting the choice of a_x.

Therefore, Claim 3 holds.

Thus, Lemma 2.5 holds. □

Lemma 2.6 $\sigma(\overline{P_6}) = 4$.

Proof: Let $V = \{a_x, a_{x+1}, a_{x-1}, a_{l_1}, a_{l_2}, a_{j_0}\}$ and $a_x = \max\{a : a \in V\}$. Then $a_x + a_i \in C$ for all $a_x a_i \in E$. Firstly, Figure 11 below shows that $\sigma(\overline{P_6}) \le 4$. What we need is to prove $\sigma(\overline{P_6}) \ge 4$.

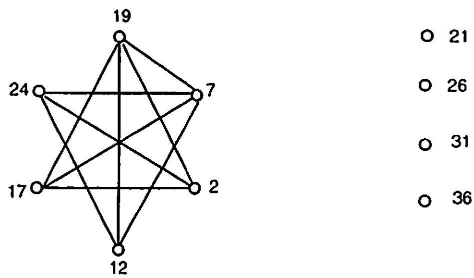


Figure 11

If a_x is an end vertex of P_n then $\sigma(\overline{P_6}) \ge 4$. Otherwise, it is clear that $\sigma(\overline{P_6}) \ge 3$. Below we will prove that $\sigma(\overline{P_6}) \ne 3$.

By contradiction. Suppose to the contrary that $\sigma(\overline{P_6}) = 3$. Then $C = \{a_x + a_{l_1}, a_x + a_{l_2}, a_x + a_{j_0}\}$. Assume that $\overline{V_x} = \{a_{x+1}, a_{x-1}\}$ and $a_i a_j \in \{a_{l_1} a_{x+1}, a_{l_2} a_{l_1}, a_{j_0} a_{l_2}, a_{x-1} a_{j_0}\}$, where a_i and a_j are two end vertices of P_6 (see Figure 12).

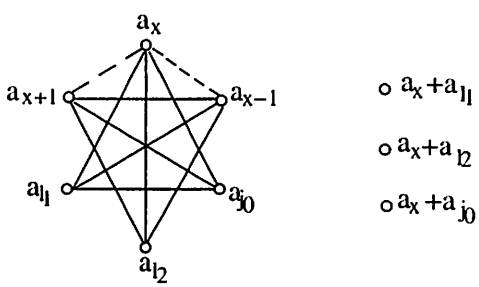


Figure 12

Since $(a_x + a_{j_0}) + a_{l_1} = a_x + (a_{j_0} + a_{l_1}) \notin S$, $a_{j_0} + a_{l_1} \in \{a_x, a_{x+1}, a_{x-1}\} \cup C$. Similarly, $a_{x-1} + a_{l_1} \in \{a_x, a_{x+1}\} \cup C$; $a_{x-1} + a_{l_2} \in \{a_x, a_{x+1}\} \cup C$; $a_{x+1} + a_{l_2} \in \{a_x, a_{x-1}\} \cup C$; $a_{x+1} + a_{j_0} \in \{a_x, a_{x-1}\} \cup C$.

(I) If $a_{j_0} + a_{l_1} = a_x$ then $a_x + a_{x-1} = (a_{j_0} + a_{l_1}) + a_{x-1} = (a_{x-1} + a_{l_1}) + a_{j_0} \notin S$, which implies $a_{x-1} + a_{l_1} \in \{a_{j_0}\} \cup \overline{V_{j_0}} \cup C$. By the above, $a_{x-1} + a_{l_1} \in C$. So

$(a_{x-1}+a_{l_1})+a_{l_2} = a_{l_1}+(a_{x-1}+a_{l_2}) \notin S$, which implies $a_{x-1}+a_{l_2} \in \{a_{x+1}\}\cup C$. Similarly, $a_{x-1}+a_{x+1} \in \{a_{l_1}\}\cup \overline{V_{l_1}} \cup C$ with $\overline{V_{l_1}} \subseteq \{a_{x+1}, a_{l_2}\}$.

(I.1) If $a_{x-1}+a_{l_2} = a_{x+1}$ then $a_{x-1}+a_{x+1} \in \{a_{l_1}\}\cup C$. Furthermore, $a_{x-1}+a_{x+1} \in C$. (If not, then $a_{x-1}+a_{x+1} = a_{l_1}$. Then $a_x = a_{j_0}+a_{l_1} = a_{j_0}+(a_{x-1}+a_{x+1}) = (a_{j_0}+a_{x+1})+a_{x-1} \in S$. So $a_{j_0}+a_{x+1} = a_{l_1}(= a_{x-1}+a_{x+1})$, which implies $a_{j_0}+a_{l_2} = a_{x+1}$, a contradiction with $a_{x-1}+a_{l_2} = a_{x+1}$.). So $(a_{x-1}+a_{x+1})+a_{l_2} = a_{x-1}+(a_{x+1}+a_{l_2}) \notin S$. By the above, $a_{x+1}+a_{l_2} \in \{a_x\}\cup C$.

(I.1.1) If $a_{x+1}+a_{l_2} = a_x$ then $a_x+a_{j_0} = (a_{x+1}+a_{l_2})+a_{j_0} = (a_{x+1}+a_{j_0})+a_{l_2} \in S$, which implies $a_{x+1}+a_{j_0} \in V$. By the above, $a_{x+1}+a_{j_0} = a_{x-1}$. Uniting $a_{x+1} = a_{x-1}+a_{l_2}$, we have $a_{l_2}+a_{j_0} = 0$, a contradiction.

(I.1.2) If $a_{x+1}+a_{l_2} \in C$ then $(a_{x+1}+a_{l_2})+a_{j_0} = (a_{x+1}+a_{j_0})+a_{l_2} \notin S$, which implies $a_{x+1}+a_{j_0} \in \{a_{l_2}\}\cup\overline{V_{l_2}}\cup C$. By the above, $a_{x+1}+a_{j_0} \in C$.

(I.1.2.1) If $a_{x+1}a_{l_1} \in E$ then $a_{x+1}+a_{l_1} \in S$. So $a_{x+1}+a_{l_1} = (a_{x-1}+a_{l_2})+a_{l_1} = a_{l_2}+(a_{x-1}+a_{l_1}) \in S$, contradicting $a_{x-1}+a_{l_1} \in C$.

(I.1.2.2) If $a_{l_1}a_{l_2} \in E$ then $a_{l_1}+a_{l_2} \in S$. Uniting $a_{x+1}+a_{l_1} = (a_{x-1}+a_{l_2})+a_{l_1} = a_{x-1}+(a_{l_1}+a_{l_2}) \notin S$ and $a_x+a_{l_2} = (a_{j_0}+a_{l_1})+a_{l_2} = a_{j_0}+(a_{l_1}+a_{l_2}) \in S$, we have $a_{l_1}+a_{l_2} = a_{j_0}$. So $(a_x+a_{l_2})+a_{l_1} = a_x+(a_{l_1}+a_{l_2}) = a_x+a_{j_0} \in S$, contradicting $a_x+a_{l_2} \in C$.

(I.1.2.3) If $a_{l_2}a_{j_0} \in E$ then $a_{l_2}+a_{j_0} \in S$. Since $a_x+a_{l_2} = (a_{j_0}+a_{l_1})+a_{l_2} = (a_{j_0}+a_{l_2})+a_{l_1} \in S$, $a_{j_0}+a_{l_2} \in \{a_{l_1}, a_{x-1}\}$. Uniting $(a_{j_0}+a_{l_2})+a_x = (a_x+a_{l_2})+a_{j_0} \notin S$, we have $a_{j_0}+a_{l_2} = a_{x-1}$. So $a_{x+1}+a_{x-1} = a_{x+1}+(a_{j_0}+a_{l_2}) = (a_{x+1}+a_{l_2})+a_{j_0} \notin S$, a contradiction.

(I.1.2.4) If $a_{x-1}a_{j_0} \in E$ then $a_{x-1}+a_{j_0} \in S$. Since $a_{x+1}+a_{j_0} = (a_{x-1}+a_{l_2})+a_{j_0} = (a_{x-1}+a_{j_0})+a_{l_2} \in S$, $a_{x-1}+a_{j_0} = a_{l_2}$. So $a_x+a_{l_2} = a_x+(a_{x-1}+a_{j_0}) = (a_x+a_{j_0})+a_{x-1} \in S$, contradicting $a_x+a_{j_0} \in C$.

(I.2) If $a_{x-1}+a_{l_2} \in C$ then $a_{x-1}+a_{x+1} \in \{a_{l_2}\}\cup\overline{V_{l_2}}\cup C$. Since $a_{x-1}+a_{l_1} \in C$, $a_{x-1}+a_{x+1} \in \{a_{l_1}\}\cup\overline{V_{l_1}}\cup C$. So $a_{x-1}+a_{x+1} \in \{a_{l_1}, a_{l_2}\}\cup C$. Since $a_{x+1}+a_{j_0} \neq a_x = a_{j_0}+a_{l_1}$, $a_{x+1}+a_{j_0} \in \{a_{x-1}\}\cup C$. Since $\{a_{x-1}+a_{l_1}, a_{x-1}+a_{l_2}\} \subset C = \{a_x+a_{l_1}, a_x+a_{l_2}, a_x+a_{j_0}\}$, $a_{x-1}+a_{l_1} \in \{a_x+a_{l_2}, a_x+a_{j_0}\}$ and $a_{x-1}+a_{l_2} \in \{a_x+a_{l_1}, a_x+a_{j_0}\}$.

(I.2.1) If $a_{x-1}+a_{l_1} = a_x+a_{l_2}(= (a_{j_0}+a_{l_1})+a_{l_2})$ then $a_{x-1} = a_{j_0}+a_{l_2}$ and $a_{x-1}+a_{l_2} = a_x+a_{j_0}$. So $2a_{l_2} = a_{l_1}+a_{j_0} = a_x$. Since $a_{x+1}+a_{j_0} \neq a_{x-1} = a_{j_0}+a_{l_2}$, $a_{x+1}+a_{j_0} \in C$. Uniting $a_x+a_{l_2} = a_{x-1}+a_{l_1}$ and $a_x+a_{j_0} = a_{x-1}+a_{l_2}$, we have $a_{x+1}+a_{j_0} = a_x+a_{l_1}$. So $a_{x+1} = 2a_{l_1}$ and $a_{x-1}+a_{x+1} \in C$, but $a_{x-1}+a_{x+1} \notin \{a_x+a_{l_1}, a_x+a_{l_2}, a_x+a_{j_0}\}$, a contradiction.

(I.2.2) If $a_{x-1}+a_{l_1} = a_x+a_{j_0}(= (a_{j_0}+a_{l_1})+a_{j_0})$ then $a_{x-1}+a_{l_2} = a_x+a_{l_1}$ and $a_{x-1} = 2a_{j_0}$. So $a_{x+1}+a_{j_0} \in \{a_x+a_{l_1}, a_x+a_{l_2}\} \subset C$.

(I.2.2.1) If $a_{x+1}+a_{j_0} = a_x+a_{l_1}(= (a_{j_0}+a_{l_1})+a_{l_1} = 2a_{l_1}+a_{j_0} = a_{l_2}+a_{x-1})$ then $2a_{l_1} = a_{x+1}$. If $a_{x+1}+a_{j_0} = a_{x-1}$ then $2a_{l_1}+a_{j_0} = a_{x-1} = 2a_{j_0}$. So $2a_{l_1} = a_{j_0}$, contradicting $a_{x+1} = 2a_{l_1}$. So $a_{x+1}+a_{j_0} = a_x+a_{l_1} = a_{l_2}+a_{x-1} \in C$. By the choice of a_x, $a_{x+1}+a_{x-1} \in C$. Since $a_{x+1}+a_{x-1} = 2a_{l_1}+2a_{j_0} = 2(a_{l_1}+a_{j_0}) = 2a_x \in S$, contradicting the choice of a_x.

(I.2.2.2) If $a_{x+1}+a_{j_0} = a_x+a_{l_2}(= (a_{j_0}+a_{l_1})+a_{l_2})$ then $a_{l_1}+a_{l_2} = a_{x+1}$. So $a_{x-1}+a_{x+1} \in C$, but, $a_{x-1}+a_{x+1} \notin \{a_x+a_{l_1}, a_x+a_{l_2}, a_x+a_{j_0}\} = \{a_{x-1}+a_{l_2}, a_{x+1}+a_{j_0}, a_{x-1}+a_{l_1}\}$, a contradiction.

(II) If $a_{j_0} + a_{l_1} = a_{x+1}$ then $a_{x+1} + a_{x-1} = (a_{j_0} + a_{l_1}) + a_{x-1} = (a_{j_0} + a_{x-1}) + a_{l_1} \in S$. Then $a_{x-1} + a_{j_0} = a_x$. Since $a_x + a_{l_2} = (a_{x-1} + a_{j_0}) + a_{l_2} = (a_{x-1} + a_{l_2}) + a_{j_0} \in S$, $a_{x-1} + a_{l_2} = a_{x+1}(= a_{j_0} + a_{l_1})$.

Uniting $a_{x-1} + a_{j_0} = a_x$ and $a_{x-1} + a_{l_2} = a_{x+1}$, we have $a_x + a_{l_2} = a_{x+1} + a_{j_0} \in C$.

Uniting $a_{x-1} + a_{j_0} = a_x$ and $a_{j_0} + a_{l_1} = a_{x+1}$, we have $a_x + a_{l_1} = a_{x+1} + a_{x-1} \in C$.

Uniting $(a_{x+1} + a_{j_0}) + a_{l_2} = (a_{x+1} + a_{l_2}) + a_{j_0} \notin S$ and $a_{x-1} + a_{l_2} = a_{x+1}$, we have $a_{x+1} + a_{l_2} \in C$. Then $a_{x+1} + a_{l_2} = a_x + a_{j_0}$, that is, $(a_{x-1} + a_{j_0}) + a_{j_0} = (a_{x-1} + a_{l_2}) + a_{l_2}$. So $2a_{j_0} = 2a_{l_2}$, a contradiction.

A similar argument works for $a_{j_0} + a_{l_1} = a_{x+1}$.

(III) If $a_{j_0} + a_{l_1} \in C$ then $a_{j_0} + a_{l_1} = a_x + a_{l_2}$.

Since $(a_{j_0} + a_{l_1}) + a_{x-1} = (a_{l_1} + a_{x-1}) + a_{j_0} \notin S$, $a_{l_1} + a_{x-1} \in C$. So $a_{l_1} + a_{x-1} = a_x + a_{j_0}$.

Since $a_{j_0} + a_{l_1} \in C$, $(a_{j_0} + a_{l_1}) + a_{x+1} = (a_{j_0} + a_{x+1}) + a_{l_1} \notin S$, $a_{j_0} + a_{x+1} = a_x + a_{l_1} \in C$.

Since $a_{j_0} + a_{x+1} \in C$, $(a_{j_0} + a_{x+1}) + a_{x-1} = a_{j_0} + (a_{x+1} + a_{x-1}) \notin S$. So $a_{x+1} + a_{x-1} = a_x + a_{l_2} = a_{j_0} + a_{l_1} \in C$.

Uniting $a_{j_0} + a_{x+1} = a_x + a_{l_1}$ and $a_{x+1} + a_{x-1} = a_x + a_{l_2}$, we have $a_{x-1} + a_{l_1} = a_{j_0} + a_{l_2}$. Then $a_{j_0}a_{l_2} \in E$.

Uniting $a_{x+1} + a_{x-1} = a_x + a_{l_2}$ and $a_{l_1} + a_{x-1} = a_x + a_{j_0}$, we have $a_{l_1} + a_{l_2} = a_{x+1} + a_{j_0} \in S$. Then $a_{l_1}a_{l_2} \in E$, a contradiction.

So $\sigma(\overline{P_6}) \neq 3$.

Thus, Lemma 2.6 holds. □

Lemma 2.7 Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Then $a_x + a_p \in C$ for all $a_xa_p \in E$ with $n = 7$.

Proof: Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $V = \{a_x, a_{x+1}, a_{x-1}, a_{j_0}, a_{l_1}, a_{l_2}, a_{l_3}\}$ and $a_x > 0$ (A similar argument works for $a_x < 0$). By Lemma 2.4, there exists one edge $a_xa_{j_0} \in E_x$ such that $a_x + a_{j_0} \in C$. Since $(a_x + a_{j_0}) + a_l = (a_x + a_l) + a_{j_0} \notin S$, $a_x + a_l \in \{a_{j_0}\} \cup \overline{V_{j_0}} \cup C$ for all $a_xa_l \in E_x - a_xa_{j_0}$.

Claim There exists at least one edge $a_xa_l \in E_x - a_xa_{j_0}$ such that $a_xa_l \in C$.

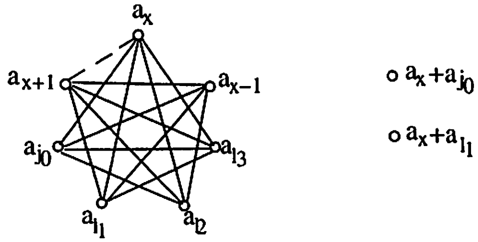
In fact, if $|\overline{V_x}| = 2$ then $|E_x| = 5$. Since $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 3$, Claim 1 holds. If $|\overline{V_x}| = 1$ then suppose to the contrary that $a_x + a_l \in V$ for all $a_xa_l \in E_x - a_xa_{j_0}$. Then $a_l < 0$ and $a_x + a_l > 0$. So there are at least eight distinct vertices, contradicting $n = 7$.

Thus, Claim holds.

Assume that $a_xa_{x+1} \notin E$ and $a_x + a_{l_1} \in C$ with $a_xa_{l_1} \in E_x - a_xa_{j_0}$. Since $a_xa_{l_1} \in C$, $a_x + a_{l_i} \in \{a_{j_0}\} \cup \overline{V_{j_0}} \cup C$ for all $a_xa_{l_i} \in E_x - \{a_xa_{j_0}, a_xa_{l_1}\}$ with $i = 2, 3$. So $a_x + a_{l_i} \in [(\{a_{j_0}\} \cup \overline{V_{j_0}}) \cap (\{a_{l_1}\} \cup \overline{V_{l_1}})] \cup C$.

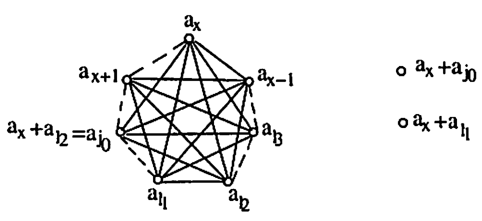
By contradiction. Suppose to the contrary that $a_xa_{l_2} \in V$. By the above, $a_x + a_{l_2} \in (\{a_{j_0}\} \cup \overline{V_{j_0}}) \cap (\{a_{l_1}\} \cup \overline{V_{l_1}})$. There are at most two cases (I) (II) in all (see Figure 13,15). Let a_i and a_j be two end vertices of P_7.

(I) If $a_ia_j \in \{a_{x+1}a_{j_0}, a_{j_0}a_{l_1}, a_{l_1}a_{l_2}, a_{l_2}a_{l_3}, a_{l_3}a_{x-1}, a_{x-1}a_x\}$ then $a_x + a_{l_2} \in \{a_{j_0}, a_{l_1}\}$ (see Figure 13).



I: Figure 13

(I.1) If $a_x + a_{l_2} = a_{j_0}$ then $a_{j_0} + a_{l_3} = (a_x + a_{l_3}) + a_{l_2} \in S$. So $a_x + a_{l_3} \in V$. Uniting $a_x + a_{j_0} \in C$ and $a_x + a_{l_1} \in C$, we have $a_x + a_{l_3} = a_{l_1}$, which implies $a_{l_1} + a_{l_2} = a_{l_3} + a_{j_0}$ and $a_{l_3} < 0$. Then $a_{l_1}a_{l_2} \in E$ (see Figure 14).



I.1: Figure 14

Since $a_{j_0} + a_{x-1} = (a_x + a_{l_2}) + a_{x-1} = a_x + (a_{l_2} + a_{x-1}) \in S$, $a_{l_2} + a_{x-1} \in \{a_{l_1}, a_{l_3}\}$.

Since $a_{j_0} + a_{x+1} = (a_x + a_{l_2}) + a_{x+1} = a_x + (a_{l_2} + a_{x+1}) \notin S$, $a_{l_2} + a_{x+1} \in \{a_{x-1}\} \cup C$.

If $a_{l_2} + a_{x+1} \in C$ then $(a_{l_2} + a_{x+1}) + a_{x-1} = (a_{l_2} + a_{x-1}) + a_{x+1} \notin S$, contradicting $a_{l_2} + a_{x-1} \in \{a_{l_1}, a_{l_3}\}$. So $a_{l_2} + a_{x+1} = a_{x-1}$.

Since $a_{l_3} + a_{x-1} = a_{l_3} + (a_{x+1} + a_{l_2}) = a_{l_2} + (a_{l_3} + a_{x+1}) \notin S$ and $a_{l_3} + a_{x+1} \in \{a_{j_0}, a_{l_2}\}$, $a_{l_3} + a_{x+1} = a_{l_2}$. Note that $a_{l_1} + a_{l_2} = a_{l_3} + a_{j_0} \in \{a_{x+1}, a_{x-1}\} \cup C$, then $(a_{l_1} + a_{l_2} =)a_{l_3} + a_{j_0} \in C$. So $(a_{l_3} + a_{j_0}) + a_{x+1} = (a_{l_3} + a_{x+1}) + a_{j_0} = a_{l_2} + a_{j_0} \in S$, contradicting $a_{l_3} + a_{j_0} \in C$.

(I.2) If $a_x + a_{l_2} = a_{l_1}$ then $a_x + a_{l_3} = a_{j_0}$, which implies $a_{l_1} + a_{l_3} = a_{l_2} + a_{j_0}$. So $a_x + a_{l_3} = a_{j_0}$ (If not, then $a_x + a_{l_3} \in C$. So $(a_x + a_{l_3}) + a_{l_2} = (a_x + a_{l_2}) + a_{l_3} = a_{l_1} + a_{l_3} \in S$, contradicting $a_x + a_{l_3} \in C$).

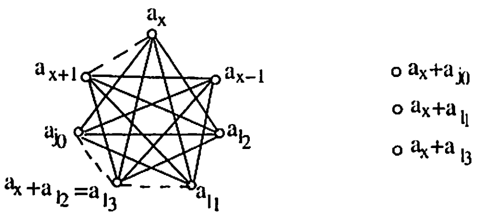
Since $a_{l_1} + a_{x-1} = (a_x + a_{l_2}) + a_{x-1} = a_x + (a_{x-1} + a_{l_2}) \in S$, $a_{x-1} + a_{l_2} \in \{a_{j_0}, a_{l_3}\}$.

(I.2.1) If $a_{x-1} + a_{l_2} = a_{j_0}$ then $a_x + a_{j_0} = a_{l_1} + a_{x-1} \in C$. So $a_{l_1} + a_{j_0} = a_{l_1} + (a_{x-1} + a_{l_2}) = (a_{l_1} + a_{x-1}) + a_{l_2} \in S$, contradicting $a_{l_1} + a_{x-1} \in C$.

(I.2.2) If $a_{x-1} + a_{l_2} = a_{l_3}$ then $a_{x-1} + a_{l_1} = a_{j_0}$ (since $a_{l_1} + a_{l_3} = a_{l_2} + a_{j_0}$).

So $a_x + a_{j_0} = a_x + (a_{x-1} + a_{l_1}) = (a_x + a_{l_1}) + a_{x-1} \in S$, contradicting $a_x + a_{l_1} \in C$.

(II) If $a_i a_j \in \{a_{x+1}a_{j_0}, a_{j_0}a_{l_3}, a_{l_3}a_{l_1}, a_{l_1}a_{l_2}, a_{l_2}a_{x-1}, a_{x-1}a_x\}$ then $a_x + a_{l_2} = a_{l_3}$ and $a_{l_1}a_{l_3} \notin E$ and $a_{j_0}a_{l_3} \notin E$. According to the choice of a_x, $a_x + a_{l_3} \in C$ and $a_{l_3} > 0$ (see Figure 15).



II: Figure 15

Since $a_{l_3} + a_{x+1} = (a_x + a_{l_2}) + a_{x+1} = a_x + (a_{l_2} + a_{x+1}) \in S$, $a_{l_2} + a_{x+1} \in \{a_{x-1}, a_{j_0}, a_{l_1}\}$.

Since $(a_x + a_{l_3}) + a_{x-1} = a_x + (a_{l_3} + a_{x-1}) \notin S$, $a_{l_3} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C$. Similarly, $a_{x+1} + a_{l_3} \in \{a_{x-1}\} \cup C$; $a_{l_1} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C$; $a_{j_0} + a_{x-1} \in \{a_x, a_{x+1}\} \cup C$; $a_{x+1} + a_{l_1} \in \{a_x, a_{x-1}\} \cup C$; $a_{l_2} + a_{l_3} \in \{a_{x+1}, a_{x-1}\} \cup C$.

(II.1) If $a_{x+1} + a_{l_3} = a_{x-1}$ then $a_{x+1} + a_{l_3} = a_{x+1} + (a_x + a_{l_2}) = (a_{x+1} + a_{l_2}) + a_x (= a_{x-1})$, contradicting $a_{x+1} + a_{l_2} \in \{a_{x-1}, a_{j_0}, a_{l_1}\}$.

(II.2) If $a_{x+1} + a_{l_3} \in C$ then $a_{x+1} + a_{l_1} \in C$ (since $(a_{x+1} + a_{l_3}) + a_{l_1} = (a_{x+1} + a_{l_1}) + a_{l_3} \notin S$). Then $(a_{x+1} + a_{l_1}) + a_{l_2} = (a_{x+1} + a_{l_2}) + a_{l_1} \notin S$. Since $a_{x+1} + a_{l_3} \in C$, $(a_{x+1} + a_{l_3}) + a_{l_2} = (a_{x+1} + a_{l_2}) + a_{l_3} \notin S$. Uniting $a_{l_2} + a_{x+1} \in \{a_{x-1}, a_{j_0}, a_{l_1}\}$, we have $a_{x+1} + a_{l_2} = a_{l_1}$. So $a_{l_1} + a_{l_3} = (a_{x+1} + a_{l_2}) + a_{l_3} = a_{x+1} + (a_{l_2} + a_{l_3}) \notin S$, which implies $a_{l_2} + a_{l_3} \in \{a_{x+1}\} \cup C$. So $a_{l_1} + a_{j_0} = (a_{x+1} + a_{l_2}) + a_{j_0} = a_{x+1} + (a_{l_2} + a_{j_0}) \in S$, which implies $a_{l_2} + a_{j_0} \in \{a_x\} \cup \overline{V_x}$. Then $a_{x+1} + (a_{l_1} + a_{j_0}) = (a_{x+1} + a_{l_1}) + a_{j_0} \notin S$, which implies that $a_{l_1} + a_{j_0} \in \{a_x\} \cup C$.

(II.2.1) If $a_{l_1} + a_{j_0} \in C$ then $(a_{l_1} + a_{j_0}) + a_{l_2} = a_{l_1} + (a_{l_2} + a_{j_0}) \notin S$, a contradiction $a_{l_2} + a_{j_0} \in \{a_x\} \cup \overline{V_x}$.

(II.2.2) If $a_{l_1} + a_{j_0} = a_x$ then $a_x + a_{l_2} = (a_{l_1} + a_{j_0}) + a_{l_2} = a_{l_1} + (a_{l_2} + a_{j_0}) \in S$, which implies $a_{l_2} + a_{j_0} \in \{a_{x-1}, a_{x+1}\}$.

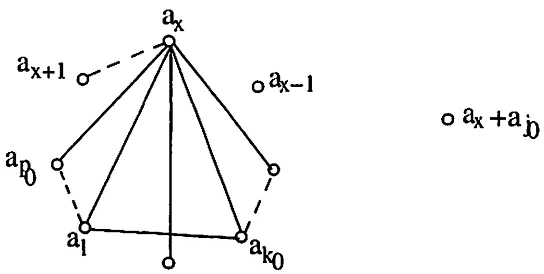
(II.2.2.1) If $a_{l_2} + a_{l_3} \in C$ then $(a_{l_2} + a_{l_3}) + a_{j_0} = (a_{l_2} + a_{j_0}) + a_{l_3} \notin S$, contradicting $a_{l_2} + a_{j_0} \in \{a_{x-1}, a_{x+1}\}$.

(II.2.2.2) If $a_{l_2} + a_{l_3} = a_{x+1}$ then $a_{l_2} + a_{j_0} = a_{x-1}$. Since $a_{x-1} + a_{l_1} = (a_{l_2} + a_{j_0}) + a_{l_1} = a_{l_2} + (a_{l_1} + a_{j_0}) = a_{l_2} + a_x = a_{l_3}$, $a_{x-1} + a_{l_1} = a_{l_3}$, a contradiction.

Thus, Lemma 2.7 holds. □

Lemma 2.8 If $a_x \in V$ with $|a_x| = \max\{|a| : a \in V\}$, then $a_x + a_p \in C$ for any $a_x a_p \in E$ with $n \geq 8$.

Proof: Let $|a_x| = \max\{|a| : a \in V\}$ with $a_x \in V$. Assume that $a_x > 0$ (A similar argument works for $a_x < 0$). By contradiction. Suppose to the contrary that there exist $a_{p_0} \in V$ and $a_{k_0} \in V - \{a_{p_0}, a_x\}$ such that $a_x + a_{p_0} = a_{k_0}$. According to the choice of a_x, $a_x + a_{p_0} > 0$ and $a_{p_0} < 0$. Let $V_0 = \{a_{k_0}, a_x\} \cup \overline{V_{k_0}} \cup \overline{V_x}$. Then $a_x a_l \in E$ and $a_{k_0} a_l \in E$ for all $a_l \in V - V_0$. So $a_{k_0} + a_l = (a_x + a_{p_0}) + a_l = (a_x + a_l) + a_{p_0} \in S$. Thus, $a_x + a_l \in V - \{a_x, a_{k_0}, a_{p_0}\}$ with $a_x + a_l > 0$ and $a_l < 0$. Since $n \geq 8$, there exists at least one such vertex a_l above (see Figure 16).



II. Figure 16

On the other hand, by Lemma 2.2, there exists one edge $a_x a_{j_0} \in E$ such that $a_x + a_{j_0} \in C$ for $n \geq 8$. Then $a_x a_j \in E$ for all $a_j \in V - (\{a_x\} \cup \overline{V_x})$. So $(a_x + a_{j_0}) + a_j = (a_x + a_j) + a_{j_0} \notin S$. Thus, $a_x + a_j \in \{a_{j_0}\} \cup \overline{V_{j_0}} \cup C$ for all $a_j \in V - (\{a_x\} \cup \overline{V_x})$.

For all $a_l \in V - V_0$, $a_x + a_l \in \{a_{j_0}\} \cup \overline{V_{j_0}}$. Since $|\overline{V_i}| \in \{1, 2\}$ for all $a_i \in V$, $n - 6 \leq |V - V_0| \leq |\{a_{j_0}\} \cup \overline{V_{j_0}}| \leq 3$, that is, $n \leq 9$. So we only consider $n = 9$ and $n = 8$. If $|\overline{V_i}| = 2$ then let $\overline{V_i} = \{a_{i-1}, a_{i+1}\}$ for any $a_i \in V$. If $|\overline{V_i}| = 1$ then let $\overline{V_i} = \{a_{i+1}\}$ for any $a_i \in V$.

Case 1 $n = 9$

(I) If a_{j_0} is an end vertex of P_n then $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 2$, contradicting $|V - V_0| \geq 3$.

(II) If a_{j_0} is not an end vertex of P_n then $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 3$. So a_x is not an end vertex of P_n (If not, $|\{a_x\} \cup \overline{V_x}| = 2$. So $|V - V_0| \geq 4$, contradicting $|\{a_{j_0}\} \cup \overline{V_{j_0}}| \leq 3$). Thus, only $|\{a_x\} \cup \overline{V_x}| = |\{a_{k_0}\} \cup \overline{V_{k_0}}| = |\{a_{j_0}\} \cup \overline{V_{j_0}}| = 3$. Note: $n = 9$ and none of the vertices in $\{a_x, a_{j_0}, a_{k_0}\}$ is an end vertex of P_n.

If $a_{k_0} a_{j_0} \in E$ then $a_{k_0} + a_{j_0} = (a_x + a_{p_0}) + a_{j_0} = (a_x + a_{j_0}) + a_{p_0} \in S$, contradicting $a_x + a_{j_0} \in C$.

If $a_{k_0}a_{j_0} \notin E$ then there exists one vertex $a_y \in V - V_0$ such that $a_x + a_y = a_{j_0+1}$ with $a_{j_0}a_{j_0+1} \notin E$ and $a_{j_0+1} \in V$. So $a_{k_0} + a_{j_0+1} = a_{k_0} + (a_x + a_y) = (a_x + a_{k_0}) + a_y \in S$, contradicting $a_x + a_{k_0} \in C$.

Case 2 $n = 8$

(I) If a_{j_0} is an end vertex of P_n then $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 2$. Let $\overline{V_{j_0}} = \{a_{j_0+1}\}$.
(I.1) If a_x is the other end vertex of P_n then we consider the below.
If $a_{j_0} = a_{k_0} = a_x + a_{p_0}$ then $|V_0| = 4$, contradicting $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 2$.
If $a_x + a_{p_0} = a_{k_0} \notin \{a_{j_0}, a_{j_0+1}\}$ then $a_{j_0}a_{k_0} \in E$. So $a_{j_0} + a_{k_0} = a_{j_0} + (a_x + a_{p_0}) = (a_x + a_{j_0}) + a_{p_0} \in S$, contradicting $a_x + a_{j_0} \in C$ (See Figure 17).

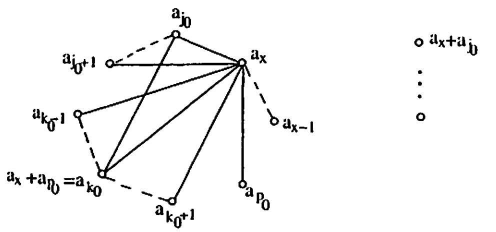


Figure 17

If $a_{j_0} \neq a_{k_0}$ and $a_{j_0+1} = a_{k_0} = a_x + a_{p_0}$ then $|V - V_0| = 3$, contradicting $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 2$ (See Figure 18).

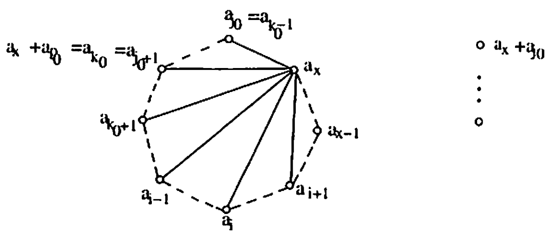


Figure 18

(I.2) If a_x is not the other end vertex of P_n then $|\{a_x\} \cup \overline{V_x}| = 3$, so $|V - V_0| \geq n - 6 = 2$. If $|V - V_0| > 2$ then it is a contradiction with $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 2$, so only $|V_0| = 6$ and $|\{a_{k_0}\} \cup \overline{V_{k_0}}| = 3$. There are only two subcases in the following.

(I.2.1) If $a_{j_0} = a_{k_0-1}$ with $a_{j_0}a_{j_0+1} \notin E$, then there exist two distinct vertices $a_y, a_{j_0-1} \in V - V_0$, then $\{a_x + a_y, a_x + a_{j_0-1}\} = \{a_{j_0}, a_{j_0+1}\}$. Since $a_x + a_{j_0} \in C$, $(a_x + a_{k_0+1}) + a_{j_0} = (a_x + a_{j_0}) + a_{k_0+1} \notin S$, then $a_x + a_{k_0+1} \in C$. Select any vertex $a_z \in \{a_y, a_{j_0-1}\}$ and then $a_{j_0} + a_{k_0+1} = (a_x + a_z) + a_{k_0+1} = (a_x + a_{k_0+1}) + a_z \in S$, contradicting $a_x + a_{k_0+1} \in C$ (See Figure 19).

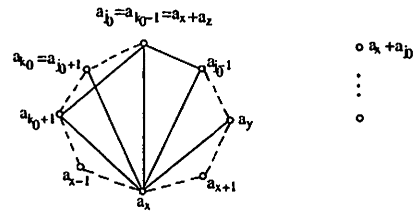


Figure 19

(I.2.2) If $V = \{a_{x-1}, a_x, a_{x+1}, a_{k_0}, a_{k_0+1}, a_{k_0-1}, a_{j_0}, a_{j_0+1}\}$, then a_{x+1} is the other end vertex of P_n. So $a_x a_{x+1} \notin E$. Since $a_x + a_{p_0} = a_{k_0} > 0$, we have $a_x + a_{k_0}\} \in C$. Since $\{a_x + a_{k_0}, a_x + a_{j_0}\} \subseteq C$, we have $\{a_x + a_{j_0+1}, a_x + a_{k_0-1}\} \subseteq C$. So only $a_x + a_{k_0+1} = a_{k_0}$. Thus, $a_{k_0} + a_{j_0} = (a_x + a_{k_0+1}) + a_{j_0} = (a_x + a_{j_0}) + a_{k_0+1} \in S$, contradicting $a_x + a_{j_0} \in C$ (See Figure 20).

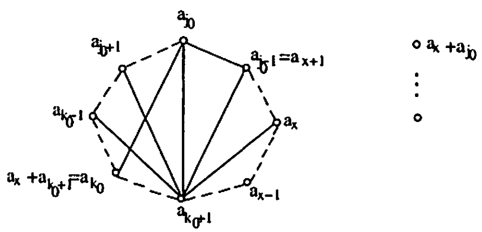


Figure 20

(II) If a_{j_0} is not an end vertex of P_n then $|\{a_{j_0}\} \cup \overline{V_{j_0}}| = 3$.

(II.1) If there exist two distinct vertices $a_{l_1}, a_{l_2} \in V - V_0$ such that $a_x + a_{l_1} = a_{j_0-1} > 0$ and $a_x + a_{l_2} = a_{j_0+1} > 0$ then $a_x + a_{j_0-1} \in C$ and $a_x + a_{j_0+1} \in C$. So $a_{j_0-1} + a_{j_0+1} = (a_x + a_{l_1}) + a_{j_0+1} = (a_x + a_{j_0+1}) + a_{l_1} \in S$, contradicting $a_x + a_{j_0+1} \in C$ (See Figure 21).

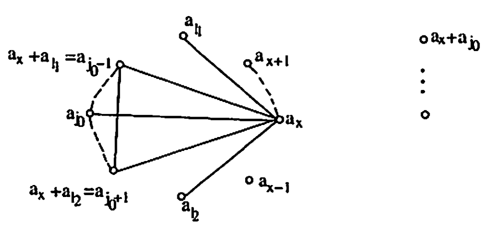


Figure 21

(II.2) Let $\{a_{y_1}, a_{y_2}\} = \{a_{j_0-1}, a_{j_0+1}\}$. If there exists at most one vertex $a_{y_1} \in \{a_{j_0-1}, a_{j_0+1}\}$ such that $a_x + a_{l_1} = a_{y_1} > 0$, then we can consider a_{y_1} as a_{k_0} in the following.

(II.2.1) If $a_x a_{y_2} \in E$ then $a_x + a_{y_2} \in V \cup C$.

If $a_x + a_{y_2} \in C$ then $a_{y_1} + a_{y_2} = (a_x + a_{l_1}) + a_{y_2} = (a_x + a_{y_2}) + a_{l_1} \in S$, contradicting $a_x + a_{y_2} \in C$.

If $a_x + a_{y_2} \in V$ and $a_{l_1} a_{y_1} \notin E$ then $a_x + a_{y_2} \in \{a_{j_0}\} \cup \overline{V_{j_0}}$, a contradiction (See Figure 22).

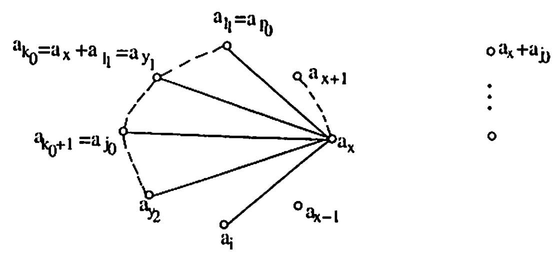


Figure 22

If $a_x + a_{y_2} \in V$ and $a_{l_1} a_{y_1} \in E$ then exists one vertex $a_z \in V - V_0$ such that $V - V_0$ such that $a_x + a_z \in \{a_{y_1}, a_{y_2}, a_{j_0}\}$. But it is impossible (See Figure 23).

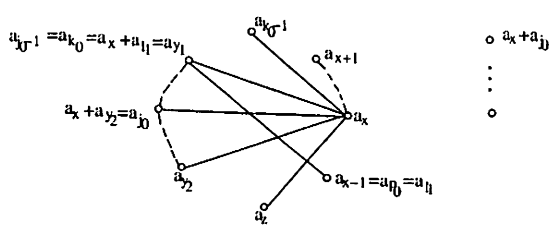


Figure 23

(II.2.2) If $a_x a_{y_2} \notin E$ then $a_x a_{y_1} \in E$ and there exist two distinct vertices $a_{x_1}, a_{x_2} \in V - V_0$ such that $a_x a_{x_1} \in E$ and $a_x a_{x_2} \in E$. Since $a_x + a_{l_1} = a_{y_1}$, we have $\{a_x + a_{x_1}, a_x + a_{x_2}\} = \{a_{y_2}, a_{j_0}\}$. Assume that $a_x + a_{x_1} = a_{y_2}$. Then $a_{y_1} + a_{y_2} = a_{y_1} + (a_x + a_{x_1}) = a_{x_1} + (a_x + a_{y_1}) \in S$, contradicting $a_x + a_{y_1} \in C$ (see Figure 24).

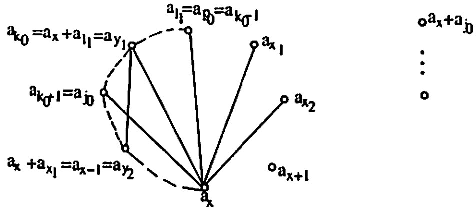


Figure 24

Thus, Lemma 2.8 holds. □

Lemma 2.9 Let $|a_x| = \max\{|a| : a \in V\}$ and $E_x = \{a_x a_i | a_x a_i \in E\}$ with $a_x \in V$. Then $a_k + a_l \in C$ for any $a_k a_l \in E - E_x$ for $n \geq 7$.

Proof: Let $|a_x| = \max\{|a| : a \in V\}$ and $E_x = \{a_x a_i | a_x a_i \in E\}$ with $a_x \in V$. If $|\overline{V_i}| = 2$ then we may assume that $\overline{V_i} = \{a_{i-1}, a_{i+1}\}$ for $a_i \in V$. Assume $a_x > 0$ (A similar argument works for $a_x < 0$). By lemma 2.3, $a_x + a_i \in C$ for any $a_x a_i \in E_x$. For all $a_k a_l \in E - E_x$, either there exists one vertex in $\{a_k, a_l\}$ (we may assume a_k) such that $a_k a_x \in E$, or $a_k a_x \notin E$ and $a_l a_x \notin E$.

Claim 1 If a_k and a_y are the end vertices of P_n then all the sums of the edges adjacent to a_k or a_y belong to C.

In fact, if a_k and a_y are the end vertices of P_n then $a_k + a_y \in E$ and $d_G(a_k) = d_G(a_y) = n - 2$. By lemma 2.3, $a_x + a_k \in C$. For all $a_k a_l \in E - E_x$, $(a_x + a_k) + a_l = a_x + (a_k + a_l) \notin S$. So $a_k + a_l \in \{a_x\} \cup \overline{V_x}$ or $a_k + a_l \in C$. Then there are at most three edges $a_k a_{l_i} \in E - E_x$ such that $a_k + a_{l_i} \in \{a_x\} \cup \overline{V_x}$ for $i = 1, 2, 3$ (Since $|\{a_x\} \cup \overline{V_x}| \leq 3$). And others belong to C.

If there exist three edges $a_k a_{l_i} \in E - E_x$ such that $a_k + a_{l_i} \in V$ then we may assume $a_k + a_{l_i} = a_{z_i}$ with $a_{z_i} \in \{a_x\} \cup \overline{V_x}$ and $i \in \{1, 2, 3\}$. Since $n \geq 7$ and $d_G(a_k) = n - 2 \geq 5$, there exists one edge $a_k a_{l_4} \in E - E_x - \{a_k a_{l_1}, a_k a_{l_2}, a_k a_{l_3}\}$ such that $a_{z_{i_0}} a_{l_4} \in E$ with $i_0 \in \{1, 2, 3\}$. Then $a_k + a_{l_4} \in C$ and $a_{z_{i_0}} + a_{l_4} \in S$ (see Figure 25).

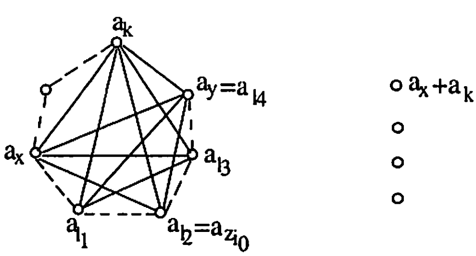


Figure 25

So $a_{z_{i_0}} + a_{l_4} = (a_k + a_{l_{i_0}}) + a_{l_4} = (a_k + a_{l_4}) + a_{l_{i_0}} \in S$, contradicting the fact $a_k + a_{l_4} \in C$.

It is more easy to get contradictions when there exist two or one edge $a_k a_{l_i} \in E - E_x$ such that $a_k + a_{l_i} \in V$ for $i \in \{1,2\}$. Thus, all the sums of the edges adjacent to a_k belong to C.

A similar argument works for a_y.

Thus, Claim 1 holds.

Claim 2 If $a_h + a_{l'} \in C$ and $a_h + a_{l''} \in C$ with $a_{l'} a_{l''} \in E$ then $a_h + a_l \in C$ for any $a_h a_l \in E$.

In fact, since $a_h + a_{l'} \in C$, $(a_h + a_{l'}) + a_l = (a_h + a_l) + a_{l'} \notin S$ for all $a_h a_l \in E - \{a_h a_{l'}, a_h a_{l''}\}$. Then $a_h + a_l \in \{a_{l'}\} \cup \overline{V_{l'}} \cup C$. Similarly, $a_h + a_l \in \{a_{l''}\} \cup \overline{V_{l''}} \cup C$ (see Figure 26).

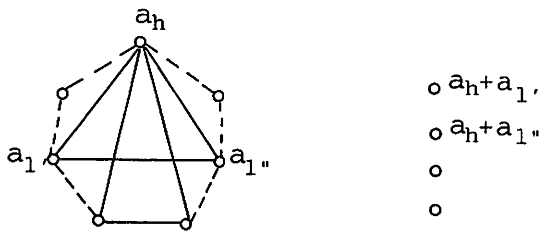


Figure 26

Since $a_h a_l \in E$, $(\{a_{l'}\} \cup \overline{V_{l'}}) \cap (\{a_{l''}\} \cup \overline{V_{l''}}) = \emptyset$. Thus, $a_h + a_l \in C$.

Thus, Claim 2 holds.

By Claim 1, if a_x is not an end vertex for $n \geq 7$ then Claim 2 works for any vertex in $V - \{a_x, a_k, a_y\}$ (see Figure 27,28,29).

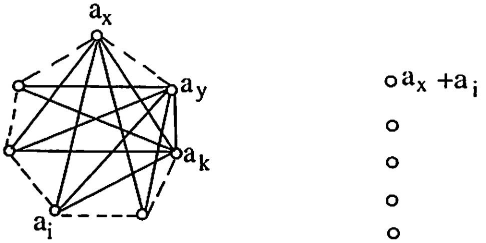


Figure 27

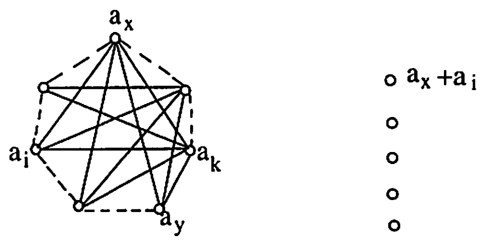


Figure 28

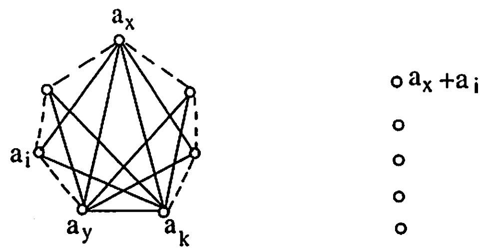


Figure 29

If a_x is an end vertex then assume that $a_x a_{x+1} \notin E$ and $a_k a_{k-1} \notin E$ (see Figure 30).

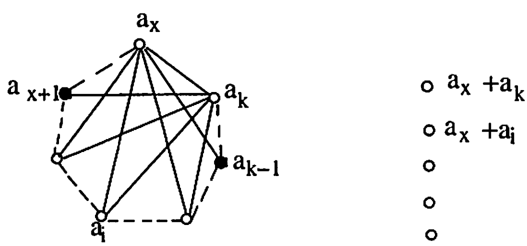


Figure 30

Firstly, Claim 2 works for every vertex in $V - \{a_x, a_{x+1}, a_k, a_{k-1}, a_y\}$. Secondly, Claim 2 works for a_{x+1} and a_{k-1}. Thus, $a_k + a_l \in C$ for any $a_k a_l \in E - E_x$ for $n \geq 7$.

Thus, Lemma 2.9 holds. □

Lemma 2.10 $\overline{P_n}$ is exclusive for $n \geq 7$. □

Lemma 2.11 $\zeta(\overline{P_n}) \geq 2n - 7$ for $n \geq 7$. □

Proof: Let $V = \{b_1, b_2, ..., b_n\}$. Without loss of generality, we can assume that $b_1 < b_2 < ... < b_n$. So $b_1 + b_2 < b_1 + b_3 < b_1 + b_4 < ... < b_1 + b_n < b_2 + b_n < b_3 + b_n < ... < b_{n-1} + b_n$. Let $C_0 = \{b_1+b_2, b_1+b_3, ..., b_1+b_n, b_2+b_n, ..., b_{n-1}+b_n\}$. Then there are at most four numbers which are not in S, but in C_0. On the other hand, the others in C_0 are the isolated vertices by Lemma 2.10. Thus, $\zeta(\overline{P_n}) \geq 2n - 7$ for $n \geq 7$. □

Lemma 2.12 $\sigma(\overline{P_n}) \leq 2n - 7$ for $n \geq 7$.

Proof: Let $V = \{a_1, a_2, ..., a_n\}$ and $S = V \cup C$, where C is the isolated set.

case 1: $n = 2k$ $(k \geq 4)$.
$a_i = (i - 1) \times 10 + 1, \quad i = 1, 2, 3, ..., n,$
$c_j = (j + 2) \times 10 + 2, \quad j = 1, 2, 3, ..., n - 3, n - 1, n + 1, n + 2, ..., 2n - 5,$
$C = \{c_1, c_2, ..., c_{n-3}, c_{n-1}, c_{n+1}, c_{n+2}, ..., c_{2n-5}\}$.
Let us verify this labelling is a sum labelling in detail.
(1) The vertices in S are distinct.
(2) For $\forall i, j \in \{1, 2, ..., n\}$ and $i \neq j$, $a_i + a_j = [(i + j - 4) + 2] \times 10 + 2$. Since $1 \leq i, j \leq n$ and $i \neq j$, $-1 \leq i + j - 4 \leq 2n - 5$. So $a_i a_j \notin E \iff a_i + a_j \notin C \iff a_i + a_j \in \{12, 22, 10n + 2, (n + 2) \times 10 + 2\} \iff i + j - 4 \in \{-1, 0, n - 2, n\}$. That is, $i + j - 4 = -1 \iff i + j = 3 \iff (i, j) \in \{(1, 2), (2, 1)\} \iff a_1 a_2 \notin E$; $i + j - 4 = 0 \iff i + j = 4 \iff (i, j) \in \{(1, 3), (3, 1)\} \iff a_1 a_3 \notin E$; $i + j - 4 \in \{n - 2, n\} \iff i + j \in \{n + 2, n + 4\} \iff \{(i, j), (j, i)\} \subseteq \{(\frac{n}{2} + 2, \frac{n}{2}), (\frac{n}{2}, \frac{n}{2} + 4), (\frac{n}{2} + 4, \frac{n}{2} - 2), (\frac{n}{2} - 2, \frac{n}{2} + 6), (\frac{n}{2} + 6, \frac{n}{2} - 4), ..., (4, n), (n, 2), ..., (3, n - 1), (n - 1, 5), (5, n - 3), (n - 3, 7), (7, n - 5), ..., (\frac{n}{2} - 1, \frac{n}{2} + 3), (\frac{n}{2} + 3, \frac{n}{2} + 1)\}$. So $P_n = a_{\frac{n}{2}+2} a_{\frac{n}{2}} a_{\frac{n}{2}+4} a_{\frac{n}{2}-2} a_{\frac{n}{2}+6} a_{\frac{n}{2}-4} \dots a_4 a_n a_2 a_1 a_3 a_{n-1} a_5 a_{n-3} a_7 a_{n-5} \dots a_{\frac{n}{2}-1} a_{\frac{n}{2}+3} a_{\frac{n}{2}+1}$.
Hence, for any $a_i a_j \notin E$, $a_i + a_j \notin S$; for any $a_i a_j \in E$, $a_i + a_j \in S$. Therefore, the labelling is a sum labelling of $\overline{P_n} \cup (2n - 7)K_1$ for $n = 2k$ and $k \geq 3$.

case 2: $n = 2k + 1$ $(k \geq 3)$.
$a_i = (i - 1) \times 10 + 1, \quad i = 1, 2, 3, ..., n;$
$c_j = (j + 2) \times 10 + 2, \quad j = 1, 2, 3, ..., n - 3, n - 1, n, n + 2, ..., 2n - 5;$
$C = \{c_1, c_2, ..., c_{n-3}, c_{n-1}, c_n, c_{n+2}, ..., c_{2n-5}\}$(For example Figure 31).
Let us verify this labelling is a sum labelling in detail.
(1) The vertices in S are distinct.
(2) For $\forall i, j \in \{1, 2, ..., n\}$ and $i \neq j$, $a_i + a_j = [(i + j - 4) + 2] \times 10 + 2$. Since $1 \leq i, j \leq n$ and $i \neq j$, $-1 \leq i + j - 4 \leq 2n - 5$. So $a_i a_j \notin E \iff a_i + a_j \notin C \iff a_i + a_j \in \{12, 22, 10n + 2, (n + 3) \times 10 + 2\} \iff i + j - 4 \in \{-1, 0, n - 2, n + 1\}$. That is, $i + j - 4 = -1 \iff i + j = 3 \iff (i, j) \in \{(1, 2), (2, 1)\} \iff a_1 a_2 \notin E$; $i + j - 4 = 0 \iff i + j = 4 \iff (i, j) \in \{(1, 3), (3, 1)\} \iff a_1 a_3 \notin E$; $i + j - 4 \in \{n - 2, n + 1\} \iff i + j \in \{n + 2, n + 5\} \iff \{(i, j), (i, j)\} \subseteq \{(\frac{n+3}{2} + 1, \frac{n-3}{2} + 1), (\frac{n-3}{2} + 1, \frac{n+3}{2} + 4), (\frac{n+3}{2} + 4, \frac{n-3}{2} - 2), (\frac{n-3}{2} -$

$2, \frac{n+3}{2} + 7), ..., (5, n), (n, 2), (2, 1), (1, 3), (3, n-1), (n-1, 6), (6, n-4)\}, ..., (n-2, 4)\}$. So $P_n = a_{\frac{n+3}{2}+1} a_{\frac{n-3}{2}+1} a_{\frac{n+3}{2}+4} a_{\frac{n-3}{2}-2} a_{\frac{n+3}{2}+7} \dots a_8 a_{n-3} a_5 a_n a_2 a_1 a_3 a_{n-1}$ $a_6 a_{n-4} \dots a_{n-2} a_4$.

Hence, for any $a_i a_j \notin E$, $a_i + a_j \notin S$; for any $a_i a_j \in E$, $a_i + a_j \in S$. Therefore, the labelling is a sum labelling of $\overline{P_n} \cup (2n-7)K_1$ for $n = 2k+1$ and $k \geq 3$. □

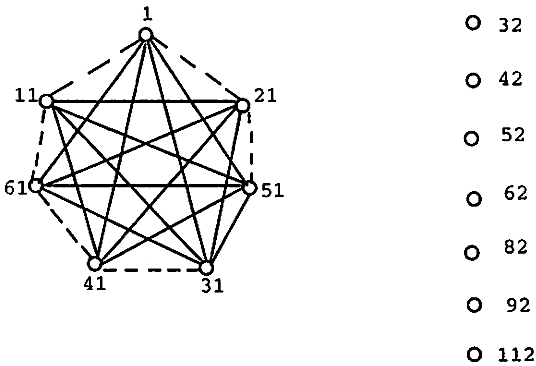


Figure 31

Theorem 2.1
$$\begin{cases} 0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1; \\ 1 = \zeta(\overline{P_5}) < \sigma(\overline{P_5}) = 2; \\ 3 = \zeta(\overline{P_6}) < \sigma(\overline{P_6}) = 4; \\ \zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 0, \quad n = 1, 2, 3; \\ \zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 2n - 7, \quad n \geq 7. \end{cases}$$

Corollary 2.1
$$\begin{cases} 0 = \zeta(\overline{F_5}) < \sigma(\overline{F_5}) = 1; \\ 2 = \zeta(\overline{F_6}) < \sigma(\overline{F_6}) = 3; \\ \zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 0, \ n = 3, 4; \\ \zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 2n - 8, n \geq 7. \end{cases}$$

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